Tilings of the sphere with right triangles I: The asymptotically right families

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Abstract

Sommerville [10] and Davies [2] classified the spherical triangles that can tile the sphere in an edge-to-edge fashion. Relaxing this condition yields other triangles, which tile the sphere but have some tiles intersecting in partial edges. This paper determines which right spherical triangles within certain families can tile the sphere.

Keywords: spherical right triangle, monohedral tiling, non-normal, non-edge-to-edge, asymptotically right

1 Introduction

A tiling is called *monohedral* (or *homohedral*) if all tiles are congruent, and *edge-to-edge* (or *normal*) if every two tiles that intersect do so in a single vertex or an entire edge. In 1923, D.M.Y. Sommerville [10] classified the edge-to-edge monohedral tilings of the sphere with isosceles triangles, and those with scalene triangles in which the angles meeting at any one vertex are congruent. H.L. Davies [2] completed the classification of edge-to-edge monohedral tilings by triangles in 1967 (apparently without knowledge of Sommerville's work) , allowing any combination of angles at a vertex. (Coxeter[1] and Dawson[5] both

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erred in failing to note that Davies does include triangles - notably the half- and quarter-lune families - that Sommerville did not consider.)

There are, of course, reasons why the edge-to-edge tilings are of special interest; however, non-edge-to-edge tilings do exist. Some use tiles that can also tile in an edge-to-edge fashion; others use tiles that admit no edge-to-edge tilings [3, 4, 5]. In [3] a complete classification of isosceles spherical triangles that tile the sphere was given. In [5] a special class of right triangles was considered, and shown to contain only one triangle that could tile the sphere.

This paper and its companion papers [6, 7, 8] continue the program of classifying the triangles that tile the sphere, by giving a complete classification of the right triangles with this property. Non-right triangles will be classified in future work.

2 Basic results and definitions

In this section we gather together some elementary definitions and basic results used later in the paper. We will represent the measure of the larger of the two non-right angles of the triangle by β and that of the smaller by γ. (Where convenient, we will use α to represent a 90° angle.) The lengths of the edges opposite these angles will be B and C respectively, with H as the length of the hypotenuse. (Note that it may be that $\beta > 90^\circ$ and $B > H$.) We will make frequent use of the well known result

$$90^\circ < \beta + \gamma < 270^\circ, \quad \beta - \gamma < 90^\circ \tag{1}$$

We will denote the number of tiles by N; this is of course equal to $720^\circ/(\beta+\gamma-90^\circ)$.

Let $\mathcal{V} = \{(a,b,c) \in \mathbb{Z}^3 : a\alpha + b\beta + c\gamma = 360^\circ,\ a,b,c \geq 0\}$. We call the triples (a,b,c) the *vertex vectors* of the triangle and the equations *vertex equations.* The vertex vectors represent the possible (unordered) ways to surround a vertex with the available angles.

We call $\mathcal{V}$ itself the *vertex signature* of the triangle. For right triangles $\mathcal{V}$ is always nonempty, containing at least $(4,0,0)$. Any subset of $\mathcal{V}$ that is linearly independent over $\mathbb{Z}$ and generates $\mathcal{V}$ is called a *basis* for $\mathcal{V}$. All bases for $\mathcal{V}$ have the same number of elements; if bases for $\mathcal{V}$ have $n+1$ elements we will define the *dimension* of $\mathcal{V}$, $\dim(\mathcal{V})$, to be n. An oblique triangle could in principle have $\mathcal{V} = \emptyset$ and $\dim(\mathcal{V}) = -1$; but such a triangle could not even tile the neighborhood of a vertex. The dimension of $\mathcal{V}$ may be less than the dimension of the lattice $\{(a,b,c) \in \mathbb{Z}^3 : a\alpha + b\beta + c\gamma = 360^\circ\}$ that contains it, but it cannot be greater.

If a triangle can tile the sphere in a non-edge-to-edge fashion, it must have one or more *split vertices* at which one or more edges ends at a point in the relative interior of another edge. The angles at such a *split vertex* must add to 180°, and two copies of this set of angles must give a vertex vector in which a,b, and c are all even. We shall call (a,b,c) *even* and $(a,b,c)/2$ a *split vector.* We will call $(a,b,c)/2$ a β (resp. γ) *split* if b (resp. c) is nonzero. If both are nonzero we will call the split vector a $\beta\gamma$ split.

It is easily seen that if $(3,b,c) \in \mathcal{V}$, then also $(0,4b,4c) \in \mathcal{V}$; and if $(2,b,c) \in \mathcal{V}$, then also $(0,2b,2c) \in \mathcal{V}$. A vertex vector with $a = 0$ or 1 will be called *reduced.* If $\mathcal{V}$ contains a vector (a, b, c) such that $(a, b, c)/2$ is a β split, a γ split, or a $\beta\gamma$ split, then it must have a reduced vector corresponding to a split of the same type, with $a = 0$.

The following result was proved in [5]:

Proposition 1 *The only right triangle that tiles the sphere, does not tile in an edge-to-edge fashion, and has no split vector apart from* $(4, 0, 0)/2$ *is the* $(90°, 108°, 54°)$ *triangle.*

In fact, this triangle tiles in exactly three distinct ways. One is illustrated in Figure 1; the others are obtained by rotating one of the equilateral triangles, composed of two tiles, that cover the polar regions.

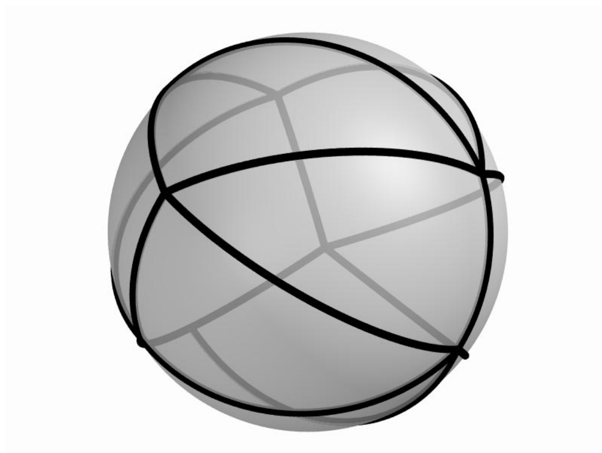

Figure 1: A tiling with the $(90°, 108°, 54°)$ triangle

This lets us prove:

Proposition 2 *For any right triangle that tiles the sphere but does not tile in an edge-to-edge fashion,* $\dim(\mathcal{V}) = 2$.

Proof: A right triangle with $\dim(\mathcal{V}) = 0$ would have $(4, 0, 0)$ as its only vertex vector, which means that the neighborhood of a β or γ corner could not be covered. Moreover, the lattice $\{(a, b, c) \in \mathbb{Z}^3 : a\alpha + b\beta + c\gamma = 360°\}$ is at most two-dimensional; so $\dim(\mathcal{V}) \leq 2$.

If $\dim(\mathcal{V}) = 1$, the other basis vector $V_1 = (a_1, b_1, c_1)$ must have $b_1 = c_1$, or the total numbers of β and γ angles in the tiling would differ. If $a_1 = 2, b_1 = c_1 \geq 2$ or $a_1 = 0, b_1 = c_1 \geq 4$, we would have $\beta + \gamma \leq 90°$; and if $a_1 = 0, b_1 = c_1 = 2$ the triangle is of the form $(90°, \theta, 180° - \theta)$ and tiles in an edge-to-edge fashion. Thus, under our hypotheses, there is no second split, and the only such triangle that tiles but not in an edge-to-edge fashion is (by the previous proposition) the $(90°, 108°, 54°)$ triangle. However, this has $\mathcal{V} = \{(4, 0, 0), (1, 2, 1), (1, 1, 3), (1, 0, 5)\}$, and $\dim(\mathcal{V}) = 2$. ■

Corollary 1 *There are no continuous families of right triangles that tile the sphere but do not tile in an edge-to-edge fashion.*

Proof: As $\dim(\mathcal{V}) = 2$, the system of equations

$$\begin{aligned} 4\alpha + 0\beta + 0\gamma &= 360^\circ \quad (2) \\ a_1\alpha + b_1\beta + c_1\gamma &= 360^\circ \quad (3) \\ a_2\alpha + b_2\beta + c_2\gamma &= 360^\circ \quad (4) \end{aligned}$$

has a unique solution (α, β, γ) whose angles (in degrees) are rational. ■

Note: Both requirements (that the triangle is right, and that it allows no edge-to-edge tiling), are necessary. Consider the $(\frac{360}{n}^\circ, 180^\circ - \theta, \theta)$ triangles where n is, in the first case, odd, and, in the second case, equal to 4. In each case $\dim(\mathcal{V}) = 1$ for almost every θ and the family is continuous. We may also consider the triangles with $\alpha + \beta + \gamma = 360^\circ$; four of any such triangle tile the sphere, almost every such triangle has $\dim \mathcal{V} = 0$, and they form a continuous two-parameter family.

2.1 The irrationality hypothesis

With a few well-known exceptions such as the isosceles triangles, and the half-equilateral triangles with angles $(90^\circ, \theta, \theta/2)$, it seems natural to conjecture that a spherical triangle with rational angles will always have irrational ratios of edge lengths. This "irrationality hypothesis" is probably not provable without a major advance in transcendence theory. However, for our purposes it will always suffice to rule out identities of the form $pH + qB + rC = p'H + q'B + r'C$ where p, q, r, p', q', r' are positive and the sums are less than 360°. For any specified triangle for which the hypothesis holds, this can be done by testing a rather small number of possibilities, and without any great precision in the arithmetic. This will generally be done without comment.

Note: The possibility that some linear combination $pA + qB + rC$ of edge lengths will have a rational measure in degrees is *not* ruled out, and in fact this is sometimes the case. For instance, the $(90^\circ, 60^\circ, 40^\circ)$ triangle has $H + 2B + 2C = 180^\circ$.

Note: It will be seen below that, while edge-to-edge tilings tend to have mirror symmetries, the symmetry groups of non-edge-to-edge tilings are usually chiral. The irrationality hypothesis offers an explanation for this. Frequently there will only be one way (up to reversal) to fit triangles together along one side of an extended edge of a given length without obtaining an immediately impossible configuration. If the configuration on one side of an extended edge is the reflection in the edge of that on the other, the tiling will be locally edge-to-edge. A non-edge-to-edge tiling must have an extended edge where this does not happen; the configuration on one side must either be completely different from that on the other or must be its image under a 180° rotation about the center of the edge.

Note: It may be observed that all known tilings of the sphere with congruent triangles have an even number of elements. This is easily seen for edge-to-edge tilings, as $3N = 2E$

(where E is the number of edges.) The irrationality hypothesis, if true, would explain this observation in general.

A maximal arc of a great circle that is contained in the union of the edges will be called an *extended edge*. Each side of an extended edge is covered by a sequence of triangle edges; the sum of the edges on one side is equal to that on the other. In the absence of any rational dependencies between the sides, it follows that one of these sequences must be a rearrangement of the other, so that $3N$ is again even.

In light of this, one might wonder whether in fact every triangle that tiles the sphere admits a tiling that is invariant under point inversion and thus corresponds to a tiling of the projective plane; however, while some tiles do admit such a tiling, others do not. For instance, it is shown below that the $(90°, 75°, 60°)$ triangle admits, up to reflection, a unique tiling; and the symmetry group of that tiling is a Klein 4-group consisting of the identity and three 180° rotations.

2.2 Classification of β sources

It follows from Proposition 2 that the vertex signature of every triangle that tiles but does not do so in an edge-to-edge fashion must contain at least one vector with $b > a, c$ and at least one with $c > a, b$. We will call such vectors β *sources* and γ *sources* respectively; and we may always choose them to be reduced. Henceforth, then, we will assume $\mathcal{V}$ to have a basis consisting of three vectors $V_0 = (4, 0, 0)$, $V_1 = (a, b, c)$, and $V_2 = (a', b', c')$, with $a, a' < 2$, $b > c$, and $b' < c'$. (For some triangles, more than one basis satisfies these conditions; this need not concern us.)

The restrictions that $\beta > \gamma$ and $b > c$ leave us only finitely many possibilities for V_1. In particular, if $a = 0$ and $b + c > 7$, then $360° = b\beta + c\gamma > 4\beta + 4\gamma$ and $\beta + \gamma < 90°$, which is impossible. Similarly, if $a = 1$ we must have $b + c \leq 5$. We can also rule out the vectors $(0, 2, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$, all of which force $\beta \geq 180°$. (In fact, there are *degenerate* triangles with $\beta = 180°$, but these are of little interest and easily classified.)

We are left with 22 possibilities for V_1. We may divide them into three groups, depending on whether $\lim_{c' \to \infty} \beta$ is acute, right, or obtuse.

- The *asymptotically acute* V_1 are $(0, 7, 0)$, $(0, 6, 1)$, $(0, 6, 0)$, $(0, 5, 2)$, $(0, 5, 1)$, $(0, 5, 0)$, $(1, 5, 0)$, $(1, 4, 1)$, and $(1, 4, 0)$. As for large enough c' these yield Euclidean or hyperbolic triangles, there are only finitely many vectors V_2 that can be used in combination with each of these.

- The *asymptotically right* V_1 are $(0, 4, 3)$, $(0, 4, 2)$, $(0, 4, 1)$, $(0, 4, 0)$, $(1, 3, 2)$, $(1, 3, 1)$, and $(1, 3, 0)$. Each of these vectors forms part of a basis for $\mathcal{V}$ for infinitely many spherical triangles.

- The *asymptotically obtuse* V_1 are $(0, 3, 2)$, $(0, 3, 1)$, $(0, 3, 0)$, $(0, 2, 1)$, $(1, 2, 1)$, and $(1, 2, 0)$. For large enough c' these yield triples of angles that do not satisfy the second inequality of (1); so again there are only finitely many possible V_2 to consider.

In the remainder of this paper, we will classify the triangles that tile the sphere and have vertex signatures with asymptotically right V_1 (referring to [2] for those which tile edge-to-edge, and [3] for the remaining isosceles cases). One particularly lengthy subcase is dealt with in a companion paper [6]. The aymptotically obtuse case is dealt with in the preprint [7]; and a paper now in preparation [8] will classify the right triangles that tile the sphere and have vertex signatures with asymptotically acute V_1, completing the classification of right triangles that tile the sphere.

3 The main result

The main result of this paper is the following theorem, the proof of which will be deferred until the next section.

Theorem 1 *The right spherical triangles which have vertex signatures with asymptotically right V_1 and tile the sphere (including those which tile edge-to-edge and those which are isosceles) are*

i). $(90°, 90°, \frac{360}{n}°)$,

ii). $(90°, 60°, 45°)$,

iii). $(90°, 90° - \frac{180}{n}°, \frac{360}{n}°)$ *for even* $n \geq 6$,

iv). $(90°, 90° - \frac{180}{n}°, \frac{360}{n}°)$ *for odd* $n > 6$,

v). $(90°, 75°, 60°)$,

vi). $(90°, 60°, 40°)$,

vii). $(90°, 75°, 45°)$, *and*

viii). $(90°, 78\frac{3}{4}°, 33\frac{3}{4}°)$.

The first three of these tile in an edge-to-edge fashion, though they also admit non-edge-to-edge tilings. The remaining five have only non-edge-to-edge tilings.

We now examine the tiles listed above in more detail.

i-iii) The three edge-to-edge cases

Both Sommerville and Davies included the $(90°, 90°, \frac{360}{n}°)$ and $(90°, 60°, 45°)$ triangles in their lists; but Sommerville did not include the $(90°, 90° - \frac{180}{n}°, \frac{360}{n}°)$ triangles, which are not isosceles and do not admit a tiling with all the angles equal at each vertex.

Sommerville and Davies give two edge-to-edge tilings with the first family of triangles when n is even, and Davies gives a second edge-to-edge tiling with the $(90°, 60°, 45°)$ triangle. In each case these are obtained by "twisting" the tiling shown along a great

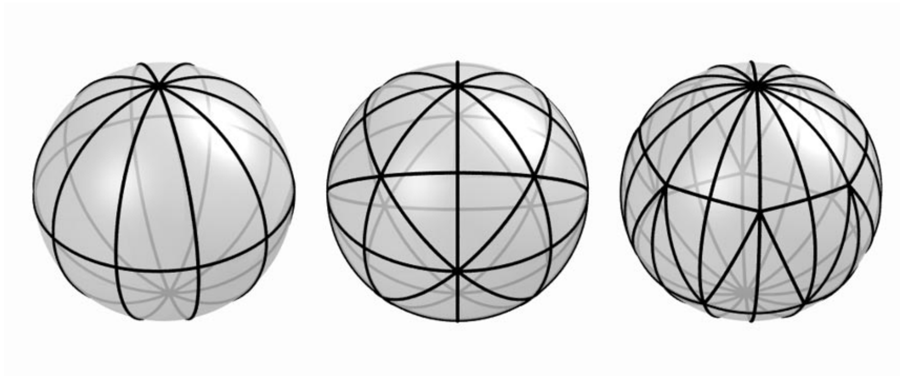

Figure 2: Examples of edge-to-edge tilings

circle composed of congruent edges,until vertices match up again. (For a clear account of these the reader is referred to Ueno and Agaoka [11].) There are also a large number of non-edge-to-edge tilings with these triangles, which we shall not attempt to enumerate here; some of the possibilities are described in [3].

iv) The $(90°, 90° - \frac{180}{n}°, \frac{360}{n}°)$ quarterlunes (n odd)

When n is odd, there is no edge-to-edge tiling with the $(90°, 90° - \frac{180}{n}°, \frac{360}{n}°)$ triangle. However, there are tilings, in which the sphere is divided into n lunes with polar angle $\frac{360}{n}°$, each of which is subdivided into four $(90°, 90° - \frac{180}{n}°, \frac{360}{n}°)$ triangles. This may be thought of as a further subdivision of the tiling with $2n$ $(180° - \frac{360°}{n}, \frac{360}{n}°, \frac{360}{n}°)$ triangles, given in [3].

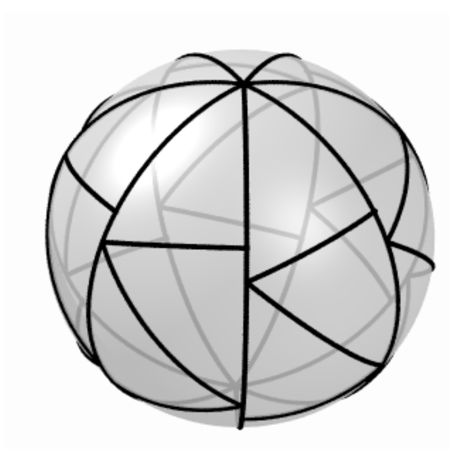

Figure 3: An odd quarterlune tiling

There are two ways to divide a lune into four triangles, mirror images of each other, and this choice may be made independently for each lune. When two adjacent dissections are mirror images, then the edges match up correctly on the common meridian; but with

n odd, this cannot be done everywhere. (However, it is interesting to note that a double cover of the sphere with $2n$ lunes can be tiled in an edge-to-edge fashion.) As shown in [3], there are appproximately $2^{2n-2}/n$ essentially different tilings of this type.

The symmetry group depends on the choice of tiling; most tilings are completely asymmetric. We have $\mathcal{V} = \{(4,0,0),(2,2,1),(1,1,\frac{n+1}{2}),(0,4,2),(0,0,n)\}$ in all cases (see section 5). It may be shown that no tiling with this tile can contain an entire great circle within the union of the edges; as the tile itself is asymmetric, no tiling can have a mirror symmetry. The largest possible symmetry group is thus the proper dihedral group of order $2n$.

We do not at present know whether there are other tilings with these triangles, as there are when n is even. Despite the existence of two vertex vectors not used in any of the known tilings, we conjecture that there are not.

v) The $(90°, 75°, 60°)$ triangle

This triangle subdivides the $(150°, 60°, 60°)$ triangle. It was shown in [3] that eight copies of the latter triangle tile the sphere; thus, sixteen $(90°, 75°, 60°)$ triangles tile.

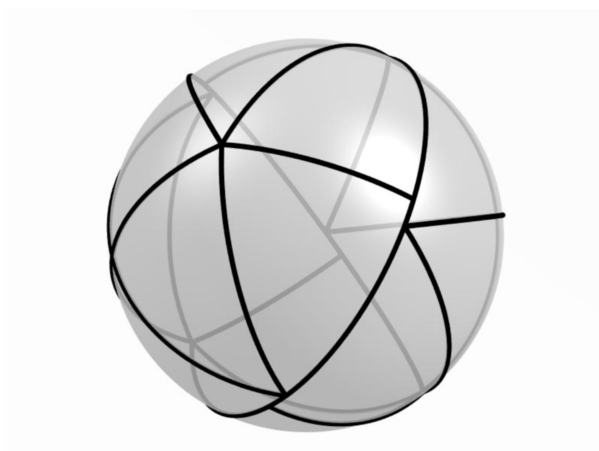

Figure 4: The tiling with the $(90°, 75°, 60°)$ triangle

This tiling is unique up to mirror symmetry (Proposition 26). Its symmetry group is the Klein 4-group, represented by three 180° rotations and the identity. (As this does not include the point inversion, we conclude that the $(90°, 75°, 60°)$ triangle fails to tile the projective plane.) An interesting feature of this tiling (and the one it subdivides) is the long extended edge, of length $226.32+°$, visible in the figure.

vi) The $(90°, 60°, 40°)$ triangle

This triangle tiles the sphere ($N = 72$) in many ways. Two copies make one $(80°, 60°, 60°)$ triangle, which was shown in [4] to tile the sphere in three distinct ways. Moreover, four copies yield the $(120°, 60°, 40°)$ triangle, and six copies yield the $(140°, 60°, 40°)$ triangle.

Both of these tile as semilunes, giving tilings of the 40° and 60° lunes respectively (the latter already non-edge-to-edge).

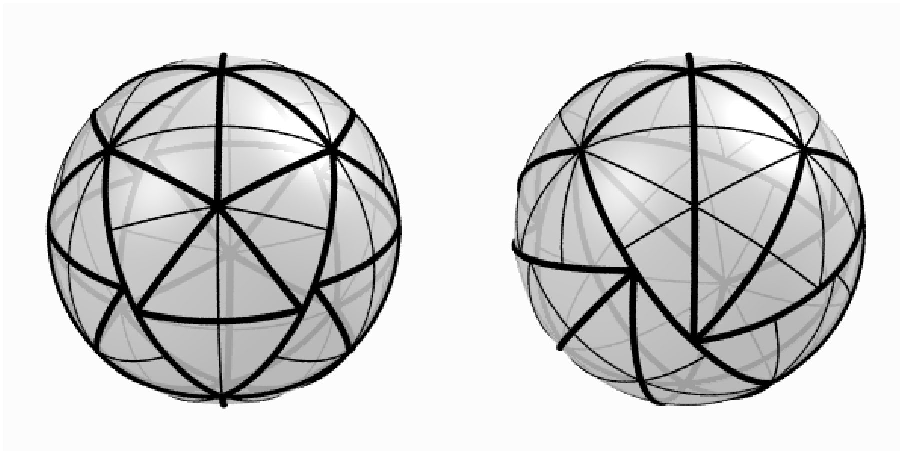

Figure 5: Some tilings with the $(90°, 60°, 40°)$ triangle

Five copies yield the $(90°, 100°, 40°)$ triangle, and seven yield the $(90°, 120°, 40°)$ triangle. While neither of these tiles, either combines with the $(140°, 60°, 40°)$, yielding the $(90°, 140°, 60°)$ and $(90°, 140°, 80°)$ triangle respectively; and combining all three gives a 90° lune (Figure 6), which does tile. It is interesting to note that this (unique; we leave this as an exercise to the reader!) tiling of the 90° lune has no internal symmetries; usually when a lune can be tiled it may be done in a centrally symmetric fashion..

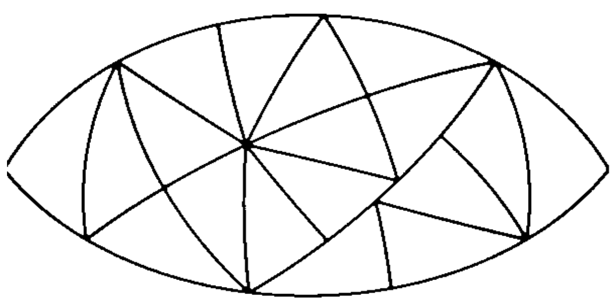

Figure 6: The unique tiling of the 90° lune with the $(90°, 60°, 40°)$ triangle

Furthermore, six tiles can also be assembled into an $(80°, 80°, 80°)$ triangle, which, while it does not tile on its own, yields tilings in combination with three 100° lunes, each assembled out of one 40° and one 60° lune.

It seems probable that the most symmetric tiling is the one with nine 40° lunes, with a symmetry group of order 18 and 4 orbits; various other symmetries are possible, including completely asymmetric tilings. Some tilings (such as the one on the left in Figure 5) have central symmetry, so this triangle tiles the projective plane as well as the sphere.

A complete enumeration of the tilings with this tile remains an interesting open problem.

***vii)*: The $(90°, 75°, 45°)$ triangle**

Eight copies of this triangle tile a 120° lune, in a rotationally symmetric fashion (Figure 7). There are exactly two distinct ways to fit three such lunes together, forming non-edge-to-edge tilings with $N = 24$. Either of the three lunes have the same handedness, in which case edges do not match on any of the three meridian boundaries and the symmetry group of the tiling is of order 6; or one lune has a different handedness than the other, edges match on two of the three meridians, and the symmetry group has order 2. It is conjectured that there are no other tilings.

A double cover of the sphere exists with 48 tiles in six lunes, alternating handedness; this double cover is edge-to-edge.

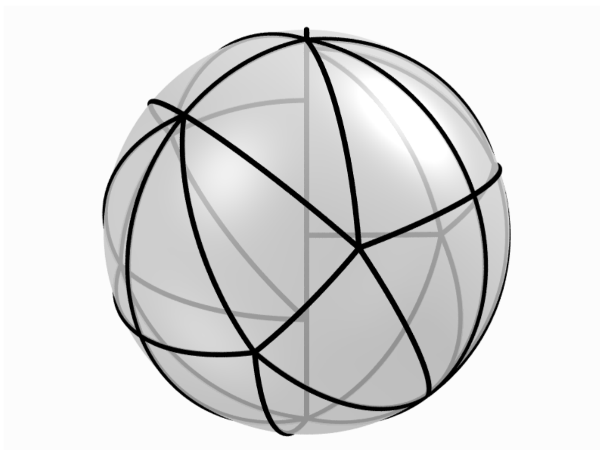

Figure 7: A tiling with the $(90°, 75°, 45°)$ triangle

***viii)*: The $(90°, 78\frac{3}{4}°, 33\frac{3}{4}°)$ triangle**

This triangle is conjectured to tile uniquely (N=32) up to reflection (Figure 8). The symmetry group of the only known tiling is the Klein 4-group, represented by three 180° rotations and the identity. The tiles are partitioned into eight orbits under this symmetry group; this appears to be the largest possible number of orbits for a maximally symmetric tiling. This tiling, like the previous one, is also noteworthy for having a rather small number of split vertices; in a sense, such tilings are "nearly edge-to-edge".

4 Proof of Theorem 1

The proof of Theorem 1 breaks up naturally into a sequence of propositions, dealing separately with each possible V_1. The nontrivial asymptotically right V_1 are $(0, 4, 3)$, $(0, 4, 2)$, $(0, 4, 1)$, $(1, 3, 2)$, and $(1, 3, 1)$; there are also the trivial (and equivalent) cases $(0, 4, 0)$ and $(1, 3, 0)$ for which the triangle is isosceles with two right angles. It is shown in [3] that these triangles tile the sphere precisely when the third angle divides 360°; and

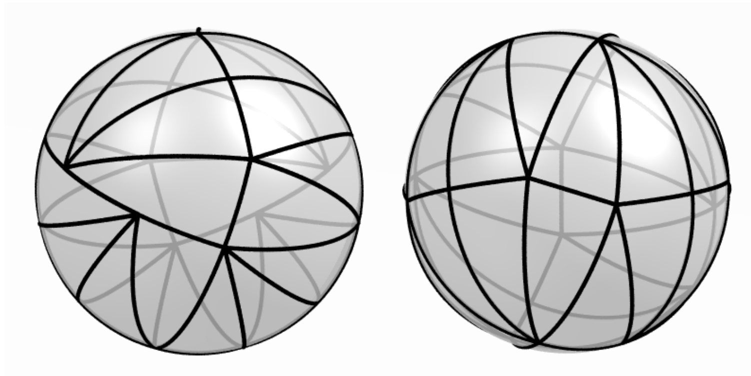

Figure 8: A tiling with the $(90°, 78\frac{3}{4}°, 33\frac{3}{4}°)$ triangle

in these cases there is always an edge-to-edge tiling [2, 10]. For each remaining V_1, we will begin by determining an exhaustive set of V_2, and, for each of these, find the rest of $\mathcal{V}$. In some cases the lack of a split vector other than $(4,0,0)/2$ will then eliminate the triangle from consideration; in other cases we will need to examine the geometry explicitly.

4.1 The $(0,4,3)$ family

Proposition 3 *If a right triangle tiles the sphere and has $V_1 = (0,4,3)$, then without loss of generality $V_2 = (0,0,c')$ or $(1,1,c')$.*

Proof: Consider any reduced γ source $V = (a_V, b_V, c_V)$; by definition, $a_V = 0$ or 1. If $a_V = 1$ and $1 < b_V$, we have $c_V \geq 3$. Then $W = 4V - 2(0,4,3) - (4,0,0)$ has $a_W = 0$ and $c_W > b_W > 0$ and is again a reduced γ source in $\mathcal{V}$. If $a_V = 1$ and $b_V = 0$, then $W = \frac{4}{3}V - \frac{1}{3}(4,0,0))$ has $a_W = b_W = 0$ and is also a reduced γ source in $\mathcal{V}$. Thus, without loss of generality, $a_V = 0$ or $a_V = b_V = 1$.

Now suppose $a_V = 0$ and $b_V > 0$. As V is a γ source, we must have $b_V = 1, 2$, or 3. If $b_V = 2$, then $W = 2V - (0,4,3)$ is a reduced γ source in $\mathcal{V}$ and has $a_W = b_W = 0$. If $b_V = 3$, then $W = 4V - 3(0,4,3)$ is a reduced γ source in $\mathcal{V}$ with $a_W = b_W = 0$. Finally, if $b_V = 1$, we solve the system of equations

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 1 & c_V \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 360° \\ 360° \\ 360° \end{pmatrix} \tag{5}$$

to obtain

$$\begin{aligned} \alpha &= 90° \\ \beta &= \left(\frac{360c_V - 1080}{4c_V - 3}\right)^{\circ} \\ \gamma &= \left(\frac{1080}{4c_V - 3}\right)^{\circ} \end{aligned}$$

so that

$$N = \frac{720^\circ}{\alpha+\beta+\gamma-180^\circ} = \frac{32c_V}{3} - 8\ ;$$

but this is only an integer when $3|c_V$. As we have assumed that the triangle tiles, this must be the case; and $W = \frac{4}{3}V - \frac{1}{3}(0,4,3)$ is a reduced γ source in $\mathcal{V}$ with $a_W = b_W = 0$. ■

Proposition 4 *If a right triangle tiles the sphere and has $V_1 = (0,4,3)$ and $V_2 = (0,0,c')$, then $c' \geq 8$ and $\mathcal{V}$ consists of the vectors in the appropriate set below that have all components positive:*

$$\left\{(4,0,0),(0,4,3),(0,0,c'),(1,0,\tfrac{3c'}{4}),(2,0,\tfrac{c'}{2}),(3,0,\tfrac{c'}{4}),(1,4,3-\tfrac{c'}{4})\right\} \quad \textit{if } c' \equiv 0 \textit{ mod } 4$$

$$\begin{array}{ll}\big\{(4,0,0),(0,4,3),(0,0,c'), & \\ \quad (0,2,\frac{c'+3}{2}),(2,1,\frac{c'+3}{4}),(2,3,\frac{9-c'}{4}),(0,6,\frac{9-c'}{2})\big\} & \textit{if } c' \equiv 1 \textit{ mod } 4\end{array}$$

$$\left\{(4,0,0),(0,4,3),(0,0,c'),(2,0,\tfrac{c'}{2}),(1,2,\tfrac{c'+6}{4})\right\} \quad \textit{if } c' \equiv 2 \textit{ mod } 4$$

$$\begin{array}{ll}\big\{(4,0,0),(0,4,3),(0,0,c'), & \\ \quad (0,1,\frac{3c'+3}{4}),(0,2,\frac{c'+3}{2}),(0,3,\frac{c'+9}{4}),(0,5,\frac{15-c'}{4})\big\} & \textit{if } c' \equiv 3 \textit{ mod } 4.\end{array}$$

(To be explicit, $(1,4,3-\frac{c'}{4})$ is present for $c' = 8, 12$; $(2,3,\frac{9-c'}{4})$ and $(0,6,\frac{9-c'}{2})$ for $c' = 9$; and $(0,5,\frac{15-c'}{4})$ for $c' = 11, 15$.)

Proof: (i) Solving, as above, for β and γ, and noting that $\beta > \gamma$, we have $360c' - 1080 > 1080$ and $c' \geq 8$.

(ii) The equation of the plane $\Pi_{\mathcal{V}}$ containing $\mathcal{V}$ is

$$4c = 4c' - c'a - (c'-3)b\ . \tag{6}$$

We need to find the non-negative integer points on this plane. Substituting the lower bounds $c' \geq 8$, $c \geq 0$ into this, we obtain

$$8a + 5b \leq 32 \tag{7}$$

On the other hand, we note that, regardless of the value of c',

$$a + b \leq 4 \Rightarrow c \geq 0\ . \tag{8}$$

Reducing (6) modulo 4, we obtain

$$(a+b)c' \equiv -b \pmod 4. \tag{9}$$

The final step depends on the congruence class of c' (mod 4).

$c' \equiv 0$: In this case, (9) reduces to $b \equiv 0 \pmod 4$, and subject to (7) we have

$$(a,b) \in \{(0,0),(1,0),(2,0),(3,0),(4,0),(0,4),(1,4)\}$$

The first six of these pairs satisfy (8) and thus give rise to solutions (as listed above) for all c'; the last gives $c \geq 0$ only for $c' = 8, 12$.

$c' \equiv 1$**:** Now, (9) reduces to $a \equiv 2b \pmod 4$, and we have

$$(a, b) \in \{(0, 0), (0, 2), (0, 4), (2, 1), (4, 0), (0, 6), (2, 3)\}$$

Again, the first five of these give rise to solutions for all c'; the last two give $c \geq 0$ for $c' = 9$ only.

$c' \equiv 2$**:** This gives us $2a \equiv b \pmod 4$, and the only solutions are

$$(a, b) \in \{(0, 0), (0, 4), (1, 2), (2, 0), (4, 0)\}$$

All of these satisfy (8).

$c' \equiv 3$**:** This makes $a \equiv 0 \pmod 4$, and we have

$$(a, b) \in \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (4, 0), (0, 5)\}$$

All but the last of these satisfy (8); for $(0, 5)$ we get $c \geq 0$ only for $c' = 11, 15$.

Computing the various values of c completes the proof. ■

It was shown in [5] that the only right triangle to tile the sphere in a non-edge-to-edge fashion, with no split vertex other than $(4, 0, 0)/2$, is the $(90°, 108°, 54°)$ triangle (which has $V_1 = (1, 2, 1)$). Any other triangle, not tiling edge-to-edge, is thus shown not to tile as soon as it is shown that it has no second split. In particular, the triangles considered above with $c' \equiv 3 \pmod 4$ never tile. We also have the following:

Proposition 5 *No right triangle that has $V_1 = (0, 4, 3)$ and $V_2 = (1, 1, c')$ tiles the sphere.*

Proof: The equation of of $\Pi_{\mathcal{V}}$ is

$$8c = 16c' - 12 - (4c' - 3)a - (4c' - 9)b\ . \tag{10}$$

Computing modulo 8, we obtain

$$\begin{aligned} a + 3b &\equiv 4 \pmod 8) \qquad && (c' \text{ is odd}) \\ 3a + b &\equiv 4 \pmod 8) \qquad && (c' \text{ is even})\ . \end{aligned} \tag{11}$$

Multiplying either of these congruences by 3 gives the other, so they have the same solutions.

The requirement that $\beta > \gamma$ gives $c' \geq 5$, and substituting this and $c \geq 0$ into (10) gives us the inequality $14a + 8b \leq 56$. But the only pairs (a, b) that satisfy this inequality and the congruences (11) are $(0, 4)$,$(1, 1)$, and $(4, 0)$, so $\mathcal{V}$ never has any elements other than the given basis. Moreover, among these, only $(4, 0, 0)$ corresponds to a split, so none of these triangles tile the sphere. ■

Proposition 6 *The only right triangle that has $V_1 = (0, 4, 3)$ and tiles the sphere is the $(90°, 60°, 40°)$ triangle.*

Proof: On the strength of the previous three propositions, we may assume that $V_2 = (0, 0, c')$ with $c' \geq 8$ and $c' \not\equiv 3 \pmod 4$. All such triangles have $B < H$. When $c' = 9$ we have the $(90°, 60°, 40°)$ triangle.

If $c' \equiv 1 \pmod 4$ and $c' > 9$, the only vertex vector corresponding to a split is $(0, 2, \frac{c'+3}{2})$, in which γ angles outnumber β angles by at least 3; and the only β source is $(0, 4, 3)$. Let the vertex O be one such β source. At least one of the three triangles contributing a γ vertex to O must have its medium edge Oa paired with a hypotenuse or short edge, not another medium edge.

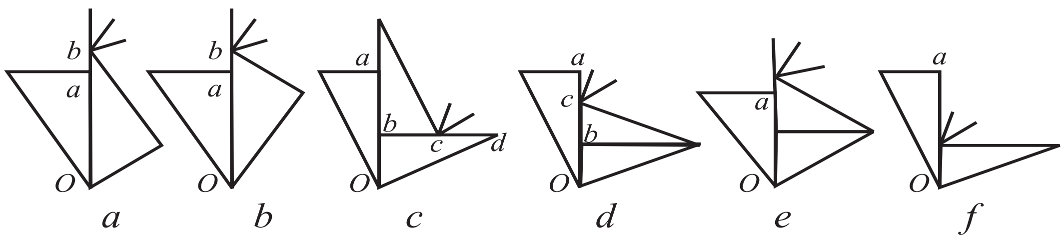

Figure 9: Configurations near a $(0, 4, 3)$ vertex

If the other edge Ob is a hypotenuse (Figure 9*a*,*b*), b is necessarily a split vertex. If Ob is short, (Figure 9*c*,*d*; note that for $c' \geq 13$, we have $B > 2C$), there must again be an associated split vertex, on the extended edge bc. In every case, the split vertex has a surplus of at least three γ angles. Examining the four configurations, we see that it is not possible for the identified split vertex to be related in any of these four ways to two $(0, 4, 3)$ vertices O, O' unless certain relations hold among the edge lengths which are easily ruled out by numerical computation - for instance, in Figure 10, only a medium edge could fill the gap $\overline{bb'}$ without a β split; but it is easily verified that $3B \neq 2H$.

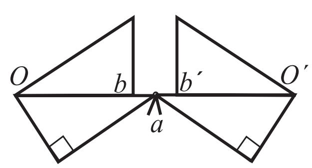

Figure 10: Two $(0, 4, 3)$ vertices attempting to share a split vertex

We conclude that every $(0, 4, 3)$ vertex is associated in a 1-1 fashion with a $(0, 2, 2n)/2$ split vertex; but this requires the number of γ angles in the whole tiling to be greater than the number of β angles, which is impossible. Thus, when $c' \equiv 1 \pmod 4$ and $c' > 9$, the triangle does not tile.

If c' is even, there are no β splits, and unless $c' = 8$ or 12, the only β source is $(0, 4, 3)$. We shall show that no tiling exists using this β source alone. As above, every such vertex must have an unpaired medium edge. If this edge is covered by a hypotenuse Ob, this must be oriented as in Figure 9a, as the β split in Figure 9b is impossible; and there is a γ split at b.

If it is covered by a short edge, there is a right-angle gap at b. In the absence of β splits, this cannot be filled by another right angle (Figure 9$c-e$ - this last configuration must be considered when $c' = 8$, 10, or 12, as then $2C > B$). The split must therefore be a right-γ split (Figure 9f).

It is easily verified that no split vertex can be related as in Figure 9a or 9f to two $(0,4,3)$ vertices; so each $(0,4,3)$ is associated with a split vertex that is not shared with any other $(0,4,3)$, and between them the number of γ angles is again greater than the number of β angles. Thus none of these triangles (including those with $c' = 8, 12$) tile using $(0,4,3)$ as the sole β source.

The only remaining possibilities for tilings involve triangles with $c' = 8$ or $c' = 12$, using $(1,4,1)$ and $(1,4,0)$ respectively as β sources. We shall show that these vertices, too, are necessarily associated with γ splits.

When $c' = 8$, we obtain the $(90°, 56\frac{1}{4}°, 45°)$ triangle. Between them, the angles meeting at a $(1,4,1)$ vertex O have five hypotenuses, at least one of which must be unpaired. The β angle of the unpaired hypotenuse must be at O, or its other end would require a β split. If we assume that the unpaired hypotenuse meets the short edge of the neighboring triangle, triangles 1,2 and 3 of Figure 11a are forced in turn by avoiding β splits. If it meets a long edge, one of Figure 11b,c is forced.

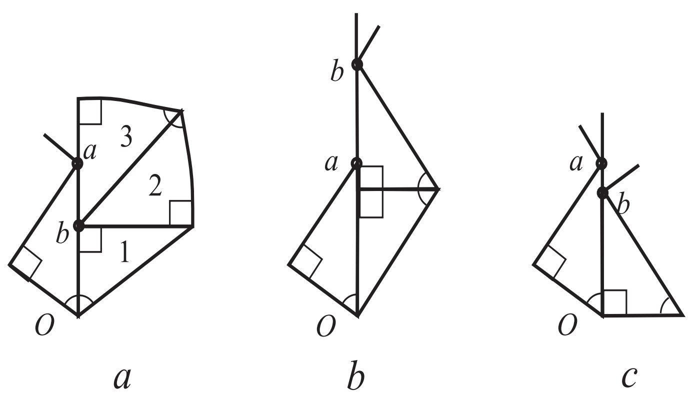

Figure 11: Configurations near $(1,4,1)$

In each of these three cases, the indicated *two* split vertices must exist. We must now consider whether at least one of these split vertices may form part of another configuration of the same type, of another type from Figure 11, or from Figure 9a or 9e (as shown above, the only two cases that can occur for a $(0,4,3)$ β source). Most of the 12 pairings are impossible unless the edge lengths satisfy simple equations that are easily ruled out, as in Figure 10.

The only three cases in which the designated split vertices can be shared are shown in Figure 12a (11c with 9a), Figure 12b (11c with another of the same type) and Figure

12c (11b with another of the same type). Figure 12a can be immediately ruled out, due to the unavoidable β split at y.

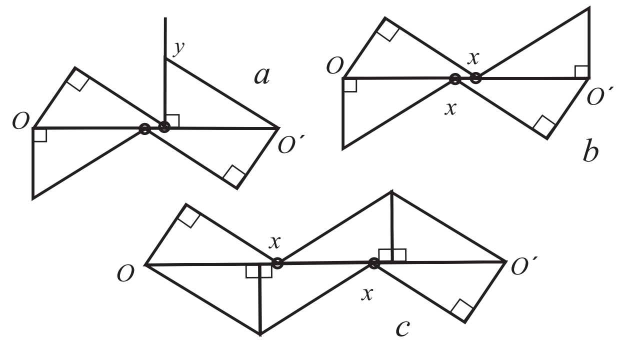

Figure 12: Two β sources O, O', sharing split vertices

In Figure 12b, c, two right-angled gaps x, x bounded by medium edges or hypotenuses exist, and these cannot be filled by a right angle without requiring a β split. The two $(1, 4, 1)$ vertices thus share split vertices containing a total of eight γ angles. In every other case, a single $(1, 4, 1)$ vertex has sole custody of two split vertices with at least four γ angles. In every case, these configurations require more γ angles than β angles; so the $(90°, 56\frac{1}{4}°, 45°)$ triangle does not tile.

We now consider the $(90°, 67\frac{1}{2}°, 30°)$ triangle, which has $c' = 12$. As shown above, it cannot tile without using vertices $(1, 4, 0)$ as β sources. Any vertex of this type has a single right angle, with an unpaired medium edge. This cannot meet the adjacent triangle (1, in Figure 13a) on a short edge, as avoiding β splits gives us triangles 2,3,4, and 5, and then a β split at x cannot be avoided. If the unpaired medium edge meets a hypotenuse, the β angle of the adjacent triangle must be at O, giving us 13b with two split vertices; and these are the only two possibilities.

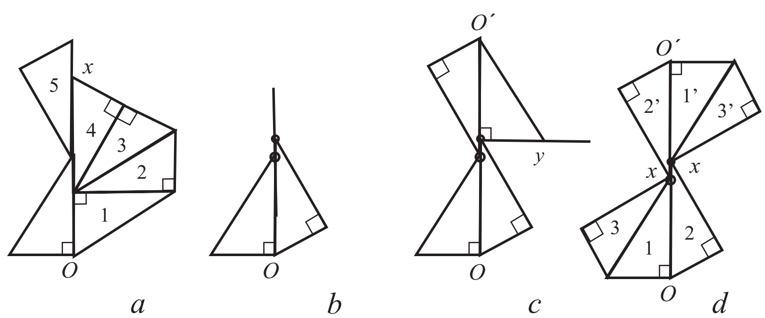

Figure 13: Configurations near $(1, 4, 0)$ vertices

Again, there are two special cases in which these γ splits may be shared with another β source. In one of these (Figure 13c) a $(1, 4, 0)$ vertex and a $(0, 4, 3)$ vertex share splits; but this can be eliminated because of the need for a β split at y. In the other(Figure 13d), two $(1, 4, 0)$ vertices O, O' share both their split vertices. The triangles 3,3$'$, are then forced; the right-angled gaps x, x cannot be filled with right angles without a β split; and so O and O' share twelve γ angles. In every case, γ angles outnumber β angles, so that the triangle cannot tile. ■

4.2 The $(0,4,2)$ family

Proposition 7 *If a right triangle has $V_1 = (0,4,2)$ and tiles the sphere, then without loss of generality $V_2 = (0,0,c')$. Conversely, every triangle with $(0,4,2),(0,0,c') \in \mathcal{V}$ tiles the sphere.*

Proof: The proof of the first part is similar to that of Proposition 3; we note that to have $\beta > \gamma$ we must have $c' \geq 6$, although for $c' = 4,5$ we have valid triangles that appear with their angles in the correct order elsewhere. Tilings with these triangles are quarterlune families (iii) (c' even) and (iv) (c' odd) described in Section 3. ■

4.3 The $(0,4,1)$ family

Proposition 8 *If a right triangle has $V_1 = (0,4,1)$ and tiles the sphere, then without loss of generality $V_2 = (0,0,c')$ or $(1,1,c')$.*

Proof: as for Proposition 3. ■

Proposition 9 *If a right triangle has $V_1 = (0,4,1)$ and $V_2 = (0,0,c')$ and tiles the sphere, then $3|c'$, $c' \geq 6$ and $\mathcal{V}$ =*

$$\begin{cases} \{(4,0,0),(0,4,1),(0,0,c'),(1,0,\frac{3c'}{4}),(2,0,\frac{c'}{2}),(3,0,\frac{c'}{4})\} & \textit{if } c' \equiv 0 \textit{ mod } 4 \\ \{(4,0,0),(0,4,1),(0,0,c'),(0,1,\frac{3c'+1}{4}),(0,2,\frac{c'+1}{2}),(0,3,\frac{c'+3}{4})\} & \textit{if } c' \equiv 1 \textit{ mod } 4 \\ \{(4,0,0),(0,4,1),(0,0,c'),(1,2,\frac{c'+2}{4}),(2,0,\frac{c'}{2})\} & \textit{if } c' \equiv 2 \textit{ mod } 4 \\ \{(4,0,0),(0,4,1),(0,0,c'),(0,2,\frac{c'+1}{2}),(2,1,\frac{c'+1}{4})\} & \textit{if } c' \equiv 3 \textit{ mod } 4. \end{cases}$$

Proof: as for Proposition 4.■

Proposition 10 *No right triangle with $V_1 = (0,4,1)$ and $V_2 = (1,1,c')$ tiles the sphere.*

Proof: as for Proposition 5; there is never any second split. ■

Proposition 11 *The only right triangle that has $V_1 = (0,4,1)$ and tiles the sphere is the $(90°,75°,60°)$ triangle.*

Proof: From Proposition 9 we see that there is no second split unless c' is divisible by 6, in which case we have $(0,0,c')/2)$; or unless $c' \equiv 3 \pmod{12}$, when we have $(0,2,\frac{c'+1}{2})$.

In the first case, we also have $(2,0,\frac{c'}{2})/2$ if $12|c'$; there are no further splits. In the case $c' = 6$ we obtain the $(90°, 75°, 60°)$ triangle, which has been shown to tile; henceforth, then, we suppose $c' \geq 12$. The possible splits are then $(0,0,2m)/2$ with $m \geq 6$ and $(2,0,4n)/2$ with $n \geq 3$.

We see also that $(0,4,1)$ is the only β source, so such a vertex must appear in any tiling with this triangle. We examine the neighborhood of any such vertex O (see Figure 14). Let triangle 1 contribute the γ angle. Consider the triangle 2, which covers the long leg of 1 near O. If the short leg of 2 meets 1 (Figure 14a, b) and the gap is filled by a right angle, then we need a β split, which is impossible. (It is easily checked that $2C \neq B$ for any triangle in this family.)

If the gap is filled by γ angles (Figure 14c), or if the long leg of 1 is covered by the hypotenuse of 2 (Figure 14d), there is a γ split at x. This split cannot be related in the same way to any other $(0,4,1)$ vertex.

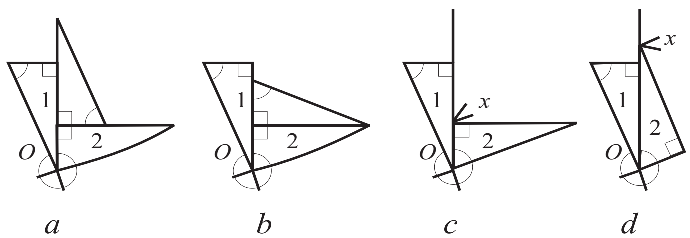

Figure 14: The split vertex associated with O

Unless the split vertex x is of the form $(2,0,6)/2$ there are more than three γ angles at x, and it follows that O and x between them have a surplus of γ angles; thus the entire tiling has a surplus of γ angles, which is impossible.

If X is $(2,0,6)/2$, there must be a right angle at x. If x is as shown in Figure 14c, triangles 3 and 4 must be as shown in Figure 15a to avoid a β split; but then whichever way we place the third triangle between them, a β split is required.

If x has the configuration of Figure 14d, and the right angle is between two γ angles (Figure 15b) , a β split is required (at y); if not (Figure 15c), we must either have a β split at z, or have $H + C = 2B$, which is easily shown not to hold for any triangle in this family.

We now consider the case in which $c' \equiv 3 \pmod{12}$. There is never any split except for $(4,0,0)/2$ and $(0,2,\frac{c'+1}{2})/2$; and, again, $(0,4,1)$ is the only β source. We will show that any such vertex is necessarily associated with enough γ angles at $(0,2,\frac{c'+1}{2})/2$ splits (note that $\frac{c'+1}{2} \geq 8$) that the tiling must have a net surplus of γ angles.

The angles at a $(0,4,1)$ vertex have, adjacent to them, a total of four short edges, a medium edge, and five hypotenuses. Either all short edges are paired, or at least two are unpaired. But, as Figure 16 shows, every configuration with an unpaired short edge requires a nearby $(0,2,\frac{c'+1}{2})/2$ split, connected to it by a chain of two short edges (either in line or perpendicular).

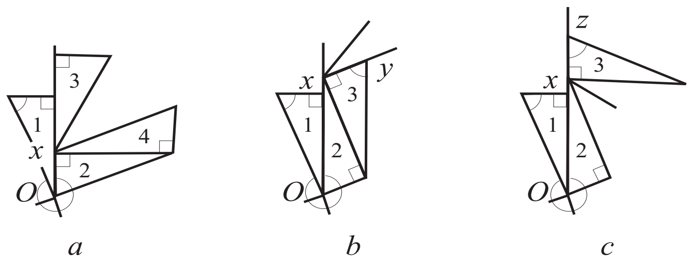

Figure 15: X cannot have a right angle

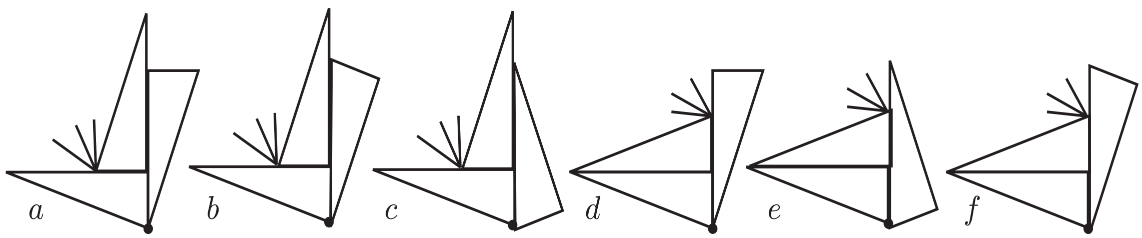

Figure 16: Split vertices associated with unpaired short edges

If all short edges are paired, the $(0, 4, 1)$ vertex O has the triangles around it postioned as 1-5 in Figure 17a. Filling the gap at p necessarily produces a $(0, 2, \frac{c'+1}{2})/2$ split at q, and a second one either at r (Figure 17b) or at s (Figure 17c). In the latter case, triangle 7 must be as shown.

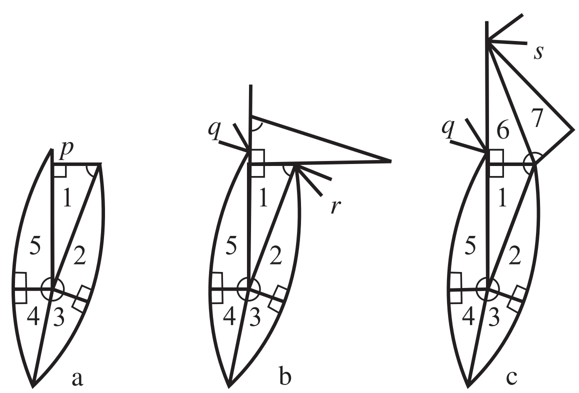

Figure 17: Split vertices associated with the remaining configuration

Each of these splits is at a distance l_1 from the end of the extended edge it lies on, and requires one or more specified edges with total length l_2 on the opposite side. For the configurations of Figure 16, the pair (l_1, l_2) can be (C, B) (Figure 16$a - c$), $(2C, B)$ (Figure 16d), or $(2C, B)$ (Figure 16e, f). In Figure 17b, the length pairs are $(H, B + C)$ and (C, B); and in Figure 17c, the length pairs are $(H, 2B)$ and $(2B, H)$. In order for two configurations to share a split, the length of the so far uncovered segment on the opposite side, $(l_1 + l_1') - (l_2 + l_2')$, must either be zero or a sum of edge lengths. As there is only one pair, $(2B, H)$, with $l_1 \geq l_2$, there are few cases to consider, and all are easily ruled out except for the case in which the vertex B of Figure 17c is paired with the vertex D of a similar configuration (Figure 18a)

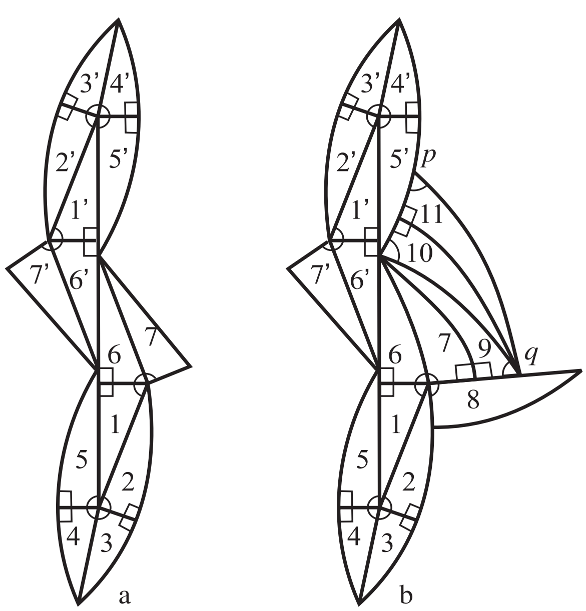

Figure 18: An impossible configuration

Positioning triangle 8 as shown in Figure 18b would force triangle 9, 10, and 11 as shown; but the hypotenuse of the latter triangle cannot be covered without a split with two β angles at either p or q. We thus have triangle 8 as shown in Figure 18c and there is an overhang as shown at r. But now the extended edges $\overline{rs}$ and $\overline{st}$ can each be covered only as shown (as there can be no right angle at r, s, or t), and this leaves a gap at s that cannot be filled.

It follows that, in this case as well, every β source is uniquely associated with enough γ angles to give a net surplus of γ angles. We conclude that, except for the $(90°, 75°, 60°)$ triangle, no triangle in the $(0, 4, 1)$ family tiles the sphere. ■

4.4 The $(1, 3, 2)$ family

Proposition 12 *If a right triangle has* $V_1 = (1, 3, 2)$ *and tiles the sphere, then without loss of generality* $V_2 = (0, 0, c')$, $(0, 1, c')$, $(0, 2, c')$, $(1, 0, c')$, *or* $(1, 1, c')$.

Proof: Consider a reduced γ source $V = (a_V, b_V, c_V)$; by definition, $a_V = 0$ or 1. Suppose $a_V = 0$. If $b_V \geq 4$, then $c_V > 4$ and $\beta + \gamma < 90°$, which is impossible. Suppose now that $b_V = 3$; then $W = 4V - 4(1, 3, 2) + (4, 0, 0) = (0, 0, 4c_V - 8)$ is also a γ source.

If on the other hand $a_V = 1$, we must (by a similar argument) have $b_V \leq 2$. If $b_V = 2$, then $c_V > 2$ and $2V - (1, 3, 2) = (1, 1, 2c' - 2)$ is also a γ source. ■

Proposition 13 *If a right triangle has* $V_1 = (1,3,2)$ *and* $V_2 = (0,0,c')$ *and tiles the sphere, then* $c' \geq 8$ *and* $\mathcal{V}$ *consists of all vectors in*

$$\Big\{(4,0,0),(1,3,2),(0,0,c'),(0,3,\frac{c'+8}{4}),(0,6,\frac{8-c'}{2}),$$
$$(1,0,\frac{3c'}{4}),(2,0,\frac{c'}{2}),(2,3,\frac{8-c'}{4}),(3,0,\frac{c'}{4})\Big\}$$

that have nonnegative integer components.

Proof: as for Proposition 4. ∎

Proposition 14 *The only right triangle that has* $V_1 = (1,3,2)$ *and* $V_2 = (0,0,c')$ *and tiles the sphere is the* $(90°, 60°, 45°)$ *triangle, which tiles edge-to-edge.*

Proof: As observed above, c' must be at least 8; and unless it is even there is no second split. When $c' = 8$ we obtain the known $(90°, 60°, 45°)$ tile, so we consider the case when $c' \geq 10$. The minimum number of γ angles at a split other than $(4,0,0)/2$ is 3, achieved by the $(2,0,6)/2$ split when $c' = 12$.

The only β source is the rather weak $(1,3,2)$; so such a vertex (call it O) must appear in any tiling. Between them, the angles at O have 4 short edges, 3 medium edges, and 5 hypotenuses. There is thus at least one unpaired hypotenuse. This cannot be covered exactly by other edges; $2C < H < B + C$, and in the absence of a β split we cannot have more than two short edges on an extended edge. The other end of this hypotenuse is therefore at a split, necessarily involving at least three γ angles.

The split vertex is contained in an extended edge which terminates at O. At most two $(1,3,2)$ vertices can be related to one split in such a way; but between them these three vertices have seven γ angles and only six β angles. Thus no such tiling is possible. ∎

Note: In fact, it is probably true that the split vertex could not be related even to a second $(1,3,2)$ vertex, but it is easier to concede the point.

Proposition 15 *No right triangle with* $V_1 = (1,3,2)$ *and* $V_2 = (0,1,c')$ *tiles the sphere.*

Proof: Calculation shows that $N = 8c' - \frac{16}{3}$, which is never an integer. ∎

Proposition 16 *No right triangle with* $V_1 = (1,3,2)$ *and* $V_2 = (0,2,c')$ *tiles the sphere.*

(The proof of this proposition is lengthy, and is carried out in the companion paper [6] .)

Proposition 17 *The only right triangle with* $V_1 = (1,3,2)$ *and* $V_2 = (1,0,c')$ *that tiles the sphere is the* $(90°, 60°, 45°)$ *triangle, which tiles edge-to-edge.*

Proof: If $c' \equiv 0 \pmod 3$ we have $(0, 0, \frac{4c'}{3}) \in \mathcal{V}$; by Proposition 14 this gives the $(90°, 60°, 45°)$ tile for $c' = 6$ and triangles that do not tile otherwise. If $c' \equiv 1$ (mod 3), we have $(0, 2, \frac{2c'+4}{3}) \in \mathcal{V}$ and by Proposition 16 none of these triangles tile. Finally, if $c' \equiv 2 \pmod 3$, every vertex vector (a, b, c) has $a \equiv 1$, and the only split is $(4, 0, 0)/2$. ■

Proposition 18 *No right triangle with $V_1 = (1, 3, 2)$ and $V_2 = (1, 1, c')$ tiles the sphere.*

Proof: If c' is even then $\frac{3}{2}(1, 1, c') - \frac{1}{2}(1, 3, 2) = (1, 0, \frac{3c'}{2} - 1)$ is also a γ source. As $\frac{3c'}{2} - 1 \equiv 2 \pmod 3$, there is (as shown in the proof of Proposition 17) no second split and these triangles do not tile. When c' is odd, a calculation similar to that of Proposition 4 shows that $\mathcal{V} = \{(4, 0, 0), (1, 3, 2), (1, 1, c')\}$ and again there is no second split. ■

4.5 The $(1, 3, 1)$ family

Proposition 19 *If a right triangle has $V_1 = (1, 3, 1)$ and tiles the sphere, then without loss of generality $V_2 = (0, 0, c')$, $(0, 2, c')$, $(1, 0, c')$ or $(1, 1, c')$.*

Proof: as for Proposition 3. ■

Proposition 20 *If a right triangle has $V_1 = (1, 3, 1)$ and $V_2 = (0, 0, c')$ and tiles the sphere, then $c' \geq 6$ and $\mathcal{V}$ consists of all vectors in*

$$\{(4, 0, 0), (1, 3, 1), (0, 0, c'), (0, 3, \frac{c'+4}{4}), (1, 0, \frac{3c'}{4}), (2, 0, \frac{c'}{2}), (3, 0, \frac{c'}{4})\}$$

that have integer components.

Proof: as for Proposition 4. ■

Proposition 21 *The only right triangle that has $V_1 = (1, 3, 1)$, $V_2 = (0, 0, c')$, and tiles the sphere is the $(90°, 75°, 45°)$ triangle.*

Proof: We note that there is no second split when c' is odd; so by Proposition 1 we may assume c' to be even. There is never a β split. As a result, there can be no $(1, 0, \frac{3c'}{4})$ or $(3, 0, \frac{c'}{4})$ vertices; the right angles at such a vertex would have between them an odd number of short edges terminating in β angles, at least one of which would be unmatched. By the same token, a $(2, 0, \frac{c'}{2})$ vertex can only exist if $4|c'$ and the two short edges of the right triangles are paired, as in Figure 19a.

We will now show that (with this interpretation) a split vertex involving γ angles always exists. (Note that Proposition 1 states that a tile other than $(90°, 108°, 54°)$ must have a second split *vector*, but not that it is necessarily used in the tiling.) Suppose, for

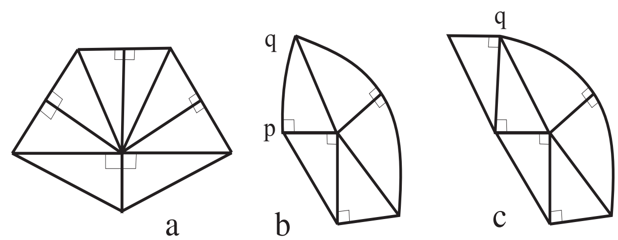

Figure 19: Split vertex near a (1,3,1) vertex

a contradiction, that there is a tiling that does not use any γ split. As $(0, 3, \frac{c'+4}{4})$ is never a β source, there must exist at least one $(1, 3, 1)$ vertex. In the absence of non-right-angle splits, the only possible configuration at such a vertex is as shown in Figure 19b. But then the edge $\overline{pq}$ must be covered by another medium edge. The right angle cannot go at q, as no vertex vector has two right angles and a β; we thus have the configuration of Figure 19c, in which q is the required $(2, 0, \frac{c'}{2})$ vertex.

However, we will see that unless $c' = 8$ no split vertex is possible. It is clear that when we put a "fan" of γ angles at a vertex, all β angles must either be at the end of the "fan" or paired, as otherwise there will be an overhang and a gap that cannot be filled without a β split. (Figure 20a). For a $(0, 0, c')/2$ split, $c' \geq 10$ this results in two pairs of adjacent edges such as $\overline{pq}$, $\overline{qr}$. In the absence of a β split, the only way to cover either of these extended edges is with another pair of triangles as shown; but this leaves an impossible four β angles at q. Similar problems occur for $(2, 0, \frac{c'}{2})/2$ splits with $c' \geq 20$ (Figure 20b).

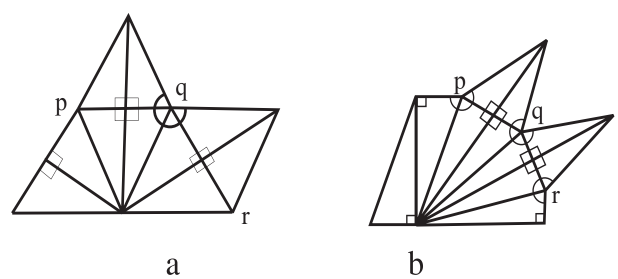

Figure 20: Fans of triangles lead to illegal configurations

There remain three cases when $c' = 6$, 12, and 16. When $c' = 6$, the only split vector other than $(4, 0, 0)/2$ is $(0, 0, 6)/2$. Avoiding overhangs at β angles forces the configuration of Figure 21a.

If the extended edge $\overrightarrow{pq}$ were extended beyond q, we would have an overhang or a fourth β angle at r; neither is permitted. It follows that $\overline{pq}$ is a complete extended edge and must be covered by another hypotenuse and medium edge on the other side; this implies a second copy of the same configuration. If the second copy were a mirror image, we would have four β angles at p; the only alternative is the configuration of Figure 21b. The edge $\overline{qr}$ must be matched, but we cannot have two right angles and a β together, so the triangle 1 (and the corresponding $1'$) must be as shown. This leaves a γ gap at r which must be filled by triangle 2 as shown, putting a β angle at s. But then the edge

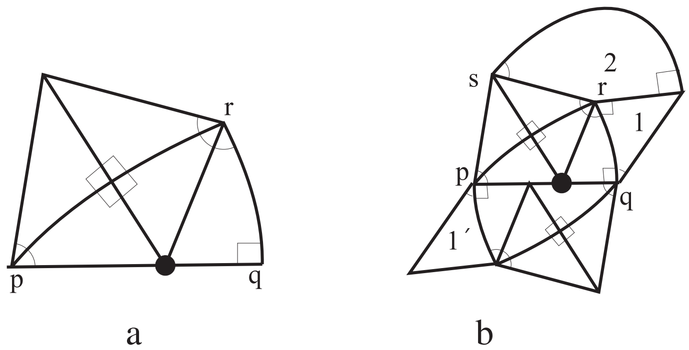

Figure 21: The $(0, 0, 6)/2$ split

$\overline{ps}$ cannot be covered without creating an illegal combination of angles at one end or the other.

When $c' = 12$, we have already ruled out $(0, 0, 12)/2$ but we must show that the $(2, 0, 6)/2$ split also leads to illegal configurations. By arguments similar to those used in the last case, we obtain the configuration of Figure 22a. If we put a right angle and a γ angle into the gap at p, the short edge at the right angle will be unpaired and will require an illegal β split; it follows that p must be a $(0, 3, 4)$ vertex.

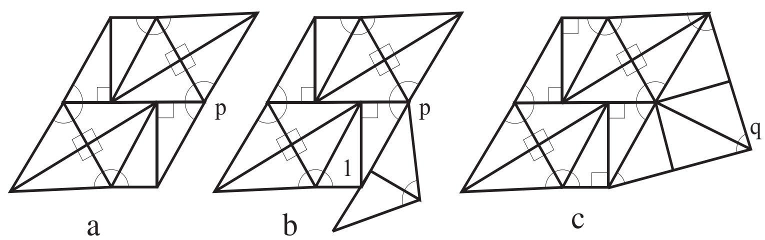

Figure 22: The $(2, 0, 6)/2$ split

There are only two ways to place the γ angles without a β split; in Figure 22b, an overhang is created that makes it impossible to cover the remaining edge of triangle 1, while in Figure 22c, we eventually get four β angles at q (as in Figure 20).

Finally, when $c' = 16$, we first note that we cannot have a $(0, 3, 5)$ vertex. The set of edges of the angles meeting at such a vertex would contain eight hypotenuses, five of them from γ angles and hence terminating in a β angle. All of these must be paired with other hypotenuses to prevent a β split, and at least two of them must be paired with each other, as triangles 1,2 are in Figure 23*a*.

We now examine the medium edges of these angles. Either one of these edges is matched (as at left), in which case there is an extended edge $\overline{pq}$ which must be matched exactly by two more short edges; or it is not, in which case there is an overhang (as at r). In any case, we end up with two more β angles at p, which is impossible.

Now we show that we cannot have a $(2, 0, 8)/2$ split. Suppose we did; by arguments similar to those used above, its neighborhood would have the configuration of Figure 23*b*. The extended edge $\overline{st}$ must be covered as shown (triangles 3,4 in Figure 23*c*). Triangle

5 is then forced, as the only way to cover $\overline{tu}$ without creating a β split or a vertex with four β angles.

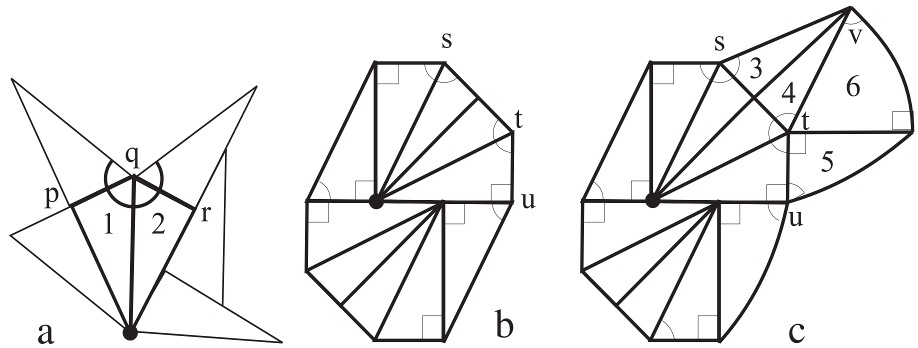

Figure 23: The $(2, 0, 8)/2$ split

Then t must be a $(1, 3, 1)$ vertex. The remaining γ angle is provided by triangle 6. The hypotenuse of that triangle must be paired with that of triangle 4 to avoid a β split; but then vertex v has at least two γ angles and a β angle. However, the only such vertex vector is $(0, 3, 5)$, and we have seen that this cannot occur in a tiling.

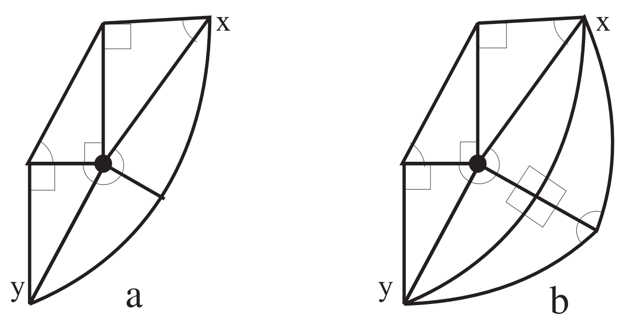

Figure 24: The $(1, 3, 1)$ vertex in the absence of the $(2, 0, 8)/2$ split

We can now show that there must be a $(0, 0, 16)/2$ split associated with each $(1, 3, 1)$ β source. Otherwise, the $(1, 3, 1)$ vertex must have the configuration of Figure 24*a*. By hypothesis, the extended edge $\overline{xy}$ is be covered exactly, and this can only be done as shown in Figure 24*b*. But then vertex x has two γ angles and a β angle, and we have seen that this is impossible. We conclude that there is a net shortage of β angles, so the triangle fails to tile the sphere. ■

Proposition 22 *If a right triangle has* $V_1 = (1, 3, 1)$ *and* $V_2 = (0, 2, c')$ *and tiles the sphere, then* $c' \geq 4$ *and* $\mathcal{V}$ *consists of all vectors in*

$$\{(4, 0, 0), (1, 3, 1), (0, 2, c'), (0, 5, \frac{4 - c'}{2}), (1, 0, \frac{3c' - 2}{2}), (2, 1, \frac{c'}{2})\}$$

that have positive integer components.

Proof: as for Proposition 4. ■

Proposition 23 *The only right triangles that have* $V_1 = (1, 3, 1)$*,* $V_2 = (0, 2, c')$*, and tile the sphere are the* $(90°, 72°, 54°)$ *triangle and the* $(90°, 78\frac{3}{4}°, 33\frac{3}{4}°)$ *triangle.*

Proof: From the previous proposition, c' must be even and greater than or equal to 4. When $c' = 4$ we get the $(90°, 72°, 54°)$ triangle, which is a quarterlune (although, atypically, it has a polar angle θ that is greater than $180° - 2\theta$, so the polar angle is β and not γ.) For $c' = 6$ we get the $(90°, 78\frac{3}{4}°, 33\frac{3}{4}°)$ triangle. This tiles the sphere with $N = 32$.

We will now show that when $c' \geq 8$, the $(0, 2, c')/2$ split is not realizable. Firstly, if there is such a split, then without loss of generality there exists one with the short edge of the β angle on the extended edge containing the split, as in Figure 25a; for if it is located as in Figure 25b,c, then there is, as shown, another split of the required form nearby, at the point marked x. (Note that, for $c' \geq 8$, both the medium edge and hypotenuse are more than twice as long as the short edge, and that in every case the only split involving right angles is $(4, 0, 0)/2$.)

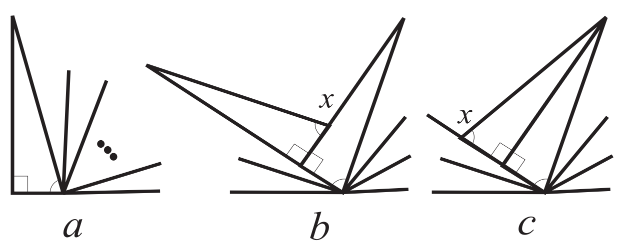

Figure 25: The configuration of the $(0, 2, c')/2$ split

We will now show by induction that all the edges in the fan of γ angles are matched. Consider the γ angle adjacent to the β angle. If this is positioned as Triangle 2 in Figure 26a,b, a right-angled split is created. If this were filled as in Figure 26a, then either way of covering the hypotenuse of triangle 2 would require a split with two β angles. The only alternative, in Figure 26b, forces the hypotenuse of triangle 2 to be matched by triangle 4, as shown. There cannot be a fourth β angle at p; we thus have either a γ angle (not shown) or a right angle next to triangle 3 at p. Any choice of angle and orientation forces triangle 5 as shown, and the overhang at q, which makes it impossible to cover the hypotenuse of triangle 5. (As $c' > 6$, this hypotenuse is not the other side of the split.)

We thus have triangle 2 positioned as shown in Figure 26c. If the medium leg of triangle 2 were not matched, we would have an overhang at r, forcing the γ angle of triangle 4. Triangle 5, with its uncoverable hypotenuse, again follows. Thus, the medium edges must match (Figure 26d).

Again, the hypotenuses of the third and fourth triangles in the split must match. Suppose not; if the short edge of triangle 4 is not matched, its hypotenuse cannot be covered (figure 27*a*). If it is matched (triangle 5 in figure 27*b*), triangles 6 and 7, the

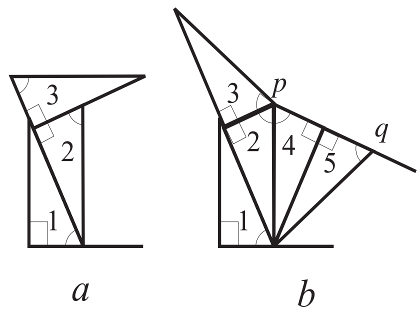

Figure 26: The configuration near the $(0, 2, c')/2$ split: the second and third triangles

γ angle 8, and triangle 9 are forced in that order. However, this creates an impossible combination of angles at v. We conclude that the first four angles of the fan must be as in figure 27c.

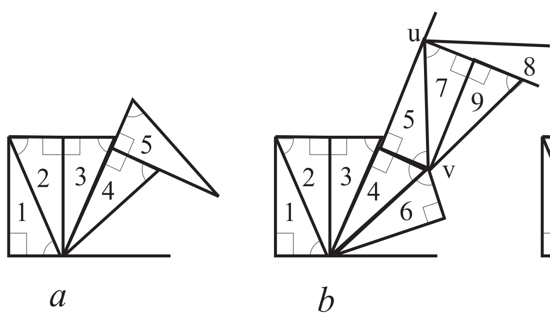

Figure 27: The fourth triangle

We can now proceed inductively to show that the other edges in the split are also matched. Suppose, for a contradiction, that the first unmatched edge to be between triangles numbered $2n + 1$ and $2n + 2$, $n \geq 2$, as in Figure 28a. The overhang and split at x and the gamma angle labelled 1 are forced, resulting in an impossible configuration at y. If, on the other hand, the first unmatched edge is between triangles $2n$ and $2n + 1$, triangle 1 of Figure 28b is forced; we then get the overhang and split at z, triangle 2, the γ angle 3, and triangle 4. However, the hypotenuse of triangle 4 cannot be covered. It follows, then, that all the edges between the angles at a $(0, 2, c')/2$ split with the β angle as shown are matched.

The next step is to show that in fact no split configuration of this type (and hence no $(0, 2, c')/2$ split whatsoever) exists in a tiling of the sphere. If $c'/2$ is even, we have a configuration something like Figure 29a. There cannot be an overhang at p, because the new split would require a triangle (as shown), which would prevent the original vertex from being a split as hypothesized. It follows that the extended edge $\overline{pq}$ is covered by the short edges of two more triangles, necessarily positioned as shown in Figure 29b.

If $c' \geq 12$, the next two short edes must be covered in the same way and we immediately have an impossible four β angles at q. If $c' = 8$, we can avoid this only by having an overhang and split at r. Completing this split gives us Figure 29c; but there is no way to

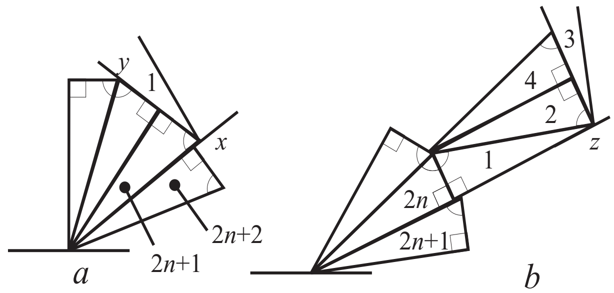

Figure 28: The inductive step

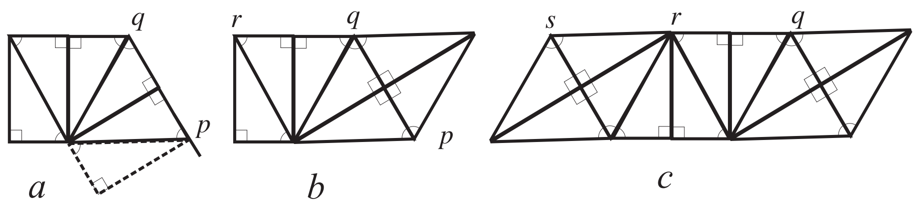

Figure 29: Nonexistence of the $(0, 2, c')/2$ split when $4|c'$

cover the extended edge $\overline{qs}$ without an illegal split or a fourth β at q. We conclude that there is no $(0, 2, c')/2$ split when $c'/2$ is even.

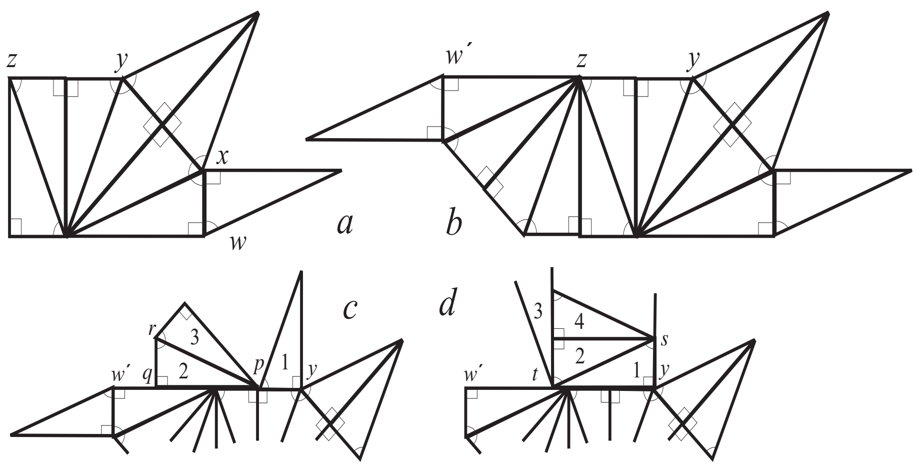

Figure 30: Nonexistence of the $(0, 2, c')/2$ split, $4 \nmid c'$

When $c'/2$ is odd, we have a configuration like that of Figure 30*a*. Again, if $c' \geq 14$, we immediately get a vertex with four β angles and we are done. When $c' = 10$, we can have a $(1, 3, 1)$ vertex at y and an overhang and split at z (Figure 30*b*); but any way of putting a right angle or a γ angle along $\overline{yw'}$ creates an impossible configuration.

A right angle with the short edge on $\overline{yw'}$ gives the configuration of Figure 30*c*; the split at p must have triangles 2 and 3 as shown, and any attempt to fill the right angle gap at q created an illegal overhang at either r or w'. On the other hand, if triangle 1 is placed with its right angle at y and its long edge along $\overline{yw'}$ (Figure 30*d*), there must be an overhang at s and its hypotenuse can only be covered as shown. After triangle 2 is placed, the remaining angles at the split at t are all γ angles, forcing angle 3 as shown; triangle 4 is then forced, and its hypotenuse cannot be covered.

The two cases with γ angles on $\overline{yw'}$ are ruled out by similar arguments. We conclude that for $c' > 6$ no $(0, 2, c')/$ split can be realized. From this it is straightforward to rule out all γ sources, and thus to show that tiling is impossible. ■

Proposition 24 *The only right triangles that have $V_1 = (1, 3, 1)$, $V_2 = (1, 0, c')$, and tile the sphere are those listed in Propositions 21 and 23.*

Proof: If $c' \equiv 0 \pmod 3$, then $(0, 0, \frac{4c'}{3}) \in \mathcal{V}$. This gives the $(90°, 75°, 45°)$ tile for $c' = 6$ and Proposition 21 shows that the triangle does not tile otherwise. If $c' \equiv 2 \pmod 3$, then $(0, 2, 2\frac{c'+1}{3}) \in \mathcal{V}$; we get the $(90°, 72°, 54°)$ tile for $c' = 5$, and the $(90°, 78\frac{3}{4}°, 33\frac{3}{4}°)$ for $c' = 8$, and Proposition 23 shows that the triangle does not tile for any other c'. Finally, if $c' \equiv 1 \pmod 3$, following the methods of Proposition 4 we find that for all $(a, b, c) \in \mathcal{V}$, $a \equiv 1$, yielding no split except $(4, 0, 0)$. ■

Proposition 25 *No right triangle with $V_1 = (1, 3, 1)$ and $V_2 = (1, 1, c')$ tiles the sphere.*

Proof: Again, every vector (a, b, c) in $\mathcal{V}$ has $a \equiv 1 \pmod 3$, so that the only split is $(4, 0, 0)$. ■

This completes the proof of Theorem 1.

5 Other results

Proposition 26 *The tiling shown in Figure 4 is (up to reflection) the only tiling of the sphere with the $(90°, 75°, 60°)$ triangle.*

Proof: Calculation shows that $\mathcal{V}_T = \{(4, 0, 0), (2, 0, 3), (1, 2, 2), (0, 4, 1), (0, 0, 6)\}$; as observed above, there is no β split, so there cannot be an overhang on either side of a β angle. We look at possible covers for the short leg $\overline{pq}$ of a triangle.

If $\overline{pq}$ is covered by a longer edge, there is an overhang as shown in Figure 31. The gap at p must be filled by a right angle, in one of two positions. As $H/C = 1.378\ldots$ and $B/C = 1.234\ldots$, either of these must result in a β split at x, which is not possible.

If $\overline{pq}$ is covered by another short leg with the opposite orientation (Figure 32), the edge $\overline{pr}$ cannot extend past r. Suppose, for the sake of contradiction, that it did; triangle 1 would be forced. The remaining angles at q would be γ's, forcing an overhang at s, so that triangle 2 must be as shown. There cannot be an overhang at t, and the extended edge $\overline{qt}$ must be covered by two more long legs, as no other combination of edges equals $2B$. However, q already has two β angles and a right angle, and cannot accept another of either type.

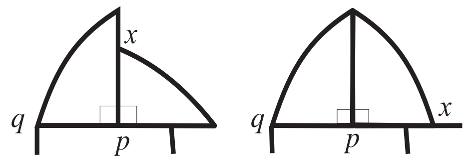

Figure 31: An impossible configuration

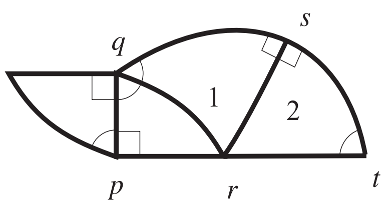

Figure 32: Another impossible configuration

Thus, if $\overline{pq}$ is covered as shown, $\overline{pr}$ must be covered by another edge of the same length; as there cannot be another right angle at p, we have the configuration of Figure 33a. The angle $\angle upv$ must be filled with no overhang at u or v; this forces (essentially) the configuration of Figure 33b, and, as in Figure 31, filling the $90°$ gap at x will force a β split.

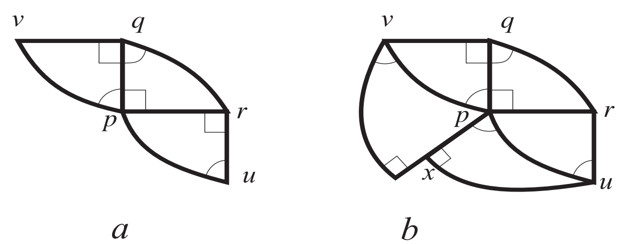

Figure 33: A third impossible configuration

We conclude that $\overline{pq}$ is paired with another short leg, oriented in the same way. It follows that the triangles in the tiling are partitioned into mirror-image pairs, forming $(150°, 60°, 60°)$ triangles; as shown in [3], these tile the sphere uniquely up to reflection. ∎

6 Conclusion

This paper lists all the right spherical triangles with asymptotically right V_1 that tile the sphere. The set of such triangles contains two infinite families and two sporadic triangles (all previously known) that tile in an edge-to-edge fashion, and one infinite family and four sporadic triangles, that only exhibit non-edge-to-edge tilings. It is a part of a sequence of papers (along with [5], [6], [7] and [8]) that will give a complete classification of the right triangles that tile the sphere.

7 Acknowledgement

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