Constructive Upper Bounds for Cycle-Saturated Graphs of Minimum Size

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Abstract

A graph G is said to be C_l-saturated if G contains no cycle of length l, but for any edge in the complement of G the graph $G+e$ does contain a cycle of length l. The minimum number of edges of a C_l-saturated graph was shown by Barefoot et al. to be between $n + c_1\frac{n}{l}$ and $n + c_2\frac{n}{l}$ for some positive constants c_1 and c_2. This confirmed a conjecture of Bollobás. Here we improve the value of c_2 for $l \geq 8$.

1 Introduction

We let $G = (V, E)$ be a graph on $|V| = n$ vertices and $|E| = m$ edges. We denote the cycle on l vertices by C_l, and the complete graph on t vertices by K^t. The graph G is said to be F-saturated if G contains no copy of F as a subgraph, but for any edge e in the complement of G, the graph $G+(e)$ contains a copy of F, where $G+(e)$ denotes the graph $(V, E \cup e)$. For a subgraph F we will denote the minimum size of an F-saturated graph by $sat(n, F)$. In 1964 Erdős, Hajnal and Moon [10] determined the *minimum* number of edges in a graph that is K^t-saturated. This number, $sat(n, K^t)$, is $(t-2)(n-1) - \binom{t-2}{2}$ and arises from the graph $K^{t-2} + \overline{K}^{n-t+2}$, where $+$ denotes the join. Determining the exact value of this function for a given graph F has been quite difficult, and is known for relatively few graphs. Kászonyi and Tuza in [12] proved the best known general upper bound for $sat(n, F)$.

Cycle-saturated graphs of minimum size have been considered by various authors. The case $l = 3$ is covered by the result of Erdős, Hajnal and Moon [10]. The case $l = 4$ was first considered by L.T. Ollmann [14] where he proved that $sat(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor$ for $n \geq 5$. Later, Z. Tuza [16] gave a shortened proof of this result. Recently, the value of $sat(n, C_5)$ was announced by Y. Chen, [6]. In 1972 Bondy [5] showed that $sat(n, C_n) \geq \lceil \frac{3n}{2} \rceil$. Later results by various authors [7, 8, 9] showed that $sat(n, C_n) = \lfloor \frac{3n+1}{2} \rfloor$ for $n \geq 53$. No other exact values are known.

In 1996, Barefoot, Clark, Entringer, Porter, Székely and Tuza [1] obtained bounds for $sat(n, C_l)$ for all $l \neq 8$ or 10 and n sufficiently large. They showed that $n + c_1 \frac{n}{l} \leq sat(n, C_l) \leq n + c_2 \frac{n}{l}$ for some positive constants c_1 and c_2. This confirmed a conjecture of Bollobás from 1978. In particular, for l odd and $l \geq 9$ they showed $sat(n, C_l) \leq n\big(1 + \frac{6}{l-3}\big) + O(l^2)$. For $l = 12$ they showed that $sat(n, C_{12}) \leq n\frac{29}{22} + \frac{99}{22}$. For $l \geq 14, l \equiv 0 \mod 2$ they showed that $sat(n, C_l) \leq n\big(1 + \frac{4}{l-2}\big) + O(l^3)$. Finally, for $l \geq 20, l \equiv 4 \mod 8$ they showed that $sat(n, C_l) \leq n\big(\frac{5}{4} + \frac{3}{4l-4}\big) + \frac{l}{2}$. In terms of a lower bound, they showed for $l \geq 5$ that $sat(n, C_l) \geq n\big(1 + \frac{1}{2l+8}\big)$.

We will provide an upper bound for the function $sat(n, C_l)$ that improves the upper bound given in [1] for most values of l. We improve the upper bound via several constructions. In our first construction we consider l even and $l \geq 10$ (thus giving an upper bound for $l = 10$), and in the second construction we consider l odd and $l \geq 17$. Finally we supplement these results by a construction valid for all $l \geq 5$ which results in new upper bounds for $sat(n, C_l)$ when $l = 8, 9, 11, 13$ and 15. Table 1 summarizes all best known results.

For any undefined terms we refer the reader to [3].

2 The Generalized Wheel Construction

2.0.1 The Even Case: $W(n, 2k+2)$

The figure below will help illustrate this graph which we refer to as the Generalized Wheel (or just the wheel for short) and adopt the terminology of the bicycle wheel in describing

C_l-saturated graphs of minimum size			
l	$sat(n, C_l)$	$n \geq$	Reference
3	$= n - 1$	3	[10]
4	$\lfloor \frac{3n-5}{2} \rfloor$	5	[14, 16]
5	$\lceil \frac{10n-10}{7} \rceil$	21	[6]
6	$\leq \frac{3n}{2}$	11	[1]
7	$\leq \frac{7n+12}{5}$	10	[1]
8,9,11,13,15	$\leq \frac{3n}{2} + \frac{l^2}{2}$	$2l$	Theorem 3
≥ 10 and $\equiv 0 \mod 2$	$\leq \left(1 + \frac{2}{l-2}\right)n + \frac{5l^2}{4}$	$3l$	Theorem 1
≥ 17 and $\equiv 1 \mod 2$	$\leq \left(1 + \frac{2}{l-3}\right)n + \frac{5l^2}{4}$	$7l$	Theorem 2
n	$\lfloor \frac{3n+1}{2} \rfloor$	20	[7, 8, 9, 13]

Table 1: A Summary of Results for $sat(n, C_l)$

the graph.

To construct a C_{2k+2}-saturated graph $W(n, 2k+2)$ ($k \geq 4$), we proceed as follows. We begin with a set of k vertices, $\{h_1, h_2, \ldots h_k\}$, that form a clique, and refer to this clique as the **hub**. Surrounding the hub exists a cycle, R, of length sk for some $s \geq 4$. We will refer to this cycle as the **rim**. Each k^{th} vertex of the rim will be joined by an edge, called a **spoke**, to the hub. Thus the number of spokes is equal to s. The vertex on the rim that is adjacent to a spoke will be referred to as a **spoke-nut**. We label the vertices of the rim as follows, $R = \{n_{1,\alpha}, r_{1,1}, r_{1,2}, \ldots r_{1,k-1}, n_{2,\beta}, r_{2,1}, r_{2,2}, \ldots n_{s,\omega}, r_{s,1}, r_{s,2}, \ldots r_{s,k-1}\}$. Here we have listed the vertices in a clockwise fashion with spoke-nut vertices denoted by $n_{i,\kappa}$, and the remaining vertices by $r_{p,q}$. For vertices denoted $n_{i,\kappa}$ the subscript i refers to its placement on the wheel and the subscript κ denotes the subscript of the vertex in the hub to which it is connected - i.e. $n_{i,\kappa} \sim h_\kappa$. For vertices denoted $r_{p,q}$, the subscript p denotes the spoke-nut, $n_{p,\kappa}$, preceding it and the subscript q the distance along the rim from $n_{p,\kappa}$. We place the following restriction on the spokes of the wheel, indicating this through the subscripts of the spoke-nuts.

Rule 1 Given four consecutive spoke-nut vertices $n_{i,\alpha}, n_{i+1,\beta}, n_{i+2,\kappa}, n_{i+3,\delta}$ we require that $\alpha, \beta, \kappa, \delta$ are all distinct.

We will call spokes s_i, s_{i+1} *consecutive* if s_i has an end-vertex $n_{i,\alpha}$ and s_{i+1} has an end-vertex $n_{i+1,\beta}$.

If $k \geq 7$, Rule 1 may be observed regardless of the number of spokes used, and thus the graph just described has $n \equiv 0 \mod k$ vertices. When $n \equiv a \mod k$ we make the following adjustment to the graph just described. We select a set of a vertices from the

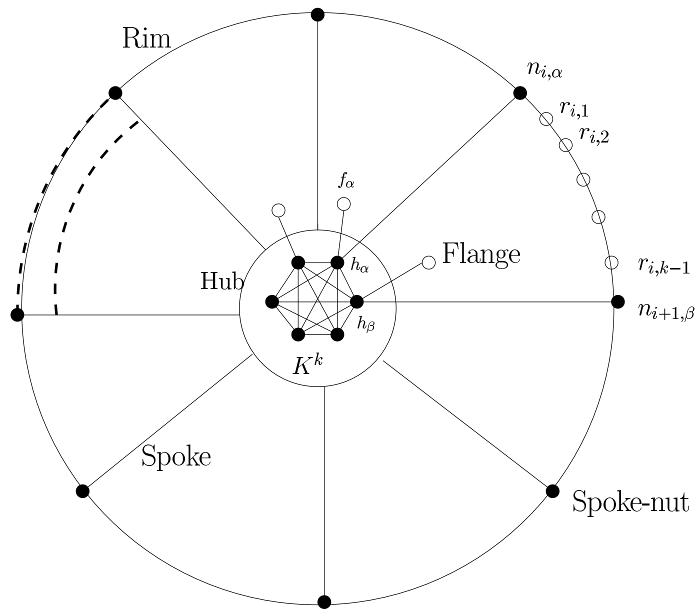

Figure 1: The Even Generalized Wheel - Cycle-Saturated Graph

hub and to each of these vertices, $\{h_1, h_2, \ldots h_a\}$, in the hub we attach a pendant edge, referred to as a **flange**, with end vertices $\{f_1, f_2, \ldots f_a\}$. Thus $h_i f_i$ is an edge for all i, $0 \le i \le a$. We will refer to these vertices as **flange vertices**. (Thus, when $a = 0$ no adjustment is made.)

If $4 \le k \le 6$, Rule 1 may force the number of spokes to be a multiple of four, and thus the number of vertices not in the hub is a multiple of $4k$, and thus the graph just described has $n - k \equiv 0 \mod 4k$ vertices on the rim. If $n - k \equiv a \mod 4k$ we make the following adjustment to the graph. We evenly distribute the a vertices into k flange sets, $F_1, F_2, \ldots, F_k$, of size $a_1, a_2, \ldots, a_k$ and on each set, F_i, we construct a clique and completely join it to the vertex in the hub labeled h_i. See Figure 1.

We now show that this graph is C_{2k+2}-saturated.

Lemma 1 *For $k \ge 4$ the graph $W(n, 2k+2)$ contains no cycle of length $l = 2k+2$.*

Proof: First note that a flange vertex may not lie on a cycle of length l as the corresponding hub vertex is a cut-vertex and no flange set contains more than k vertices. As $s \ge 4$, there is no cycle of length l comprised of edges solely from the rim. This, together with the fact that the hub contains only k vertices, implies that if such a cycle exists, it must use a spoke. As the set of spokes form an edge-cut of the graph $W(n, 2k+2)$, such a cycle must in fact use an even number of spokes. If the number of spokes used is four or more then the number of vertices involved in any cycle will be strictly greater than l. To see this note that upon using four, or more, spokes we use a corresponding number of spoke-nuts. The number of vertices used along the rim between any two distinct spoke-nuts is at least $k - 1$ and thus the number of vertices used from the rim in such a cycle is at least $2k + 2$ in addition to a positive number of vertices from the hub.

Thus, the number of spokes used in such a cycle must be exactly two. If the two spokes used are consecutive then the cycle contains $k+1$ vertices from the rim and at most k from the hub. Thus such a cycle has length at most $2k+1 < l$. If the spokes are more than one apart then any cycle containing them must use at least $3k+1 > l$ vertices from the rim. Thus the two spokes used are exactly one apart, say s_i, s_{i+2}. Notice that any cycle containing them uses exactly $2k+1$ vertices from the rim. Thus to create a cycle of length $2k+2$ we must use only one vertex from the hub, which would imply that the two spokes meet in a common vertex. However, in constructing $W(n, 2k+2)$ we have forbidden this to occur for spokes this close. Thus no cycle of length l exists. $\square$

Lemma 2 *For any edge e in the complement of $W(n, 2k+2)$ and $k \geq 4$, the graph $W(n, 2k+2)+e$ contains a cycle of length $l = 2k+2$.*

Proof: We divide the proof into the appropriate cases and in each case demonstrate the cycle of length l. Recall that we have four types of vertices - spoke-nut, hub, rim and flange.

1. Suppose $e = n_{i,\alpha} n_{j,\beta}$; that is spoke-nut to spoke-nut (different indices).

   If for $n_{j+1,\kappa}$ and $n_{i,\alpha}$ the indices $\kappa \neq \alpha$, then

   $$C_l = n_{i,\alpha} \overbrace{n_{j,\beta} r_{j,1} r_{j,2} \ldots n_{j+1,\kappa}}^{k+1} \overbrace{h_\kappa \ldots h_\alpha}^{k} n_{i,\alpha}.$$

   Hence, $|C_l| = 2k+2$.

   Otherwise, for $n_{j+1,\kappa}$ and $n_{i,\alpha}$ the indices $\kappa = \alpha$. Thus, by our construction we are guaranteed that for $n_{j-1,\delta}$ the indices $\delta \neq \alpha$ and we have

   $$C_l = n_{i,\alpha} \overbrace{n_{j,\beta} r_{j-1,k-1} r_{j,k-2} \ldots n_{j-1,\delta}}^{k+1} \overbrace{h_\delta \ldots h_\alpha}^{k} n_{i,\alpha}.$$

   Again, $|C_l| = 2k+2$.

2. Suppose $e = n_{i,\alpha} n_{j,\alpha}$; spoke-nut to spoke-nut (same indices). Then by our construction we are guaranteed that for $n_{j+1,\beta}$ the indices $\alpha \neq \beta$ and we have

   $$C_l = n_{i,\alpha} \overbrace{n_{j,\alpha} r_{j,1} r_{j,2} \ldots n_{j+1,\beta}}^{k+1} \overbrace{h_\beta \ldots h_\alpha}^{k} n_{i,\alpha}.$$

The remaining cases are shown in the Appendix. $\square$

Together Lemmas 1 and 2 imply that $W(n, 2k+2)$ is C_{2k+2}-saturated.

We now count the number of edges in the graph $W(n, 2k+2)$.

Let $n \equiv a \mod k$. The number of edges on the rim is thus $n-k-a$. The number of spokes is equal to $\frac{n-k-a}{k}$. The number of flange vertices is a and each is adjacent to one vertex of the hub. Furthermore, if k is small then we have partitioned these a vertices into flange sets of size $a_1, a_2, \dots a_k$ each of which induces a clique and thus $\Sigma_{i=1}^{k}\binom{a_i}{2}$ edges. Finally, the hub contributes $\binom{k}{2}$ edges.

Thus, when $k \geq 7$ and $n \equiv a \mod k$ we have:

$$\begin{aligned} |E(W(n, 2k+2))| &= (n-k-a) + \frac{n-k-a}{k} + a + \binom{k}{2} \qquad (1) \\ &= n\Big(1 + \frac{1}{k}\Big) + \frac{k^2 - 3k - 2}{2} - \frac{a}{k}. \qquad (2) \end{aligned}$$

By a similar count, when $4 \leq k \leq 6$ and $n \equiv a \mod 4k$ we have:

$$|E(W(n, 2k+2))| = n\Big(1 + \frac{1}{k}\Big) + \frac{k^2 - 3k - 2}{2} - \frac{a}{k} + \Sigma_{i=1}^{k}\binom{a_i}{2}. \qquad (3)$$

This immediately implies the following.

Theorem 1 *For $k \geq 4,\ l = 2k+2,$ and $n \geq 3l$,*

$$sat(n, C_l) \leq n\Big(1 + \frac{2}{l-2}\Big) + \frac{5l^2}{4}. \qquad (4)$$

2.0.2 The Odd Case: $W(n, 2k+3)$

We proceed in a similar fashion as in the even case. The graph we now define, $W(n, 2k+3)$, will differ slightly from $W(n, 2k+2)$, however we will use the same terminology given above.

To construct a C_{2k+3}-saturated graph, $k \geq 7$ we proceed as follows. To construct $W(n, 2k+3)$ we begin by placing $k+1$ vertices into the hub. These $k+1$ vertices will induce the following split graph $K^{k-3} + \overline{K}^4$. We label the four vertices of the copy of $\overline{K}^4$ by h_1, h_2, h_3, h_4 and the remaining vertices by $h_5, \dots h_{k+1}$. Surrounding the hub exists a cycle, R_o - the rim, of length sk for some $s \geq 4$. Each k^{th} vertex of the rim will be joined by a spoke to one of the four vertices h_1, h_2, h_3, h_4 of the hub. We will, in the same fashion as above, label the vertices of the rim.

Surrounding the hub exists a cycle, R, of length sk for some sufficiently large s. We will refer to this cycle as the **rim**. Each k^{th} vertex of the rim will be joined by an edge, called a **spoke**, to the hub. The vertex on the rim that is adjacent to a spoke will be referred to as a **spoke-nut**. Thus,

$$R = \{n_{1,\alpha}, r_{1,1}, r_{1,2}, \dots r_{1,k-1}, n_{2,\beta}, r_{2,1}, r_{2,2}, \dots n_{s,\omega}, r_{s,1}, r_{s,2}, \dots r_{s,k-1}\}.$$

Here we have listed the vertices in a clockwise fashion with spoke-nut vertices denoted by $n_{i,\kappa}$, and the remaining vertices by $r_{p,q}$. For vertices denoted $n_{i,\kappa}$ the subscript i refers to

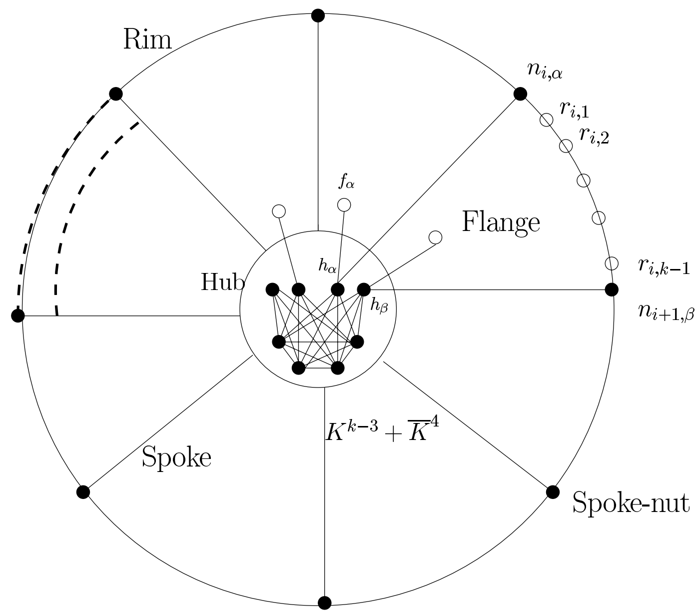

Figure 2: The Odd Generalized Wheel - Cycle-Saturated Graph

its placement on the wheel and κ denotes the subscript of the vertex in the hub to which it is connected, that is $n_{i,\kappa} \sim h_\kappa$. For vertices denoted $r_{p,q}$, the subscript p denotes the spoke-nut, $n_{p,\kappa}$, preceding it in the clockwise orientation and q the distance along the rim from $n_{p,\kappa}$. We place the following restriction on the spokes of the wheel, indicating this through the subscripts of the spoke-nuts.

Rule 2: Given three consecutive spoke-nut vertices $n_{i,\alpha}, n_{i+1,\beta}, n_{i+2,\gamma}$ we require that α, β, γ are all distinct. Furthermore, we require that for each pair α, β where $1 \leq \alpha < \beta \leq 4$ there exist spoke-nut vertices of the form $n_{i,\alpha}, n_{i+2,\beta}$ and spoke-nut vertices of the form $n_{j,\alpha}, n_{j+1,\beta}$.

Rule 2 may be observed when the number of spokes used is a multiple of four and at least twelve. This can be done by labeling the first twelve spoke nut vertices in the following manner: $\{n_{1,\alpha}, n_{2,\beta}, n_{3,\gamma}, n_{4,\delta}, n_{5,\alpha}, n_{6,\gamma}, n_{7,\beta}, n_{8,\delta}, n_{9,\alpha}, n_{10,\beta}, n_{11,\delta}, n_{12,\gamma}\}$, and each additional four spoke-nut vertices are labeled by repeating the labeling of the first four of these vertices. The graph just described has $n \equiv 0 \mod 4k$ vertices. When $n \equiv a \mod 4k$ we make the following adjustment to the graph just described. We select these vertices, $\{h_1, h_2, h_3, h_4\}$ in the hub, and evenly distribute the a vertices into 4 flange sets, F_1, F_2, F_3, F_4, of size a_1, a_2, a_3, a_4 (thus $a_i \leq k$) and on each set, F_i, we construct a clique and completely join it to the vertex in the hub labeled h_i. (Thus when $a = 0$ no adjustment is made.) See Figure 2.

We now show that this graph is C_{2k+3}-saturated.

Lemma 3 *For $k \geq 7$ the graph $W(n, 2k+3)$ contains no cycle of length $l = 2k+3$.*

Proof: First note that a flange vertex may not lie on a cycle of length l as the corresponding hub vertex is a cut-vertex and no flange set contains more than k vertices. As

$s \geq 12$ there is no cycle of length l comprised of edges solely from the rim. This, together with the fact that the hub contains only $k+1$ vertices, implies that if such a cycle exists it must use a spoke. As the set of spokes form an edge-cut of the graph $W(n, 2k+3)$, such a cycle must in fact use an even number of spokes. If the number of spokes used is four or more then the number of vertices involved in any cycle will be strictly greater than l. To see this note that upon using four or more spokes we use a corresponding number of spoke-nuts. The number of vertices used along the rim between any two spoke-nuts is at least $k-1$ and thus the number of vertices used from the rim in such a cycle is at least $2k+2$ in addition to at least two vertices from the hub. Thus $l > 2k+3$.

Hence the number of spokes used in such a cycle must be two. If the two spokes used are consecutive then the cycle contains $k+1$ vertices from the rim and at most $k+1$ from the hub. Thus such a cycle has length at most $2k+2 < l$. If the spokes are more than one apart then any cycle containing them must use at least $3k+1 > l$ vertices from the rim. Hence the two spokes used are exactly one apart, s_i, s_{i+2}. Notice that any cycle containing them uses exactly $2k+1$ vertices from the rim. Thus to create a cycle of length $2k+3$ we must use exactly two vertices from the hub. These two vertices would need to be adjacent and both would need to be the end vertex of some spoke. However, by our construction, no such pair of vertices exists in the hub. Thus, no cycle of length l exists. $\square$

Lemma 4 *For any edge e in the complement of $W(n, 2k+3)$ and $k \geq 7$, the graph $W(n, 2k+3) + e$ contains a cycle of length $l = 2k+3$.*

PROOF: We divide the proof into the appropriate cases and in each case demonstrate the cycle of length l. Recall that we have four types of vertices - spoke-nut, hub, rim and flange.

1. Suppose $e = n_{i,\alpha} n_{j,\beta}$; spoke-nut to Spoke-nut (different indices, that is $\alpha \neq \beta$).

   If for $n_{j+1,\gamma}$ and $n_{i,\alpha}$ the indices $\gamma \neq \alpha$, then

   $$C_l = n_{i,\alpha} \overbrace{n_{j,\beta} r_{j,1} r_{j,2} \ldots n_{j+1,\gamma}}^{k+1} \overbrace{h_\gamma \ldots h_\alpha}^{k+1} n_{i,\alpha}.$$

   Hence, $|C_l| = 2k+3$. Otherwise, for $n_{j+1,\kappa}$ and $n_{i,\alpha}$ the indices $\kappa = \alpha$. Hence,

   $$C_l = n_{i,\alpha} \overbrace{n_{j,\beta} r_{j-1,k-1} r_{j,k-2} \ldots n_{j-1,\delta}}^{k+1} \overbrace{h_\delta \ldots h_\alpha}^{k+1} n_{i,\alpha}.$$

   Hence, $|C_l| = 2k+3$.

2. Suppose $e = n_{i,\alpha} n_{j,\alpha}$; spoke-nut to spoke-nut (same indices). We then have

   $$C_l = n_{i,\alpha} \overbrace{n_{j,\alpha} r_{j,1} r_{j,2} \ldots n_{j+1,\beta}}^{k+1} \overbrace{h_\beta \ldots h_\alpha}^{k+1} n_{i,\alpha}.$$

The remaining cases are shown in the Appendix. □

Together Lemmas 3 and 4 imply that $W(n, 2k+3)$ is C_{2k+3}-saturated.

We now count the number of edges in the graph $W(n, 2k+3)$. Let $n \equiv a \mod 4k$. The number of edges on the rim is thus $n-(k+1)-a$. The number of spokes is equal to $\frac{n-(k+1)-a}{k}$. The number of flange edges is equal to $a + \Sigma_{i=1}^{4}\binom{a_i}{2}$. Finally, the hub contributes $\binom{k+1}{2} - 6$ edges.

Thus,

$$
\begin{aligned}
|E(W(n, 2k+3))| &= (n-k-1-a) + \frac{n-k-1-a}{k} + a + \Sigma_{i=1}^{4}\binom{a_i}{2} && (5)\\
&\quad + \binom{k+1}{2} - 6 && (6)\\
&= n\left(1+\frac{1}{k}\right) + \frac{k^2-k-16-2a}{2} - \frac{a+1}{k} + \Sigma_{i=1}^{4}\binom{a_i}{2}. && (7)
\end{aligned}
$$

This immediately implies the following.

Theorem 2 *For $k \geq 7,\ l = 2k+3,\ n \equiv a \mod 4k$ and $n \geq 7l \geq 13k+1$,*

$$
\begin{aligned}
sat(n, C_l) &\leq n\left(1+\frac{1}{k}\right) + \frac{k^2-k-16-2a}{2} - \frac{a+1}{k} + \Sigma_{i=1}^{4}\binom{a_i}{2} && (8)\\
&\leq n\left(1+\frac{2}{l-3}\right) + \frac{5l^2}{4}. && (9)
\end{aligned}
$$

3 Another Construction

We now construct a graph, $F(n, l)$, on $n \geq 2l$ vertices that is C_l-saturated for all $l \geq 5$. We begin with constructing a cycle on $l+1$ vertices, $\{c_1, c_2, \ldots c_{l+1}, c_1\}$. To vertices c_1, c_{l+1} we join a clique on $l-4$ vertices, and label these vertices $\{h_1, h_2, \ldots h_{l-4}\}$. On the remaining $n-2l+3$ vertices, $\{x_1, y_1, x_2, y_2, \ldots x_t, y_t, x_{t+1}\}$, we place a perfect, or near-perfect if this number is odd, matching so that $x_i y_i$ is an edge for all $i,\ 1 \leq i \leq \lfloor\frac{n-2l+3}{2}\rfloor$. To complete the construction we add all edges of the type $x_i c_1$ and $x_i c_{l+1}$. Figure 3 helps to illustrate this.

Lemma 5 *$F(n, l)$ contains no cycle of length $l \geq 5$.*

PROOF: First note that no vertex labeled y_i is contained in a (non-trivial) cycle. If a cycle of length l were to exist using some x_i and x_j with $i \neq j$ the vertices c_1 and c_{l+1} must also be used, hence the cycle can be at most length four. Thus at most one x_i may be used in such a cycle. If x_i were used in such a cycle then the cycle must contain the path $c_1 x_i c_{l+1}$, and thus there would need to exist a path of length $l-1$ connecting c_1 and c_{l+1}. However, no such path exists and thus no x_i is on a cycle of length l. It is now easy to observe that no cycle of length l exists on the vertices $\{c_1, \ldots c_{l+1}, h_1, \ldots h_{l-4}\}$. □

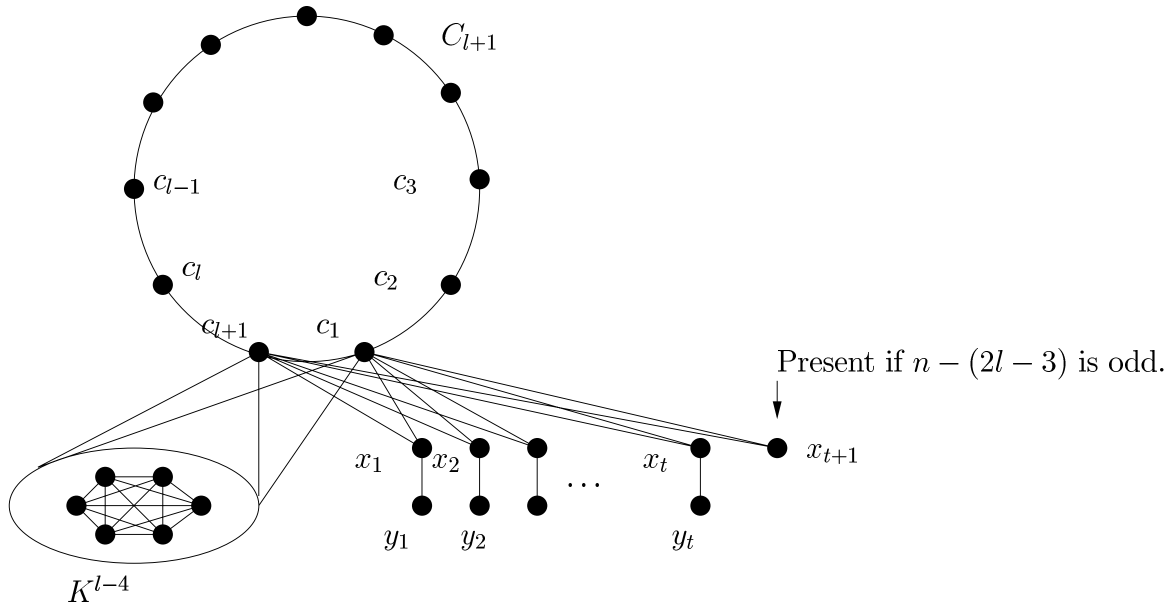

Figure 3: Another Cycle-Saturated Graph

Lemma 6 *For any edge e in the complement of $F(n,l)$ and $l \geq 5$, the graph $F(n,l)+e$ contains a cycle of length l.*

Proof: We divide the proof into the appropriate cases and in each case demonstrate the cycle of length l.

1. Suppose $e = y_i y_j,\ i \neq j$. Then

$$C_l = y_i \overbrace{y_j x_j c_{l+1}}^{3} \overbrace{h_1 h_2 \dots h_{l-6}}^{l-6} \overbrace{c_1 x_i}^{2} y_i.$$

2. Suppose $e = y_i x_j,\ i \neq j$. Then

$$C_l = y_i \overbrace{x_j c_{l+1}}^{2} \overbrace{h_1 h_2 \dots h_{l-5}}^{l-5} \overbrace{c_1 x_i}^{2} y_i.$$

The remaining cases are shown in the Appendix. □

Together Lemmas 5 and 6 imply that $F(n,l)$ is C_l-saturated. We now count the number of edges in $F(n,l)$. First, there are $l+1$ edges on the cycle C_{l+1}. The number of edges in the clique and those joining the clique and the cycle total $\binom{l-2}{2}-1$. The matching contains $\lfloor\frac{n-2l+3}{2}\rfloor$ edges and there are $2\lceil\frac{n-2l+3}{2}\rceil$ edges joining c_1, c_{l+1} to the vertices labeled x_i. Thus,

$$
\begin{aligned}
|E(G)| &= (l+1) + \binom{l-2}{2} - 1 + \lfloor \frac{n-2l+3}{2} \rfloor + 2\lceil \frac{n-2l+3}{2} \rceil && (10)\\
&= \left\lceil \frac{3(n-2l+3)}{2} \right\rceil + \frac{l^2-3l+6}{2} && (11)\\
&= \left\lceil \frac{3n+l^2-9l+15}{2} \right\rceil. && (12)
\end{aligned}
$$

This construction gives an improvement of the upper bound for $sat(n, C_l)$ for a few particular cases, as noted in the following theorem.

Theorem 3 *For $l = 8, 9, 11, 13$ or 15 and $n \geq 2l$*

$$
\begin{aligned}
sat(n, C_l) &\leq \left\lceil \frac{3n+l^2-9l+15}{2} \right\rceil && (13)\\
&\leq \lceil \frac{3n}{2} \rceil + \frac{l^2}{2}. && (14)
\end{aligned}
$$

4 Other Graphs

Other than cycles, there are many other instances of determining F-saturated graphs of minimum size. Some instances that have been considered, outside of those mentioned in the introduction, include paths and stars [12], complete hypergraphs [2], and more recently non-traceable graphs [11]. For a survey of further results we refer the reader to [4]. For a list of interesting open problems we refer the reader to [15].

5 Appendix

We complete the lemmas that demonstrate the l-cycle in $G+e$ for each of the graphs that we have constructed.

Proof of Lemma 2 continued:

1. Suppose $e = n_{i,\alpha} r_{j,q}$; spoke-nut to rim.

   If $q \neq 1$ and for $n_{j+1,\beta} \neq n_{i-1,\kappa}$, $\beta \neq \kappa$, then let

$$C_l = n_{i,\alpha} \overbrace{r_{j,q} r_{j,q+1} \ldots n_{j+1,\beta}}^{k-q+1} \overbrace{h_\beta \ldots h_\kappa}^{q} \overbrace{n_{i-1,\kappa}, r_{i-1,1} \ldots r_{i-1,k-1}}^{k} n_{i,\alpha}.$$

   If $q \neq 1$ and $\beta = \kappa$ then we must have $\beta \neq \delta$ for $n_{j+1,\beta} \neq n_{i+1,\delta}$, hence we have

$$C_l = n_{i,\alpha} \overbrace{r_{j,q} r_{j,q+1} \ldots n_{j+1,\beta}}^{k-q+1} \overbrace{h_\beta \ldots h_\delta}^{q} \overbrace{n_{i+1,\delta}, r_{i,k-1} \ldots r_{i,1}}^{k} n_{i,\alpha}.$$

If $q = 1$ and for $n_{j,\gamma}, n_{i-1,\kappa}, \quad \gamma \neq \kappa$, we have

$$C_l = n_{i,\alpha} \overbrace{r_{j,1} n_{j,\gamma}}^{2} \overbrace{h_\gamma \dots h_\kappa}^{k-1} \overbrace{n_{i-1,\kappa} r_{i-1,1} \dots r_{i-1,k-1}}^{k} n_{i,\alpha}.$$

If $q = 1$ and for $n_{j,\gamma}, n_{i+1,\delta}, \quad \gamma \neq \delta$, we have

$$C_l = n_{i,\alpha} \overbrace{r_{j,1} n_{j,\gamma}}^{2} \overbrace{h_\gamma \dots h_\delta}^{k-1} \overbrace{n_{i+1,\delta} r_{i,k-1} \dots r_{i,1}}^{k} n_{i,\alpha}.$$

2. Suppose $e = n_{i,\alpha} h_\beta$; spoke-nut to hub.

   If there exists an $n_{j,\beta}$ and $n_{j+1,\kappa}$ with $\kappa \neq \alpha$ then let

$$C_l = n_{i,\alpha} \overbrace{h_\beta}^{1} \overbrace{n_{j,\beta} r_{j,1} \dots n_{j+1,\kappa}}^{k+1} \overbrace{h_\kappa \dots h_\alpha}^{k-1} n_{i,\alpha}.$$

   Otherwise, there exists an $n_{j,\beta}$ and $n_{j-1,\delta}$ with $\delta \neq \alpha$ and then let

$$C_l = n_{i,\alpha} \overbrace{h_\beta}^{1} \overbrace{n_{j,\beta} r_{j-1,k-1} \dots n_{j-1,\delta}}^{k+1} \overbrace{h_\delta \dots h_\alpha}^{k-1} n_{i,\alpha}.$$

3. Suppose $e = n_{i,\alpha} f_\beta$; spoke-nut to flange (different indices, that is $\alpha \neq \beta$). If for $n_{j+1,\kappa}$ the indices $\alpha \neq \kappa$ then let

$$C_l = n_{i,\alpha} \overbrace{f_\beta h_\beta}^{2} \overbrace{n_{j,\beta} r_{j,1} \dots n_{j+1,\kappa}}^{k+1} \overbrace{h_\kappa \dots h_\alpha}^{k-2} n_{i,\alpha}.$$

   Otherwise, we may be assured by our construction that for $n_{j-1,\gamma}$ the indices $\alpha \neq \gamma$ and thus

$$C_l = n_{i,\alpha} \overbrace{f_\beta h_\beta}^{2} \overbrace{n_{j,\beta} r_{j-1,k-1} \dots n_{j-1,\gamma}}^{k+1} \overbrace{h_\gamma \dots h_\alpha}^{k-2} n_{i,\alpha}.$$

4. Suppose $e = n_{i,\alpha} f_\alpha$; spoke-nut to flange (same indices). We are guaranteed by our construction that for $n_{i-1,\kappa}$ the indices $\alpha \neq \kappa$ and thus

$$C_l = n_{i,\alpha} \overbrace{f_\alpha h_\alpha h_\beta \dots h_\kappa}^{k+1} \overbrace{n_{i-1,\kappa} r_{i-1,1} \dots r_{i-1,k-1}}^{k} n_{i,\alpha}.$$

5. Suppose $e = r_{i,q} h_\delta$; rim to hub.

   If for $n_{i-1,\beta}$ the indices $\beta \neq \delta$ then,

$$C_l = r_{i,q} \overbrace{h_\delta \dots h_\beta}^{k+1-q} \overbrace{n_{i-1,\beta} r_{i-1,1} \dots n_{i,\alpha}}^{k+1}, \overbrace{r_{i,1} \dots r_{i,q-1}}^{q-1} r_{i,q}.$$

Otherwise, for $n_{i+2,\beta}$ the index $\beta \neq \delta$ then,

$$C_l = r_{i,q} \overbrace{h_\delta \ldots h_\beta}^{q+1} \overbrace{n_{i+2,\beta} r_{i+1,k-1} \ldots n_{i,\alpha}}^{k+1}, \overbrace{r_{i,k-1} \ldots r_{i,q+1}}^{k-q-1} r_{i,q}.$$

6. Suppose $e = r_{i,q} f_\delta$; rim to flange.

   If for $n_{i-1,\beta}$ the indices $\beta \neq \delta$ then,

$$C_l = r_{i,q} \overbrace{f_\delta h_\delta \ldots h_\beta}^{k+1-q} \overbrace{n_{i-1,\beta} r_{i-1,1} \ldots n_{i,\alpha}}^{k+1}, \overbrace{r_{i,1} \ldots r_{i,q-1}}^{q-1} r_{i,q}.$$

   Otherwise, for $n_{i+2,\gamma}$ the index $\gamma \neq \delta$ then,

$$C_l = r_{i,q} \overbrace{f_\delta h_\delta \ldots h_\gamma}^{k+1-q} \overbrace{n_{i+2,\gamma} r_{i+1,1} \ldots n_{i+1,\kappa}}^{k+1}, \overbrace{r_{i,k-1} \ldots r_{i,q+1}}^{q-1} r_{i,q}.$$

7. Suppose $e = r_{i,q} r_{i,q+s}$; rim to rim (same indices). We then have

$$C_l = r_{i,q} \overbrace{r_{i,q+s} r_{i,q+s+1} \ldots n_{i+1,\beta}}^{k-(q+s)+1} \overbrace{r_{i+1,1} \ldots n_{i+2,\kappa}}^{k} \overbrace{h_\kappa \ldots h_\alpha}^{s} \overbrace{n_{i,\alpha} \ldots r_{i,q-1}}^{q} r_{i,q}.$$

8. Suppose $e = r_{i,j} r_{i+1,q}$; rim to rim (indices differ by 1).

   If $k - j + 1$ and $k - q + 1$ sum to at least $k + 2$ (this sum is at most $2k$) then

$$C_l = r_{i,j} \overbrace{r_{i+1,q} r_{i+1,q+1} \ldots n_{i+2,\delta}}^{k-q+1} \overbrace{h_\delta \ldots h_\beta}^{q+j} \overbrace{n_{i+1,\beta} r_{i,k-1} \ldots r_{i,j+1}}^{k-j} r_{i,j}.$$

   Otherwise, it must be the case that $j + 1$ and $q + 1$ sum to at least $k + 2$. To see this note that $(k-j+1)+(j+1) = k+2$ and $(k-q+1)+(q+1) = k+2$, together a total of $2k + 4$ and if neither $k - j + 1 + k - q + 1$ or $j + 1 + q + 1$ were at least $k + 2$ we would reach a contradiction. Hence,

$$C_l = r_{i,j} \overbrace{r_{i+1,q} r_{i+1,q-1} \ldots n_{i+1,\beta}}^{q+1} \overbrace{h_\beta \ldots h_\alpha}^{2k-q-j} \overbrace{n_{i,\alpha} r_{i,1} \ldots r_{i,j-1}}^{j} r_{i,j}.$$

9. Suppose $e = r_{i,j} r_{p,q}$; rim to rim (indices differ by at least 2, that is there exists at least two spoke-nuts between $r_{i,j}$ and $r_{p,q}$).

   We will suppose that the distance from $r_{i,j}$ to $n_{i+1,\beta}$ is at least the distance from $r_{i,j}$ to $n_{i,\alpha}$, and that the distance from $r_{p,q}$ to $n_{p+1,\delta}$ is at least the distance from $r_{p,q}$ to $n_{p.\gamma}$. The other cases are similar to that shown here.

   By the supposition it follows that $k + 2 \leq (k - j + 1) + (k - q + 1) \leq 2k$. Thus, if for $n_{i+1,\beta}, n_{p+1,\delta}$, the indices $\beta \neq \delta$ then let

$$C_l = r_{i,j}\overbrace{r_{p,q}r_{p,q+1}\dots n_{p+1,\delta}}^{k-q+1}\overbrace{h_\delta\dots h_\beta}^{q+j}\overbrace{n_{i+1,\beta}r_{i,k-1}\dots r_{i,j+1}}^{k-j}r_{i,j}.$$

Thus it must be the case that $\beta=\delta$.

If $k-q+1$ and $j+1$ sum to at least $k+2$ (this sum is at most $2k$) then

$$C_l = r_{i,j}\overbrace{r_{p,q}r_{p,q+1}\dots n_{p+1,\delta}}^{k-q+1}\overbrace{h_\delta\dots h_\alpha}^{k+q-j}\overbrace{n_{i,\alpha}r_{i,1}\dots r_{i,j-1}}^{j}r_{i,j}.$$

Otherwise, it must be the case that $q+1$ and $k-j+1$ sum to at least $k+2$. To see this note that $(k-j+1)+(j+1)=k+2$ and $(k-q+1)+(q+1)=k+2$, together a total of $2k+4$ and if neither $(k-q+1)+(j+1)$ and $(k-j+1)+(q+1)$ were at least $k+2$ we would reach a contradiction. Hence,

$$C_l = r_{i,j}\overbrace{r_{p,q}r_{p,q-1}\dots n_{p,\gamma}}^{q+1}\overbrace{h_\gamma\dots h_\beta}^{k+j-q}\overbrace{n_{i+1,\beta}r_{i,k-1}\dots r_{i,j+1}}^{k-j}r_{i,j}.$$

10. Suppose $e=h_\alpha f_\beta$; hub to flange. We then have

$$C_l = h_\alpha\overbrace{f_\beta h_\beta\dots h_\kappa}^{k}\overbrace{n_{i,\kappa}r_{i,1}\dots n_{i+1,\alpha}}^{k+1}h_\alpha.$$

11. Suppose $e=f_\alpha f_\beta$; flange to flange. We then have

$$C_l = f_\alpha\overbrace{f_\beta h_\beta\dots h_\kappa}^{k-1}\overbrace{n_{i,\kappa}r_{i,1}\dots n_{i+1,\alpha}}^{k+1}\overbrace{h_\alpha}^{1}f_\alpha.$$

This completes the proof of Lemma 2. □

PROOF OF LEMMA 4 CONTINUED:

1. Suppose $e=n_{i,\alpha}r_{j,q}$; spoke-nut to rim.

   If $q\neq 1$ and for $n_{j+1,\beta}, n_{i-1,\kappa}$, if the indices $\beta\neq\kappa$, then

$$C_l = n_{i,\alpha}\overbrace{r_{j,q}r_{j,q+1}\dots n_{j+1,\beta}}^{k-q+1}\overbrace{h_\beta\dots h_\kappa}^{q+1}\overbrace{n_{i-1,\kappa}, r_{i-1,1}\dots r_{i-1,k-1}}^{k}n_{i,\alpha}.$$

   If $q\neq 1$ then it must be the case by our construction that for $n_{j+1,\beta}, n_{i+1,\delta}$, the indices $\beta\neq\delta$. We then have

$$C_l = n_{i,\alpha}\overbrace{r_{j,q}r_{j,q+1}\dots n_{j+1,\beta}}^{k-q+1}\overbrace{h_\beta\dots h_\delta}^{q+1}\overbrace{n_{i+1,\delta}, r_{i,k-1}\dots r_{i,1}}^{k}n_{i,\alpha}.$$

   If $q=1$ and for $n_{j,\gamma}, n_{i-1,\kappa}$, if $\gamma\neq\kappa$, then

$$C_l = n_{i,\alpha} \overbrace{r_{j,1} n_{j,\gamma}}^{2} \overbrace{h_\gamma \dots h_\kappa}^{k} \overbrace{n_{i-1,\kappa} r_{i-1,1} \dots r_{i-1,k-1}}^{k} n_{i,\alpha}.$$

Otherwise, $q = 1$ and for $n_{j,\gamma}, n_{i+1,\delta},$ it must be the case by our construction that $\gamma \neq \delta$. We then have

$$C_l = n_{i,\alpha} \overbrace{r_{j,1} n_{j,\gamma}}^{2} \overbrace{h_\gamma \dots h_\delta}^{k} \overbrace{n_{i+1,\delta} r_{i,k-1} \dots r_{i,1}}^{k} n_{i,\alpha}.$$

2. Suppose $e = n_{i,\alpha} h_\beta$; spoke-nut to hub.

   If there exists an $n_{j,\gamma}$ and $n_{j+1,\delta}$ with $\delta \neq \alpha$ then

$$C_l = n_{i,\alpha} \overbrace{h_\beta \dots h_\gamma}^{s+1} \overbrace{n_{j,\gamma} r_{j,1} \dots n_{j+1,\delta}}^{k+1} \overbrace{h_\delta \dots h_\alpha}^{k-s} n_{i,\alpha}.$$

   Otherwise, there exists an $n_{j,\gamma}$ and $n_{j-1,\kappa}$ with $\kappa \neq \alpha$ and we then have

$$C_l = n_{i,\alpha} \overbrace{h_\beta \dots h_\gamma}^{s+1} \overbrace{n_{j,\gamma} r_{j-1,k-1} \dots n_{j-1,\kappa}}^{k+1} \overbrace{h_\kappa \dots h_\alpha}^{k-s} n_{i,\alpha}.$$

3. Suppose $e = n_{i,\alpha} f_\beta$; spoke-nut to flange (different indices, that is $\alpha \neq \beta$).

   If there exists an $n_{j,\beta}$ and $n_{j+1,\kappa}$ with $\kappa \neq \alpha$ then

$$C_l = n_{i,\alpha} \overbrace{f_\beta h_\beta}^{2} \overbrace{n_{j,\beta} r_{j,1} \dots n_{j+1,\kappa}}^{k+1} \overbrace{h_\kappa \dots h_\alpha}^{k-1} n_{i,\alpha}.$$

   Otherwise, there exists an $n_{j,\beta}$ and $n_{j-1,\delta}$ with $\delta \neq \alpha$, and we then have

$$C_l = n_{i,\alpha} \overbrace{f_\beta h_\beta}^{2} \overbrace{n_{j,\beta} r_{j-1,k-1} \dots n_{j-1,\delta}}^{k+1} \overbrace{h_\delta \dots h_\alpha}^{k-1} n_{i,\alpha}.$$

4. Suppose $e = n_{i,\alpha} f_\alpha$; spoke-nut to flange (same indices). We then have

$$C_l = n_{i,\alpha} \overbrace{f_\alpha h_\alpha h_\beta \dots h_\kappa}^{k+2} \overbrace{n_{i-1,\kappa} r_{i-1,1} \dots r_{i-1,k-1}}^{k} n_{i,\alpha}.$$

5. Suppose $e = r_{i,q} h_\delta$; rim to hub. We then have

$$C_l = r_{i,q} \overbrace{h_\delta \dots h_\beta}^{s+1} \overbrace{n_{j,\beta} r_{j,1} \dots n_{j+1,\gamma}}^{k+1} \overbrace{h_\gamma \dots h_\kappa}^{q-s} \overbrace{n_{i+1,\kappa} r_{i,k-1} \dots r_{i,q+1}}^{k-q} r_{i,q}.$$

6. Suppose $e = r_{i,q}f_\delta$; rim to flange.

   If for $n_{i-1,\beta}$ the indices $\beta \neq \delta$ then,

$$C_l = r_{i,q}\overbrace{f_\delta h_\delta \dots h_\beta}^{k+2-q}\overbrace{n_{i-1,\beta}r_{i-1,1}\dots n_{i,\alpha},}^{k+1}\overbrace{r_{i,1}\dots r_{i,q-1}}^{q-1}r_{i,q}.$$

   Otherwise, for $n_{i+2,\gamma}$ the index $\gamma \neq \delta$ then,

$$C_l = r_{i,q}\overbrace{f_\delta h_\delta \dots h_\gamma}^{k+2-q}\overbrace{n_{i+2,\gamma}r_{i+1,1}\dots n_{i+1,\kappa},}^{k+1}\overbrace{r_{i,k-1}\dots r_{i,q+1}}^{q-1}r_{i,q}.$$

7. Suppose $e = r_{i,q}r_{i,q+s}$; rim to rim (same indices). We then have

$$C_l = r_{i,q}\overbrace{r_{i,q+s}r_{i,q+s+1}\dots n_{i+1,\beta}}^{k-(q+s)+1}\overbrace{r_{i+1,1}\dots n_{i+2,\gamma}}^{k}\overbrace{h_\gamma \dots h_\alpha}^{s+1}\overbrace{n_{i,\alpha}\dots r_{i,q-1}}^{q}r_{i,q}.$$

8. Suppose $e = r_{i,j}r_{i+1,q}$; rim to rim (indices differ by 1).

   If $k-j+1$ and $k-q+1$ sum to at least $k+2$ (this sum is at most $2k$) then

$$C_l = r_{i,j}\overbrace{r_{i+1,q}r_{i+1,q+1}\dots n_{i+2,\delta}}^{k-q+1}\overbrace{h_\delta \dots h_\beta}^{q+j+1}\overbrace{n_{i+1,\beta}r_{i,k-1}\dots r_{i,j+1}}^{k-j}r_{i,j}.$$

   Otherwise, it must be the case that $j+1$ and $q+1$ sum to at least $k+2$. To see this note that $(k-j+1)+(j+1) = k+2$ and $(k-q+1)+(q+1) = k+2$, together a total of $2k+4$ and if neither $k-j+1+k-q+1$ or $j+1+q+1$ were at least $k+2$ we would reach a contradiction. Hence,

$$C_l = r_{i,j}\overbrace{r_{i+1,q}r_{i+1,q-1}\dots n_{i+1,\beta}}^{q+1}\overbrace{h_\beta \dots h_\alpha}^{2k-q-j+1}\overbrace{n_i,\alpha r_{i,1}\dots r_{i,j-1}}^{j}r_{i,j}.$$

9. Suppose $e = r_{i,j}r_{p,q}$; rim to rim (indices differ by at least 2, that is there exists at least two spoke nuts between $r_{i,j}$ and $r_{p,q}$).

   We will suppose that the distance from $r_{i,j}$ to $n_{i+1,\beta}$ is at least the distance from $r_{i,j}$ to $n_{i,\alpha}$, and that the distance from $r_{p,q}$ to $n_{p+1,\delta}$ is at least the distance from $r_{p,q}$ to $n_{p.\gamma}$. The other cases are similar to that shown here.

   By the supposition it follows that $k+2 \leq (k-j+1)+(k-q+1) \leq 2k$. Thus if for $n_{i+1,\beta}, n_{p+1,\delta}$ the indices $\beta \neq \delta$, then

$$C_l = r_{i,j}\overbrace{r_{p,q}r_{p,q+1}\dots n_{p+1,\delta}}^{k-q+1}\overbrace{h_\delta \dots h_\beta}^{q+j+1}\overbrace{n_{i+1,\beta}r_{i,k-1}\dots r_{i,j+1}}^{k-j}r_{i,j}.$$

   Thus it must be the case that $\beta = \delta$.

If $k-q+1$ and $j+1$ sum to at least $k+2$ (this sum is at most $2k$) then

$$C_l = r_{i,j} \overbrace{r_{p,q} r_{p,q+1} \dots n_{p+1,\delta}}^{k-q+1} \overbrace{h_\delta \dots h_\alpha}^{k+q-j+1} \overbrace{n_{i,\alpha} r_{i,1} \dots r_{i,j-1}}^{j} r_{i,j}.$$

Otherwise, it must be the case that $q+1$ and $k-j+1$ sum to at least $k+2$. To see this note that $(k-j+1)+(j+1) = k+2$ and $(k-q+1)+(q+1) = k+2$, together a total of $2k+4$ and if neither $(k-q+1)+(j+1)$ and $(k-j+1)+(q+1)$ were at least $k+2$ we would reach a contradiction. Hence,

$$C_l = r_{i,j} \overbrace{r_{p,q} r_{p,q-1} \dots n_{p,\gamma}}^{q+1} \overbrace{h_\gamma \dots h_\beta}^{k+j-q+1} \overbrace{n_{i+1,\beta} r_{i,k-1} \dots r_{i,j+1}}^{k-j} r_{i,j}.$$

10. Suppose $e = h_\alpha f_\beta$; hub to flange. We then have

$$C_l = h_\alpha \overbrace{f_\beta h_\beta \dots h_\gamma}^{s+1} \overbrace{n_{i,\gamma} r_{i,1} \dots n_{i+1,\delta}}^{k+1} \overbrace{h_\delta \dots h_\kappa}^{k+1-s} h_\alpha.$$

11. Suppose $e = f_\alpha f_\beta$; flange to flange. We then have

$$C_l = f_\alpha \overbrace{f_\beta h_\beta \dots h_\gamma}^{k} \overbrace{n_{i,\gamma} r_{i,1} \dots n_{i+1,\alpha}}^{k+1} \overbrace{h_\alpha}^{1} f_\alpha.$$

12. Suppose $e = h_\alpha h_\beta$ for $1 \le \alpha < \beta \le 4$; hub to hub.

    Rule 2 guarantees that there exists a pair of spoke-nut vertices labeled $n_{i,\beta}, n_{i+2,\alpha}$. We then have

$$C_l = h_\alpha \overbrace{h_\beta}^{1} \overbrace{n_{i,\beta} r_{i,1} \dots n_{i+1,\gamma} r_{i+1,1} \dots n_{i+2,\alpha}}^{2k+1} h_\alpha.$$

This completes the proof of Lemma 4.

Proof of Lemma 6 continued:

1. Suppose $e = y_i h_j$ for $1 \le j \le (l-4)$. Without loss of generality we may assume $j = 1$. Then

$$C_l = y_i \overbrace{h_1 h_2 \dots h_{l-4}}^{l-4} \overbrace{c_{l+1} c_1 x_i}^{3} y_i.$$

2. Suppose $e = y_i c_j$ for $1 \le j \le l-2$. Then

$$C_l = y_i \overbrace{c_j c_{j-1} \dots c_1}^{j} \overbrace{h_1 h_2 \dots h_k c_{l+1}}^{l-2-j} x_i y_i.$$

3. Suppose $e = y_i c_j$ for $l-1 \leq j \leq l+1$ and $l \geq 6$, or for $l \leq j \leq l+1$ and $l = 5$. Then

$$C_l = y_i \overbrace{c_j \dots c_{l+1}}^{l+2-j} \overbrace{h_1 h_2 \dots h_k}^{j-5} \overbrace{c_1 x_i}^{2} y_i.$$

Otherwise, for $j = l-1$ and $l = 5$ we have

$$C_l = y_i c_{l-1} c_l c_{l+1} x_i y_i.$$

4. Suppose $e = x_i x_j$. Then

$$C_l = x_i \overbrace{x_j c_{l+1}}^{2} \overbrace{h_1 h_2 \dots h_{l-4}}^{l-4} c_1 x_i.$$

5. Suppose $e = x_i h_j$ for $1 \leq j \leq (l-4)$. Without loss of generality we may assume $j = 1$. Then

$$C_l = x_i \overbrace{h_1 h_2 \dots h_{l-4}}^{l-4} \overbrace{c_{l+1} x_j c_1}^{3} c_i.$$

6. Suppose $e = x_i c_j$ for $2 \leq j \leq l-1$. Then

$$C_l = x_i \overbrace{c_j c_{j-1} \dots c_1}^{j} \overbrace{h_1 h_2 \dots h_k c_{l+1}}^{l-1-j} x_i.$$

7. Suppose $e = h_i c_j$ for $3 \leq j \leq l-1$. Without loss of generality we may assume $i = 1$. Then

$$C_l = h_1 \overbrace{c_j c_{j-1} \dots c_1}^{j} \overbrace{h_2 \dots h_k c_{l+1}}^{l-1-j} h_1.$$

8. Suppose $e = h_i c_l$. Without loss of generality we may assume $i = 1$. (The case $e = h_1 c_2$ is symmetric and we omit it here.) Then

$$C_l = h_1 \overbrace{c_l c_{l+1} x_i c_1}^{4} \overbrace{h_2 h_3 \dots h_{l-4}}^{l-5} h_1.$$

9. Suppose $e = c_i c_j$ for $1 \leq i < j \leq l+1$. Then

$$C_l = c_i \overbrace{c_j c_{j+1} \dots c_{l+1}}^{l+2-j} \overbrace{h_1 \dots h_k}^{j-i-2} \overbrace{c_1 c_2 \dots c_{i-1}}^{i-1} c_i.$$

This completes the proof of Lemma 6. $\square$

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