# A Note on the Number of Hamiltonian Paths in Strong Tournaments

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**Abstract**

We prove that the minimum number of distinct hamiltonian paths in a strong tournament of order $n$ is $5^{\frac{n-1}{3}}$. A known construction shows this number is best possible when $n \equiv 1 \mod 3$ and gives similar minimal values for $n$ congruent to 0 and 2 modulo 3.

A tournament $T = (V, A)$ is an oriented complete graph. Let $h_p(T)$ be the number of distinct hamiltonian paths in $T$ (i.e., directed paths that include every vertex of $V$). It is well known that $h_P(T) = 1$ if and only if $T$ is transitive, and Rédei [3] showed that $h_p(T)$ is always odd. More generally, if $T$ is reducible (i.e., not strongly connected), then there exists a set $A \subset V$ such that every vertex of $A$ dominates every vertex of $V \setminus A$. If we denote the subtournament induced on a set $S$ as $T[S]$, then it is easy to see that $h_p(T) = h_p(T[A]) \cdot h_p(T[V \setminus A])$. Clearly, this process can be repeated to obtain $h_p(T) = h_p(T[A_1]) \cdot h_p(T[A_2]) \cdots h_p(T[A_t])$ where $T[A_1], \ldots, T[A_t]$ are the strong components of $T$. As a result, we generally consider $h_p(T)$ for strong tournaments $T$. In particular, we wish to find the minimal value of $h_p(T)$ as $T$ ranges over all strong tournaments of order $n$. Moon [1] bounded this value above and below with the following result.

**Theorem (Moon [1]).** *Let $h_p(n)$ be the minimum number of distinct hamiltonian paths in a strong tournament of order $n \geq 3$. Then*

$$\alpha^{n-1} \leq h_p(n) \leq \begin{cases} 3 \cdot \beta^{n-3} \approx 1.026 \cdot \beta^{n-1} & \text{for } n \equiv 0 \mod 3 \\ \beta^{n-1} & \text{for } n \equiv 1 \mod 3 \\ 9 \cdot \beta^{n-5} \approx 1.053 \cdot \beta^{n-1} & \text{for } n \equiv 2 \mod 3 \end{cases}$$

*where $\alpha = \sqrt[4]{6} \approx 1.565$ and $\beta = \sqrt[3]{5} \approx 1.710$.*

This lower bound was used by Thomassen [2] to establish a lower bound for the number of hamiltonian cycles in 2-connected tournaments.

**Theorem (Thomassen [2]).** *Every 2-connected tournament of order $n$ has at least $\alpha^{(\frac{n}{32}-1)}$ distinct hamiltonian cycles.*

We shall prove that the upper bound for $h_p(n)$ by Moon is, in fact, best possible, and consequently improve the lower bound on hamiltonian cycles in 2-connected tournaments found by Thomassen.

We will call a tournament $T$ *nearly transitive* when $V(T)$ can be ordered $v_1, v_2, \ldots, v_n$ such that $v_n \to v_1$ and all other arcs are of the form $v_i \to v_j$ with $i < j$. In other words, reversing the arc $v_n \to v_1$ gives the transitive tournament of order $n$. As noted by Moon [1], there is a bijection between partitions of $V \setminus \{v_1, v_n\}$ and hamiltonian paths that include the arc $v_n \to v_1$, and there is a unique hamiltonian path of $T$ that avoids this arc. Hence, there are $2^{n-2}+1$ distinct hamiltonian paths in a nearly transitive tournament of order $n$.

**Lemma 1.** *Let $T$ be a strong tournament of order $n \geq 5$. Then, either $T$ is nearly transitive, or there exist sets $A \subset V$ and $B \subset V$ such that*

- $|A| \geq 3$ *and* $|B| \geq 3$.
- $T[A]$ *and* $T[B]$ *are both strong tournaments.*
- $|A \cap B| = 1$ *and* $A \cup B = V$.

*Proof.* First, assume that $T$ is 2-connected. Choose vertices $C = \{x_0, x_1, x_2\}$ such that $T[C]$ is strong. Since $T$ is 2-connected, every vertex of $T$ has at least two in-neighbors and at least two out-neighbors. As each vertex $x_i$ has a single in- and out-neighbor on the cycle $C$, we conclude that each $x_i$ beats some vertex in $V \setminus C$ and is beaten by a vertex in $V \setminus C$. If $T - C$ is strong, then $A = C$ and $B = V \setminus \{x_0, x_1\}$ satisfy the lemma. Otherwise, let $W_1$ (resp. $W_t$) be the set of vertices in the initial (resp. terminal) strong component of $T - C$. As $T$ is 2-connected, at least two vertices of $C$ have in-neighbors in $W_t$, and at least two vertices of $C$ have out-neighbors in $W_1$. Thus, at least one vertex of $C$ has both in-neighbors in $W_t$ and out-neighbors in $W_1$. Without loss of generality, let this vertex be $x_0$. Then $C$ and $V \setminus \{x_1, x_2\}$ satisfy the lemma.

Next, assume that $T$ contains a vertex $v$ such that $T - v$ is not strong and that no sets $A$ and $B$ satisfy the lemma. Let $t$ be the number of strong components of $T - v$ and let $W_i$ be the set of vertices in the $i^{\text{th}}$ strong component. If $|W_1| \geq 3$, then choose a vertex $w \in W_1$ such that $v \to w$. Then $A = W_1$ and $B = \bigcup_{i=2}^{t} W_i \cup \{v, w\}$ satisfy the lemma. Similarly, if $|W_t| \geq 3$, then $A = \bigcup_{i=1}^{t-1} W_i \cup \{v, w\}$ and $B = W_t$ satisfy the lemma for any $w \in W_t$ such that $w \to v$ in $T$. Hence, since there does not exist a strong tournament on two vertices, we can assume that $W_1 = \{w_1\}$ and $W_t = \{w_t\}$ with $v \to w_1$ and $w_t \to v$. Now, let $W = \bigcup_{i=2}^{t-1} W_i$. If $T[W]$ contains a cyclic triple, let $A = \{u_1, u_2, u_3\} \subseteq W$ with $T[A]$ cyclic. In this case $A$ and $B = V \setminus \{u_2, u_3\}$ are sets which satisfy the lemma. So we can assume that $T[W]$ and hence $T - v$ are both transitive.

Finally, let $W^- = W \cap N^-(v)$ and $W^+ = W \cap N^+(v)$. If $W^+ \neq \emptyset$ and $W^- \neq \emptyset$, then $A = W^- \cup \{w_1, v\}$ and $B = W^+ \cup \{w_t, v\}$ satisfy the lemma. Otherwise, either $W^+ = \emptyset$ or $W^- = \emptyset$. If $W^+ = \emptyset$, then $N^+(v) = \{w_1\}$ and reversing the arc $vw_1$ gives a transitive tournament of order $n$, and if $W^- = \emptyset$, $N^-(v) = \{w_t\}$ and a transitive tournament of order $n$ is obtained by reversing the arc $w_t v$. In both cases, this implies that $T$ is nearly transitive. □

Our next lemma is probably widely known. The proof is an easy inductive extension of the well known fact that in a tournament, every vertex $v$ not on a given path $P$ can be inserted into $P$. We include the proof for completeness.

**Lemma 2.** *Let $P = v_1 \to v_2 \to \cdots \to v_k$ and $Q = u_1 \to u_2 \to \cdots \to u_m$ be vertex disjoint paths in a tournament $T$. Then there exists a path $R$ in $T$ such that*

- $V(R) = V(P) \cup V(Q)$
- *For all $1 \leq i < j \leq k$, $v_i$ precedes $v_j$ on $R$*
- *For all $1 \leq i < j \leq m$, $u_i$ precedes $u_j$ on $R$.*

*Proof.* Note that we allow the special case where $m = 0$; in this case the path $Q$ is a path on 0 vertices, and $R = P$ satisfies the lemma trivially.

The remainder of the proof is by induction on $m$. For $m = 1$, let $i$ be the minimal index such that $u_1 \to v_i$. If no such $i$ exists then $R = v_1 \to \cdots \to v_k \to u_1$. If $i = 1$, then $R = u_1 \to v_1 \to \cdots \to v_k$. In all other cases, $R = v_1 \to \cdots \to v_{i-1} \to u_1 \to v_i \to \cdots \to v_k$. So we assume the result for all paths $Q'$ of order at most $m-1$. Let $Q' = u_1 u_2 \cdots u_{m-1}$ and apply the induction hypothesis using the paths $P$ and $Q'$ to obtain a path $R'$ satisfying the lemma. Next, we repeat the above argument with the portion of $R'$ beginning at $u_{m-1}$ and the vertex $u_m$. □

**Theorem 1.** *Let $h_p(n)$ be the minimum number of distinct hamiltonian paths in a strong tournament of order $n$. Then*

$$h_p(n) \geq \begin{cases} 3 \cdot \beta^{n-3} \approx 1.026 \cdot \beta^{n-1} & \textit{for } n \equiv 0 \mod 3 \\ \beta^{n-1} & \textit{for } n \equiv 1 \mod 3 \\ 9 \cdot \beta^{n-5} \approx 1.053 \cdot \beta^{n-1} & \textit{for } n \equiv 2 \mod 3 \end{cases}$$

*where $\beta = \sqrt[3]{5} \approx 1.710$.*

*Proof.* The proof is by induction. The result is easily verified for $n = 3$ and $n = 4$, and as observed by Thomassen [2], $h_p(5) = 9$. So assume the result for all tournaments of order at most $n - 1$ and let $T$ be a strong tournament of order $n \geq 6$.

As $T$ is strong, by Lemma 1 there are two possibilities. If $T$ is a nearly transitive tournament. Then $h_p(T) = 2^{n-2} + 1$, and for $n \geq 6$, this value exceeds $9 \cdot \beta^{n-5}$. Otherwise, there exist sets $A$ and $B$ such that $T[A]$ and $T[B]$ are strong tournaments with $|A| = a \geq 3$,

$|B| = b \geq 3$, $A \cup B = V$ and $|A \cap B| = 1$. Let $\{v\} = A \cap B$, and let $H_A = P_1 v P_2$ be a hamiltonian path of $T[A]$, and $H_B = Q_1 v Q_2$ a hamiltonian path of $T[B]$. We apply Lemma 2 twice, and obtain paths $R_1$ and $R_2$ such that $V(R_i) = V(P_i) \cup V(Q_i)$, and the vertices of $P_i$ (resp. $Q_i$) occur in the same order on $R_i$ as they do on $P_i$ (resp. $Q_i$). Now $H = R_1 v R_2$ is a hamiltonian path of $T$. Furthermore, distinct hamiltonian paths of $T[A]$ (resp. $T[B]$) give distinct hamiltonian paths of $T$. Hence by the induction hypothesis,

$$h_p(T) \geq h_p(T[A]) h_p(T[B]) \geq \beta^{a-1}\beta^{b-1} \geq \beta^{n-1}$$

Furthermore, strict inequality holds unless $a \equiv 1 \mod 3$ and $b \equiv 1 \mod 3$, which implies that $n \equiv 1 \mod 3$ as well. When $n \equiv 2 \mod 3$, there are two cases, $a \equiv b \equiv 0 \mod 3$ and without loss of generality $a \equiv 2 \mod 3$ and $b \equiv 1 \mod 3$. Using the same induction arguments above, both cases give $h_p(T) \geq 9 \cdot \beta^{n-5}$. Finally, in the case that $n \equiv 0 \mod 3$, we again have two possibilities, $a \equiv b \equiv 2 \mod 3$ and without loss of generality $a \equiv 1 \mod 3$ and $b \equiv 0 \mod 3$. In this case we find that $h_p(T) \geq \min(81 \cdot \beta^{n-9}, 3 \cdot \beta^{n-3}) = 3 \cdot \beta^{n-3}$. □

The construction utilized by Moon [1] in Theorem  gives the identical upper bound for $h_p(n)$ and equality is established.

**Corollary 1.** *Let $h_p(n)$ be the minimum number of distinct hamiltonian paths in a strong tournament of order $n$. Then*

$$h_p(n) = \begin{cases} 3 \cdot \beta^{n-3} \approx 1.026 \cdot \beta^{n-1} & \text{for } n \equiv 0 \mod 3 \\ \\ \beta^{n-1} & \text{for } n \equiv 1 \mod 3 \\ \\ 9 \cdot \beta^{n-5} \approx 1.053 \cdot \beta^{n-1} & \text{for } n \equiv 2 \mod 3 \end{cases}$$

*where $\beta = \sqrt[3]{5} \approx 1.710$.*

Additionally, this result improves Thomassen's bound on hamiltonian cycles in 2-connected tournaments.

**Corollary 2.** *Every 2-connected tournament of order $n$ has at least $\beta^{\frac{n}{32}-1}$ distinct hamiltonian cycles, with $\beta = \sqrt[3]{5} \approx 1.710$.*

# References


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