Bounding the number of edges in permutation graphs

Peter Keevash [*] Po-Shen Loh [†] Benny Sudakov [‡]

Submitted: Dec 10, 2005; Accepted: Apr 27, 2006; Published: May 5, 2006
Mathematics Subject Classification: 05C35, 20B30

Abstract

Given an integer $s \geq 0$ and a permutation $\pi \in S_n$, let $\Gamma_{\pi,s}$ be the graph on n vertices $\{1, \dots, n\}$ where two vertices $i < j$ are adjacent if the permutation flips their order and there are at most s integers k, $i < k < j$, such that $\pi = [\dots j \dots k \dots i \dots]$. In this short paper we determine the maximum number of edges in $\Gamma_{\pi,s}$ for all $s \geq 1$ and characterize all permutations π which achieve this maximum. This answers an open question of Adin and Roichman, who studied the case $s = 0$. We also consider another (closely related) permutation graph, defined by Adin and Roichman, and obtain asymptotically tight bounds on the maximum number of edges in it.

1 Introduction

We begin with standard notation. Let $\pi \in S_n$ be a permutation and let $\pi(i)$ denote the image of i, where we consider π as a bijection from $[n] = \{1, \dots, n\}$ to itself. We express permutations as linear arrangements of $[n]$ by listing images in order as $[\pi(1), \dots, \pi(n)]$. For example, $[2, 5, 1, 3, 4]$ represents the permutation that maps $1 \mapsto 2$, $2 \mapsto 5$, $3 \mapsto 1$, $4 \mapsto 3$, $5 \mapsto 4$. Note that in the context of list notation $\pi^{-1}(i)$ denotes the position of i in the permutation when π is written in a list form and $\pi^{-1}(i) < \pi^{-1}(j)$ has the simple interpretation that in π the number i appears before the number j.

Definition 1.1 *Given an integer $s \geq 0$ and a permutation $\pi \in S_n$,*

[*]Department of Mathematics, Caltech, Pasadena, CA 91125. E-mail: keevash@caltech.edu. Research supported in part by NSF grant.

[†]Department of Mathematics, Princeton University, Princeton, NJ 08544. E-mail: ploh@math.princeton.edu. Research supported in part by a Fannie and John Hertz Foundation Fellowship, an NSF Graduate Research Fellowship, and a Princeton Centennial Fellowship.

[‡]Department of Mathematics, Princeton University, Princeton, NJ 08544, and Institute for Advanced Study, Princeton. E-mail: bsudakov@math.princeton.edu. Research supported in part by NSF CAREER award DMS-0546523, NSF grant DMS-0355497, USA-Israeli BSF grant, Alfred P. Sloan fellowship, and the State of New Jersey.

- *Let $\Gamma_{\pi,s}$ be the graph on the vertex set $[n]$ in which a pair of vertices $i < j$ is adjacent if $\pi^{-1}(i) > \pi^{-1}(j)$ and there are at most s integers k, $i < k < j$, such that $\pi^{-1}(i) > \pi^{-1}(k) > \pi^{-1}(j)$.*

- *Similarly, let $G_{\pi,s}$ be the graph in which we connect a pair of vertices $i < j$ as long as there are at most s integers k, $i < k < j$, with $\pi^{-1}(k)$ lying between $\pi^{-1}(i)$ and $\pi^{-1}(j)$ (i.e., we drop the condition that $\pi^{-1}(i) > \pi^{-1}(j)$).*

These two graphs were defined by Adin and Roichman [1], who studied the maximum number of edges in $\Gamma_{\pi,s}$ and $G_{\pi,s}$ when $s = 0$. This combinatorial problem was motivated by the observation that the number of edges in $\Gamma_{\pi,0}$ equals the number of permutations σ that are covered by π in the strong Bruhat[1] order on S_n, and the number of edges in $G_{\pi,0}$ equals the degree of the vertex π in the Hasse[2] diagram corresponding to that order. $\Gamma_{\pi,0}$ also appeared in a different context, in work by Bousquet-Mélou and Butler [2], which focused on the case when $\Gamma_{\pi,0}$ is a forest; this stemmed from a question related to Schubert varieties. The second graph, $G_{\pi,0}$, was recently considered by Felsner [5], who rediscovered Adin and Roichman's bound on the maximum number of edges in $G_{\pi,0}$ while studying (2-dimensional) rectangle graphs. These graphs are actually equivalent to $G_{\pi,0}$, but defined in geometric language. Their definition naturally generalizes to higher dimensions, and Felsner's paper also contains an asymptotic bound for the maximum number of edges in 3-dimensional box graphs.

Adin and Roichman [1] noticed that there are no triangles in $\Gamma_{\pi,0}$ and therefore by Turán's Theorem [6] (see also [4]) this graph contains at most $\lfloor n^2/4 \rfloor$ edges. They also characterize all permutations that achieve this maximum. For general $s > 0$ Adin and Roichman again used Turán's Theorem together with the fact that $\Gamma_{\pi,s}$ does not contain a complete graph on $s+3$ vertices to deduce that the number of edges in $\Gamma_{\pi,s}$ is bounded by $\left(1 - \frac{1}{s+2}\right)\frac{n^2}{2}$. On the other hand, they conjectured [3] that this bound is far from being best possible and for large enough n the optimal permutations have the following form

$$\pi_n^{s,m} = [m+s+1, m+s+2, \ldots, n, m+s, m+s-1, \ldots, m+1, 1, 2, \ldots, m], \quad (1)$$

where m equals $\lfloor \frac{n-s}{2} \rfloor$ or $\lceil \frac{n-s}{2} \rceil$. These permutations have a particularly simple structure, which is most transparent in the context of the permutation diagrams in Figure 1. The corresponding graph $\Gamma_{\pi,s}$ consists of three parts of size s, $\lfloor \frac{n-s}{2} \rfloor$ and $\lceil \frac{n-s}{2} \rceil$. Each vertex in the part of size s is adjacent to all other vertices in the graph, and the edges between the parts of size $\lfloor \frac{n-s}{2} \rfloor$ and $\lceil \frac{n-s}{2} \rceil$ form a complete bipartite graph, giving a total of $\lfloor \frac{n-s}{2} \rfloor \lceil \frac{n-s}{2} \rceil + s(n-s) + \binom{s}{2}$ edges. Our first theorem proves that the permutations $\pi_m^{s,n}$ are indeed optimal, and classifies all optimal configurations.

[1]The strong Bruhat order is the transitive closure of the following partial order relations. For each pair of integers $a < b$ in $\{1, \ldots, n\}$, introduce the relation $\pi < \sigma$ if and only if $[\pi(1), \ldots, \pi(n)]$ and $[\sigma(1), \ldots, \sigma(n)]$ are identical except for the transposition of a and b, and a appears before b in $[\pi(1), \ldots, \pi(n)]$.

[2]Given a partially ordered set $(S, <)$, recall that x covers y if $x > y$ but there does not exist z for which $x > z > y$. The Hasse diagram is the graph with vertex set S and an edge between every pair of vertices $x, y \in S$ for which either x covers y or vice versa.

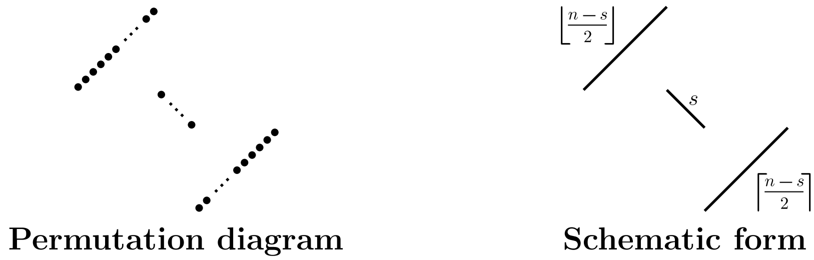

Figure 1: Optimal permutations for $\Gamma_{\pi,s}$. A permutation diagram for π, as exhibited on the left, is a plot of the n points $\{(i,\pi(i))\}$ in the lattice $[n]\times[n]$. We will often draw the plots schematically, as in the diagram on the right.

Theorem 1.2 *Suppose $s \geq 1$, $n \geq s+2$ and let $f(s,n)$ be the maximum number of edges in $\Gamma_{\pi,s}$ as π runs over all permutations in S_n. Then*

$$f(s,n) = \left\lfloor \frac{n-s}{2} \right\rfloor \left\lceil \frac{n-s}{2} \right\rceil + s(n-s) + \binom{s}{2}.$$

Moreover, the only permutations that achieve this maximum are $\pi_n^{s,m}$ from (1), where m equals $\lfloor \frac{n-s}{2} \rfloor$ or $\lceil \frac{n-s}{2} \rceil$, except when $n = s+3$. Then, there is exactly one additional optimal permutation: $[s+3, s+2, \ldots, 1]$.

We stated this theorem only for $n \geq s+2$, as for $n \leq s+2$ it is clear that there is a unique permutation $\pi = [n, n-1, \ldots, 1]$ for which $f(s,n) = \binom{n}{2}$ and $\Gamma_{\pi,s}$ is a complete graph.

When $s = 0$, Adin and Roichman also proved that the graph $G_{\pi,0}$ with $\pi = \pi_n^{0,m}$ has the maximum possible number of edges. One would naturally suspect that $\pi_n^{s,m}$ might give optimal graphs $G_{\pi,s}$ for all $s \geq 1$. Surprisingly, this is not the case and we were able to construct better permutations. This suggests that the problem of maximizing the number of edges in $G_{\pi,s}$ has a distinct flavor and is more complex than the corresponding one for $\Gamma_{\pi,s}$. Though we were not able to solve it completely we managed to obtain a good upper bound on the size of $G_{\pi,s}$.

Theorem 1.3 *Let $g(s,n)$ be the maximum number of edges in $G_{\pi,s}$ as π runs over all permutations in S_n. Then*

$$\frac{n^2}{4} + \left(\frac{23s}{12} + O(1)\right) n \leq g(s,n) \leq \frac{n^2}{4} + \frac{5s+3}{2} n.$$

For comparison, note that when $\pi = \pi_n^{s,m}$ the graph $G_{\pi,s}$ has only $\frac{n^2}{4} + \left(\frac{3s}{2} + O(1)\right) n$ edges.

The rest of this short paper is organized as follows. In the next section we study the number of edges in $\Gamma_{\pi,s}$ and prove Theorem 1.2. In Section 3 we construct permutations π which give larger $G_{\pi,s}$ graphs than those previously known and obtain an upper bound on the size of such graphs. The last section of the paper contains some concluding remarks.

2 Maximum number of edges in $\Gamma_{\pi,s}$

In this section we prove Theorem 1.2. We show by induction on n that if $\Gamma_{\pi,s}$ has at least $f(s,n)$ edges then $\pi = \pi_n^{s,m}$ for m equal to $\lfloor \frac{n-s}{2} \rfloor$ or $\lceil \frac{n-s}{2} \rceil$, or $n = s+3$ and $\pi = [s+3, s+2, \ldots, 1]$. In the base case $n = s+2$ we can make $\Gamma_{\pi,s}$ complete by choosing $\pi = \pi_n^{s,m} = [s+2, s+1, \ldots, 1]$, and this is obviously a unique maximum construction. Now we give the induction step for $n > s+2$, using the following deletion method.

Given a permutation π and an integer $1 \le v \le n$ we can write our permutation π in list form, and delete v from the list. This produces a list $\pi \setminus \{v\}$ with $n-1$ numbers, which corresponds to a permutation $\pi' \in S_{n-1}$ as follows. Let $\tau_v(j)$ be the j-th largest number in $[n] \setminus \{v\}$. Then π' is obtained by substituting j instead of $\tau_v(j)$ in the list $\pi \setminus \{v\}$. For example, if we delete 2 from the permutation $[3,2,1,4] \in S_4$, we obtain the list $[3,1,4]$, which after substitution, gives the permutation $[2,1,3] \in S_3$. It is easy to see that our operation has the following property. If the vertices $\tau_v(j)$ and $\tau_v(k)$ were adjacent in $\Gamma_{\pi,s}$, then the corresponding vertices j and k are adjacent in $\Gamma_{\pi',s}$. However, it is possible to have an edge (j,k) in $\Gamma_{\pi',s}$ without having the edge $(\tau_v(j), \tau_v(k))$ in $\Gamma_{\pi,s}$. Thus the number of edges in $\Gamma_{\pi,s}$ exceeds the number of edges in $\Gamma_{\pi',s}$ by at most the degree of vertex v, with equality only if for every edge (j,k) in $\Gamma_{\pi',s}$ the vertices $\tau_v(j)$ and $\tau_v(k)$ are adjacent in $\Gamma_{\pi,s}$. We assume that $\Gamma_{\pi,s}$ has at least $f(s,n)$ edges and by induction hypothesis the number of edges in $\Gamma_{\pi',s}$ is at most $f(s,n-1)$. This implies that every vertex v in $\Gamma_{\pi,s}$ has degree

$$d_v \ge f(s,n) - f(s,n-1) = \left\lfloor \frac{n+s}{2} \right\rfloor. \tag{2}$$

Let b be the lowest numbered vertex adjacent to n in $\Gamma_{\pi,s}$. Any common neighbor k of n and b must satisfy $b < k < n$ and lie in π as $[\ldots n \ldots k \ldots b \ldots]$. Hence the adjacency of n and b bounds the number of common neighbors by s and therefore, $d_n + d_b \le n+s$. Combining this with (2), we find that at least one of d_n or d_b must be exactly $\lfloor \frac{n+s}{2} \rfloor$. Call that vertex x. If both $d_n = d_b = \lfloor \frac{n+s}{2} \rfloor$, then choose $x = n$. Note that if $n+s$ is even, we will always choose $x = n$. Delete x from π and let π' be the new permutation, as defined above. By the above discussion, we have that the number of edges in $\Gamma_{\pi,s}$ and $\Gamma_{\pi',s}$ are $f(s,n)$ and $f(s,n-1)$ respectively. Moreover for every edge (j,k) in $\Gamma_{\pi',s}$ the corresponding vertices $\tau_x(j)$ and $\tau_x(k)$ in $\Gamma_{\pi,s}$ are adjacent as well. To determine the structure of the permutation π we consider several cases.

Case 1: $x = n$. The deletion operation is particularly easy since we are deleting the largest number, so π is obtained from π' simply by inserting n somewhere in the list form. Call this location the insertion point.

Case 1A: $n \ge s+3$, $n-s$ odd. By induction hypothesis, $\pi' \in S_{n-1}$ is precisely $\pi_{n-1}^{s,m}$ with $m = \frac{n-1-s}{2}$. Note that $\pi_{n-1}^{s,m}$ has exactly $\frac{n-1+s}{2} = d_n$ numbers listed after $n-1$, so if the insertion point is after the occurrence of $n-1$, it must be *immediately* after $n-1$ in order for n (the largest number) to be adjacent to all of them. This produces $\pi_n^{s,m}$.

On the other hand, if the insertion point is before $n-1$, then n will not be adjacent to any of the last m vertices listed in $\pi_{n-1}^{s,m}$. Yet $d_n = \frac{n-1+s}{2} = n-1-m$, so n must be adjacent to every other vertex listed in $\pi_{n-1}^{s,m}$. Since n is the largest number, this forces the insertion point to be right at the beginning of the list. Since π' is precisely $\pi_{n-1}^{s,m}$ with $m = \frac{n-1-s}{2}$, this yields the permutation $[s+3, s+2, \ldots, 1]$ when $n = s+3$, but for $n > s+3$ it produces

$$\pi = [n, m+s+1, m+s+2, \ldots, n-1, m+s, m+s-1, \ldots, m+1, 1, 2, \ldots, m],$$

where $2 \leq \frac{n-1-s}{2} = m$, so d_n is only $\max\{s+1, \frac{n-1-s}{2}\} < \lfloor \frac{n+s}{2} \rfloor$, contradiction.

Case 1B: $n \geq s+4$, $n-s$ even. Now the induction hypothesis gives us several possibilities for π': two of the form $\pi_{n-1}^{s,m}$, and if $n-1 = s+3$, also $[s+3, \ldots, 1]$. A similar argument to the above rules out the possibility that the insertion point is before $n-1$ in all three cases. If $\pi' = \pi_{n-1}^{s,m}$ with $m = \lceil \frac{n-1-s}{2} \rceil = \frac{n-s}{2}$, then the only insertion point after $n-1$ that makes $d_n = \lfloor \frac{n+s}{2} \rfloor$ is again *immediately* after $n-1$. This yields $\pi_n^{s,m}$. On the other hand, if $m = \lfloor \frac{n-1-s}{2} \rfloor$ or $\pi' = [s+3, s+2, \ldots, 1]$, then no insertion point after $n-1$ gives $d_n = \lfloor \frac{n+s}{2} \rfloor$ at all.

Case 2: $x = b$. We shall show that in fact $b = 1$. To see this, suppose $b \neq 1$. Recall that we are only in this case if $n+s$ is odd, so by induction $\pi' = \pi_{n-1}^{s,m}$ with $m = \frac{n-1-s}{2}$. We also know that if j, k are adjacent in $\Gamma_{\pi',s}$ then $\tau_b(j)$ and $\tau_b(k)$ are adjacent in $\Gamma_{\pi,s}$ as well. In particular, $\tau_b(1)$ is adjacent to $\tau_b(n-1)$, since $(1, n-1)$ is an edge in $\Gamma_{\pi',s}$. However $n > b \neq 1$ implies that $\tau_x(1) = 1$ and $\tau_x(n-1) = n$. Therefore, there is an edge between 1 and n in $\Gamma_{\pi,s}$, contradicting the choice of b as the lowest numbered vertex adjacent to n. Hence $b = 1$. Now the statement of the theorem is invariant under the transformation $[\pi(1), \ldots, \pi(n)] \leftrightarrow [n+1-\pi(n), \ldots, n+1-\pi(1)]$, so this case follows from Case 1 by this symmetry.

This completes the case analysis and the proof of Theorem 1.2. □

3 Edge bounds in $G_{\pi,s}$

In this section we prove Theorem 1.3, which gives bounds on the maximum number of edges in $G_{\pi,s}$ with $\pi \in S_n$.

Proof of Theorem 1.3. We start with the upper bound, showing by induction on n that $G_{\pi,s}$ has at most $u(s,n) = \frac{n^2}{4} + \frac{5s+3}{2}n$ edges. This trivially holds for $n \leq s+2$, as then $u(s,n) \geq \binom{n}{2}$. Suppose that $n > s+2$. We will show that there is a vertex i with degree $d_i \leq u(s,n) - u(s,n-1) = \frac{n}{2} + \frac{5s+2}{2} + \frac{1}{4}$. This will complete the induction step.

Let t be the largest numbered vertex that is adjacent to 1 in $G_{\pi,s}$. Let l and r be the respective leftmost and rightmost common neighbors of 1 and t, where "left" and "right" are defined with respect to the order in which the numbers appear when π is written in list form. Draw the permutation diagram for π, and without loss of generality, suppose that 1 appears to the right of t (a similar argument will apply to the other case). Figure 2 shows the two possible relative configurations of 1, t, l, and r in the diagram for π. Of course, there could be more diagram points to the left, right, or above the part isolated in Figure 2.

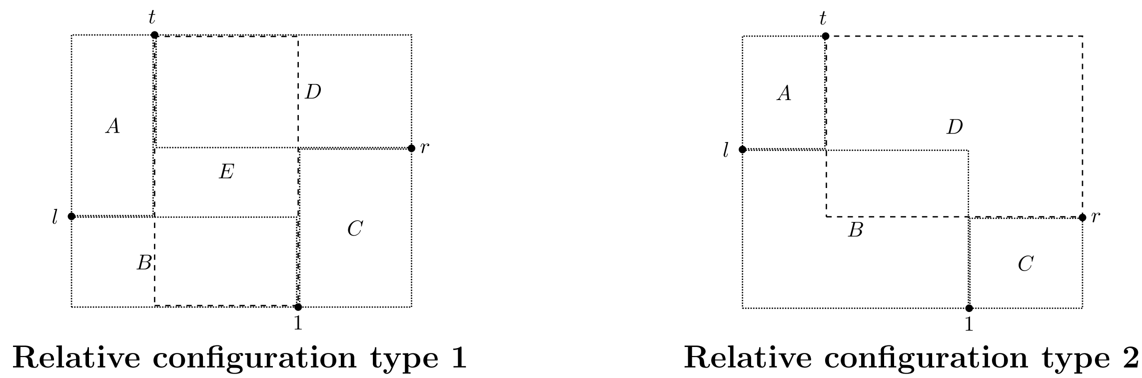

Figure 2: Diagrams for Theorem 1.3. The dotted rectangles are not part of the permutation diagram, and are drawn in for the purpose of marking regions. We only have to worry about two types of configurations, depending on the relative positions of l and r; if l is below r, then we need the additional rectangular region E to complete the cover.

Since l and r are the leftmost and rightmost common neighbors, t is the topmost neighbor of 1, and 1 is the minimal element in the set $\{1, \ldots, n\}$, all diagram points for common neighbors of 1 and t must lie in the union of the marked rectangular regions A, B, C, D, and E (but we do not need E if we are in the case of the second diagram). Since l and t are adjacent, the number of diagram points that lie in region A of the diagram is bounded by s. Similarly, the number of points in each of regions B, C, D, and E must also be bounded by s. The set of common neighbors of 1 and t must be some subset of all of these points, so its size is bounded by $5s + 2$ if we are in the first case, and $4s + 2$ if we are in the second. (The additional 2 comes from counting l and r.) It follows that $d_1 + d_t \leq n + 5s + 2$ and so one of d_1, d_t is at most $(n + 5s + 2)/2 < u(s, n) - u(s, n-1)$. This proves the upper bound.

As a lower bound we provide a construction that (asymptotically) differs from it only in the linear term. Since we are interested in asymptotics, let us make the simplifying assumptions that s is even and n is a multiple of $3s$. Consider the family of permutations that have diagrams of the form in Figure 3. We can explicitly describe these permutations in list form by defining the ordered $(3s/2)$-sublists

$$\begin{aligned} A &:= \left(\frac{s}{2}+1, \frac{s}{2}+2, \ldots, \frac{3s}{2}, \frac{s}{2}, \frac{s}{2}-1, \ldots, 1\right), \\ B &:= \left(\frac{3s}{2}, \frac{3s}{2}-1, \ldots, s+1, 1, 2, \ldots, s\right). \end{aligned}$$

Then, the permutations have list form

$$\left[A + \frac{n}{2}, A + \frac{n}{2} + \frac{3s}{2}, \ldots, A + n - \frac{3s}{2}, B, B + \frac{3s}{2}, \ldots, B + \frac{n}{2} - \frac{3s}{2}\right],$$

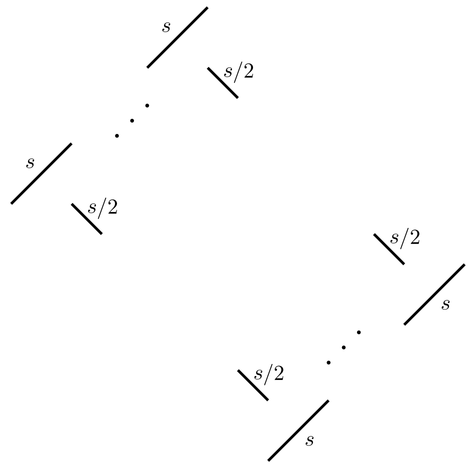

Figure 3: Construction for $\frac{n^2}{4} + \left(\frac{23s}{12} + O(1)\right) n$. The two sides of this construction are of equal size.

where we use the shorthand $A + k$ to denote the $(3s/2)$-sublist of element-wise translates of A by k.

It is clear from the diagram that every vertex corresponding to a point on the left half of the diagram will be adjacent to every vertex corresponding to a point on the right half, because these diagrams make the definition of adjacency particularly easy to visualize: two vertices i and j are adjacent exactly when the axis-parallel rectangle with corners $\{(\pi^{-1}(i), i), (\pi^{-1}(j), j)\}$ contains at most s other diagram points. Since each half has size $n/2$, this gives us the quadratic term. For the linear term, we calculate the internal degrees of vertices, i.e. the number of neighbors they have in their own half. For extreme vertices that are one of the leftmost or rightmost $3s$ points in each half, we will take the simple estimate that their internal degrees are not negative. A non-extreme vertex in an $(s/2)$-run is adjacent to the vertices diagrammed on the right side of Figure 4; this yields an internal degree of $9s/2 - 1$. Any non-extreme vertex in an s-run that is the first or last vertex of its run (call it an "endpoint vertex") is adjacent to the vertices diagrammed on the left side of Figure 4; this yields an internal degree of $7s/2 + 1$. The rest of the non-extreme vertices in s-runs have internal degree $7s/2 + 2$.

Since there are twice as many vertices in runs of length s as in runs of length $s/2$, the total number of internal edges in the two sides is at least

$$\frac{1}{2} \cdot n \cdot \left[\frac{2}{3}\left(\frac{7s}{2} + 1\right) + \frac{1}{3}\left(\frac{9s}{2} - 1\right)\right] - O(s^2) = \left(\frac{23s}{12} + O(1)\right) n,$$

as claimed. $\square$

4 Concluding remarks

Theorem 1.2 settles the $\Gamma_{\pi,s}$ problem raised in the last section of Adin and Roichman's paper, but Theorem 1.3 on $G_{\pi,s}$ still leaves a gap between the constants of 23/12 and

Adjacency of an endpoint vertex in an s-run **Adjacency of a typical vertex in an $(s/2)$-run**

Figure 4: Adjacent vertices to a non-extreme endpoint vertex in a run of length s (left hand side), and to a non-extreme vertex in a run of length $s/2$ (right hand side). In each case, the vertex in question is marked by a star.

5/2. We have some preliminary arguments based on the stability method that suggests that with more work, one could reduce the upper bound to $g(s,n) \leq \frac{n^2}{4} + (2s + O(1))\,n$. Some intuition for this bound can be obtained from the proof of Theorem 1.3, as if the structure of a permutation is 'close' to that of $\pi_n^{s,m}$ then the configuration of the first type in Figure 2 will not occur for most pairs of vertices. Then we would not need the region E in the proof, and we would replace the $5s+2$ with $4s+2$. However, the "sufficiently large" is necessary, because if $n = 5s+4$, the construction in the left side of Figure 5 gives 1 and t a total of $5s+2$ common neighbors, and it yields a complete graph, which must be optimal.

Optimal construction for $n = 5s+4$ **Asymptotically superior construction for $s = 4$**

Figure 5: On left, optimal construction for $n = 5s+4$; on right, asymptotically superior construction for $s = 4$. Note that the points in the $5s+4$ construction cluster in all 4 corners instead of condensing into the usual 2 sides.

On the other hand, it may be that our construction is not asymptotically optimal either, and that one can surpass the constant of 23/12. In particular, consider the following construction for $s = 4$, which is diagrammed in the right side of Figure 5. Define the 7-

sublists:

$$\begin{aligned} A &:= (3,4,2,6,7,5,1), \\ B &:= (7,3,1,2,6,4,5), \end{aligned}$$

suppose n is divisible by 14, and let π be

$$\left[A+\frac{n}{2}, A+\frac{n}{2}+7, \ldots, A+n-7, B, B+7, \ldots, B+\frac{n}{2}-7\right].$$

Now, a routine check finds the average internal degree tending to $118/7 \approx 16.86$ as $n \to \infty$, which beats the 16 that we get if we apply our construction in Figure 3 to the case $s = 4$. Both split the vertices into two approximately equal sides, so they have the same quadratic term $n^2/4$; hence the new construction is asymptotically better than the old one by a linear term. It is hard to tell whether this improvement indicates that the constant 23/12 can be improved, or whether it just comes under the $O(1)$ term for large s.

References

[1] R. Adin and Y. Roichman, On Degrees in the Hasse Diagram of the Strong Bruhat Order, *Séminaire Lotharingien de Combinatoire* 53 (2006), B53g.

[2] M. Bousquet-Mélou and S. Butler, Forest-like permutations, `arXiv:math.CO/0603617`, 2006.

[3] R. Adin and Y. Roichman, private communication, 2005.

[4] B. Bollobás, **Modern Graph Theory**, Graduate Texts in Mathematics 184, Springer, 1998

[5] S. Felsner, Empty Rectangles and Graph Dimension, `arXiv:math.CO/0601767`, 2006.

[6] P. Turán, Eine Extremalaufgabe aus der Graphentheorie, *Mat. Fiz Lapok* **48** (1941), 436–452.