On the number of possible row and column sums of 0,1-matrices

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Submitted: Aug 9, 2005; Accepted: Apr 4, 2006; Published: Apr 18, 2006
Mathematics Subject Classification: 05A15

Abstract

For n a positive integer, we show that the number of of $2n$-tuples of integers that are the row and column sums of some $n \times n$ matrix with entries in $\{0, 1\}$ is evenly divisible by $n+1$. This confirms a conjecture of Benton, Snow, and Wallach.

We also consider a q-analogue for $m \times n$ matrices. We give an efficient recursion formula for this analogue. We prove a divisibility result in this context that implies the $n + 1$ divisibility result.

1 Introduction

We study the number $p(m, n)$ of $(m + n)$-tuples of integers that are the row and column sums of some $m \times n$ matrix with entries in $\{0, 1\}$. For each $n \geq 1$, the sequence $\{p(m, n)\}_{m=1}^{\infty}$ is a linear recursion of degree n. Moreover, this recursion is annihilated by the polynomial $(T - (n + 1))^n$. It follows that if $1 \leq n \leq m$, then $p(m, n)$ is evenly divisible by $(n + 1)^{m-n+1}$. This confirms a conjecture of Benton, Snow, and Wallach.

For positive integers m and n, let $\mathcal{M} = \mathcal{M}_{m,n}$ be the set of $m \times n$ matrices with entries in $\{0, 1\}$. For M in $\mathcal{M}$, we write $M = (M_{ij})$.

We have two vector-valued functions on $\mathcal{M}$: the vector $x(M) = (x_1, \ldots, x_m)$ of row sums, where $x_i = \sum_{1 \leq j \leq n} M_{ij}$ for $1 \leq i \leq m$, and the vector $y(M) = (y_1, \ldots, y_n)$ of column sums, where $y_j = \sum_{1 \leq i \leq m} M_{ij}$ for $1 \leq j \leq n$.

Define $\mathcal{RC} = \mathcal{RC}_{m,n}$ to be the set of pairs of row and column sums $(x(M), y(M))$ as M ranges over $\mathcal{M}$. Our main result concerns the cardinality $p(m, n)$ of $\mathcal{RC}_{m,n}$.

Theorem 1 *We have*

1. $p(1,1) = 2$.
2. $p(m,n) = p(n,m)$ *for* $m,n \geq 1$.
3. *If* $1 \leq n \leq m$, *then* $p(m,n) = \sum_{1\leq i\leq n}(-1)^{i+1}\binom{n}{i}(n+1)^i p(m-i,n)$.

Of these statements, part (1) is clear, and part (2) follows by taking transpose, for $x(M^t) = y(M)$ and $y(M^t) = x(M)$.

Part (3) says that, for each $n \geq 1$, the sequence $\{p(m,n)\}_{m=1}^{\infty}$ is a linear recursion of degree n that is annihilated by the polynomial $(T-(n+1))^n$. Note that, for any fixed n, the recursion (3) is equivalent to $p(m,n) = r_n(m)(n+1)^m$ for some polynomial $r_n(m)$ of degree $\leq n-1$.

Part (3) implies the following corollary.

Corollary 2 *The number* $p(m,n)$ *is evenly divisible by* $(n+1)^{m-n+1}$ *if* $1 \leq n \leq m$.

Indeed each of the n terms in the sum representing $p(m,n)$ is divisible by this quantity.

A second consequence of part (3) is an efficient algorithm for computing $p(m,n)$.

Algorithm 3 *We construct a table of the values* $p(i,j)$, *for* $1 \leq i,j \leq m$ *by induction on* j. *First we fill in* $p(i,1) = 2^i$, *for* $1 \leq i \leq m$. *Next, for a given* $j \leq m$, *having filled in* $p(i,j')$ *for* $1 \leq j' < j$, *we fill in* $p(i,j)$ *by induction on* i, *using part* (2) *if* $i \leq j$ *and part* (3) *if* $i > j$.

2 A generalization

We mention a mild generalization of Theorem 1 and its corollary. Define the polynomial $P = P_{m,n}(q) = \sum_{(x,y)\in\mathcal{RC}_{m,n}} q^{|x|}$, where $|x| = x_1 + \cdots + x_m$. We recover $p(m,n)$ by evaluating the polynomial $P_{m,n}$ at $q = 1$.

Theorem 4 *We have*

1. $P_{1,1} = 1 + q$.
2. $P_{m,n} = P_{n,m}$ *for* $m,n \geq 1$.
3. *If* $1 \leq n \leq m$, *then* $P_{m,n} = \sum_{1\leq i\leq n}(-1)^{i+1}\binom{n}{i}(1+q+\cdots+q^n)^i P_{m-i,n}$.
4. *If* $1 \leq n \leq m$, *then the polynomial* $P_{m,n}$ *is evenly divisible by* $(1+q+\cdots+q^n)^{m-n+1}$ *in* $\mathbb{Z}[x]$.

Part (4) answers a conjecture of J. Benton, R. Snow, and N. Wallach in [1].

3 Start of the proof

Let $\mathbb{N} = \{0, 1, \dots\}$. Define the weight of a matrix N to be the sum of its entries, and write $|N|$ for the weight of N. With this definition, we have $|x(M)| = |M| = |y(M)|$ for $M \in \mathcal{M}$. Thus, a necessary condition for x and y to be row and column sums of a matrix is that they have the same weight.

Clearly, the row sums of a member of $\mathcal{M}$ are at most n. Conversely, if $x = (x_1, \dots, x_m)$ and $0 \leq x_i \leq n$, let $R = R(x)$ be the $m \times n$ matrix such that $R_{ij} = 1$ if $1 \leq j \leq x_i$ and $R_{ij} = 0$ otherwise. Then R lies in $\mathcal{M}$ and has row sums equal to x. This proves:

Lemma 5 *Let $x = (x_1, \dots, x_m) \in \mathbb{N}^m$. Then x is the vector of row sums of an $m \times n$ matrix with entries in $\{0, 1\}$ if and only if $x_i \leq n$ for all i.*

Let a_j be the number of rows of R that have exactly j ones. Write $a = (a_0, \dots, a_n) = a(x)$ in $\mathbb{N}^{n+1}$. We note that $|a| = m$, and write $\binom{m}{a}$ for the multinomial coefficient $\frac{m!}{a_0! \cdots a_n!}$ With this notation, we have the following lemma.

Lemma 6 *Let a in $\mathbb{N}^{n+1}$ satisfy $|a| = m$. Then the number of x in $\mathbb{N}^m$ such that $a(x) = a$ is $\binom{m}{a}$.*

Let $\lambda = (\lambda_1, \dots, \lambda_n) = \lambda(x)$ be the column sums of the matrix R constructed above. It satisfies the dominance condition:

$$\lambda_1 \geq \cdots \geq \lambda_n. \tag{1}$$

Note that a in $\mathbb{N}^{n+1}$ with $|a| = m$ determines a dominant λ in $\mathbb{N}^n$ with $m \geq \lambda_1$, and vice versa. For, given λ, set $\lambda_0 = m$ and $\lambda_{n+1} = 0$, and define $a_j = \lambda_j - \lambda_{j+1}$, for $j = 0, \dots, n$. Conversely, given a in $\mathbb{N}^{n+1}$, define $\lambda_j = a_j + \cdots + a_n$.

The weights of these vectors are related by $|x| = |\lambda| = \sum_{0 \leq j \leq n} j a_j$.

Given y, λ in $\mathbb{N}^n$ with λ dominant, we define $y \preceq \lambda$ if

$$y_1 + \cdots + y_j \leq \lambda_1 + \cdots + \lambda_j, \tag{2}$$

for all j in the range $1 \leq j \leq n$.

The symmetric group S_n acts on $\mathbb{N}^n$ by permuting coordinates. For $y \in N^n$ and $\sigma \in S_n$, we set $y\sigma = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$.

The next result, proved in [2, Corollary 6.2.5] or [3, Theorem 16.1], gives necessary and sufficient conditions for a pair of vectors to lie in $\mathcal{RC}_{m,n}$.

Lemma 7 *Let x in $\mathbb{N}^m$ be the vector of row sums of a matrix in $\mathcal{M}$, and set $\lambda = \lambda(x)$. Then $(x, y) \in \mathcal{RC}$ if and only if $y \in \mathbb{N}^n$ satisfies*

(i) $|y| = |\lambda|$, *and*

(ii) $y\sigma \preceq \lambda$ *for all* $\sigma \in S_n$.

Let $N(\lambda)$ be the number of $y \in \mathbb{N}^n$ that satisfy (i) and (ii). Then

$$P_{m,n}(q) = \sum_{x \in \{0,\ldots,n\}^m} N(\lambda(x)) q^{|x|}.$$

Combined with Lemma 6, this gives:

$$P_{m,n}(q) = \sum_{\substack{a \in \mathbb{N}^{n+1} \\ |a|=m}} \binom{m}{a} N(\lambda) q^{a_1+2a_2+\cdots+na_n}. \tag{3}$$

4 Key Lemma

Lemma 8 *Let $n \geq 1$. There is a polynomial $G = G_n$ in $\mathbb{Q}[z_1, \ldots, z_n]$ of total degree $\leq n-1$ such that $N(\lambda) = G(\lambda_1, \ldots, \lambda_n)$ for any dominant $\lambda = (\lambda_1, \ldots, \lambda_n)$ in $\mathbb{N}^n$.*

To count $N(\lambda)$, we will condition on the first term y_1 of the vector y. We will need a subsidiary function. Let $N(\lambda; t)$ be the number of solutions of (i) and (ii) with $y_1 = t$. By definition, $N(\lambda) = \sum_{t \geq 0} N(\lambda; t)$.

We need one more definition to state the next lemma. Suppose $\lambda = (\lambda_1, \ldots, \lambda_n)$ has n parts, and $\lambda_{j+1} < t \leq \lambda_j$. Then we define $\mu(t)$ with $n-1$ parts to be

$$\mu(t) = (\lambda_1, \ldots, \lambda_{j-1}, \lambda_j + \lambda_{j+1} - t, \lambda_{j+2}, \ldots, \lambda_n).$$

(In the definition of $\mu(t)$, λ_j and λ_{j+1} have been removed and $\lambda_j + \lambda_{j+1} - t$ has been inserted.) Note that if λ is dominant, then so also is $\mu(t)$ since $\lambda_j > \lambda_j + \lambda_{j+1} - t \geq \lambda_{j+1}$.

Lemma 9 *We have:*

(a) If $t < \lambda_n$ or if $t > \lambda_1$, then $N(\lambda; t) = 0$.

(b) $N(\lambda; \lambda_n) = N((\lambda_1, \ldots, \lambda_{n-1}))$.

(c) Suppose that $\lambda_{j+1} < t \leq \lambda_j$. Then $N(\lambda; t) = N(\mu(t))$.

Proof. If $y_1 > \lambda_1$ then (ii) is violated. Suppose y satisfies (i) and $y_1 < \lambda_n$. Then

$$y_2 + y_3 + \cdots + y_n > \lambda_1 + \lambda_2 + \cdots + \lambda_{n-1},$$

thus (ii) is violated if $\sigma(n) = 1$. Therefore $N(\lambda, y_1) = 0$, proving (a), and we turn to (b).

Set $\lambda' = (\lambda_1, \ldots, \lambda_{n-1})$. We claim that the correspondence

$$(y_1, y_2 \ldots, y_n) \longleftrightarrow (y_2 \ldots, y_n)$$

gives a bijection between the sets counting $N(\lambda; y_1)$ and $N(\lambda')$. One direction follows by definition: if $(y_1, \ldots, y_n)$ is counted by $N(\lambda)$, then $(y_2, \ldots, y_n)$ is counted by $N(\lambda')$.

Conversely, suppose that $(y_2, \ldots, y_n)$ is counted by $N(\lambda')$. Now (i) (for y and λ) follows since $y_1 = \lambda_n$. To prove (ii), let $\sigma \in S_n$. Set $k = \sigma^{-1}(1)$. Now

$$\begin{aligned} y_{\sigma(1)} + \cdots + y_{\sigma(j)} &\leq (\lambda_1 + \cdots + \lambda_{j-1}) + \lambda_n \\ &\leq \lambda_1 + \cdots + \lambda_j \end{aligned}$$

if $j \geq k$. The inequality is clear if $j < k$.

Part (c) is proved using the same correspondence used in part (b). The straightforward but tedious calculation is omitted. ■

Proof of Lemma 8. Suppose $n = 1$ and let $\lambda = (\lambda_1)$. Then $N(\lambda_1) = 1$, a polynomial of degree 0.

Thus the lemma holds for $n = 1$. We proceed by induction to prove it for all n. Suppose the lemma has been proved for n and we wish to prove it for $n + 1$.

We break up the sum that counts $N(\lambda)$, by conditioning on y_1. By Lemma 9(a), it is enough to consider y_1 in the range $\lambda_n \leq y_1 \leq \lambda_1$. Either $y_1 = \lambda_n$, or $\lambda_{j+1} < y_1 \leq \lambda_j$ for a unique j in the range $1 \leq j < n$, and therefore

$$N(\lambda) = N(\lambda; \lambda_n) + \sum_{1 \leq j < n} \sum_{\lambda_{j+1} < t \leq \lambda_j} N(\lambda; t).$$

In view of Lemma 9(b) and (c), this yields

$$N(\lambda) = N((\lambda_1, \ldots, \lambda_{n-1})) + \sum_{1 \leq j < n} \sum_{\lambda_{j+1} < t \leq \lambda_j} N(\mu(t)). \tag{4}$$

To see that $N(\lambda)$ is a polynomial of degree at most n, it suffices to show that each term on the right is a polynomial of total degree at most n. This is true for the first term $N((\lambda_1, \ldots, \lambda_{n-1}))$ by the inductive hypothesis.

Each of the subsequent terms is itself a sum. By the inductive hypothesis, each summand in each term is a polynomial of degree $\leq n - 1$. But, for any polynomial f, we have that $\sum_{x<t\leq y} f(t)$ is a polynomial in x and y of degree $\leq \deg f + 1$.

By induction and (4) it follows that the coefficients of G are rational numbers. This proves the lemma. ■

5 End of the proof

Since G is a polynomial of degree $\leq n - 1$ by Lemma 8, so also is H defined by $H(a_0, a_1, \ldots, a_n) = G_n(\lambda_1, \ldots, \lambda_n)$, since the transformation from λ to a is linear.

By (3) we have

$$P_{m,n} = \sum_{\substack{a \in \mathbb{N}^{n+1} \\ |a| = m}} \binom{m}{a} H(a_0, \ldots, a_n) q^{a_1 + \cdots + n a_n}. \tag{5}$$

Proof of Theorem 4. We are free to assume $n \leq m$.

We define the function E of the variables $z_0, \ldots, z_n$ by

$$E(z_0, \ldots, z_n) = \sum_{\substack{a \in N^{n+1} \\ |a|=m}} \binom{m}{a} H(a_0, \ldots, a_n) e^{a_0 z_0 + \cdots + a_n z_n} \ . \tag{6}$$

By (5) and (6), we have $P_{m,n}(q) = E(0, \log(q), 2\log(q), \ldots, n\log(q))$.

The following lemma is proved by induction.

Lemma 10 *Let $H \in \mathbb{Q}[z_0, \ldots, z_n]$ be a polynomial. Write $z = (z_0, \ldots, z_n)$ and $a = (a_0, \ldots, a_n)$, and set $a \cdot z = a_0 z_0 + \cdots + a_n z_n$. Then there is a linear differential operator D in $z_0, \ldots, z_n$ such that $H(z)e^{a \cdot z} = De^{a \cdot z}$. Moreover,* $\deg(D) = \deg(H)$.

By the lemma, we have

$$E(z) \ = \ \sum_{\substack{a \in N^{n+1} \\ |a|=m}} \binom{m}{a} De^{a \cdot z} = D\left(\sum \binom{m}{a} e^{a \cdot z}\right) \ .$$

By the multinomial theorem

$$\sum_{\substack{a \in N^{n+1} \\ |a|=m}} \binom{m}{a} e^{a \cdot z} = (e^{z_0} + \cdots + e^{z_n})^m,$$

whence E is $(e^{z_0} + \cdots + e^{z_n})^{m-n+1}$ times a polynomial $f_1(m, e^{z_0}, \ldots, e^{z_n})$ whose degree in m is $\leq n - 1$.

Set $f(m, q) = f_1(m, 1, q, \ldots, q^n)$. When evaluated at $z_i = i\log(q)$, $e^{z_0} + \cdots + e^{z_n}$ becomes $(1 + q + \cdots + q^n)$, whence $P_{m,n} = f(m, q)(1 + q + \cdots + q^n)^{m-n+1}$. Since $f(m, q)$ is a polynomial in m of degree at most $n - 1$, part (3) follows immediately.

Set $\pi = (1 + q + \cdots + q^n)^{n-m+1}$. Finally, to prove part (4), it remains to show that, for each m, the coefficients of $f(m, q)$, as a polynomial in q, are integers.

One way to see this is to regard $f = P_{m,n}/\pi$ as a power series identity and formally equate coefficients of q^i, because π is a polynomial in q with constant term 1. Theorem 4 is proved. ■

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