Relaxations of Ore's condition on cycles

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Abstract

A simple, undirected 2-connected graph G of order n belongs to class $\mathcal{O}(n,\varphi)$, $\varphi \geq 0$, if $\sigma_2 = n - \varphi$. It is well known (Ore's theorem) that G is hamiltonian if $\varphi = 0$, in which case the 2-connectedness hypothesis is implied. In this paper we provide a method for studying this class of graphs. As an application we give a full characterization of graphs G in $\mathcal{O}(n,\varphi)$, $\varphi \leq 3$, in terms of their dual hamiltonian closure.

Keywords: Hamiltonian Cycle, Dual Closure.

1 Introduction

We consider throughout only simple 2-connected graphs $G = (V, E)$. We let $\alpha(G)$, $\nu(G)$, $\omega(G)$ denote respectively the independence number, the matching number and the number of components of the graph G. A graph G is 1-tough if $|S| \geq \omega(G-S)$ is true for any subset $S \subset V$ with $\omega(G - S) > 1$. For $k \leq \alpha(G)$ we set $\sigma_k = \min \left\{ \sum_{x \in S} d(x) \mid S \text{ is a stable set} \right\}$. We use the term stable to mean independent set. A graph G of order n belongs to class $\mathcal{O}(n, \varphi)$, $\varphi \geq 0$ if $\sigma_2 = n - \varphi$. It is well known ([13]) that G is hamiltonian if $G \in \mathcal{O}(n, 0)$, in which case the 2-connectedness hypothesis is implied. Jung ([8]) proved that a 1-tough graph $G \in \mathcal{O}(n \geq 11, 4)$ is hamiltonian. Indeed this is a strong assumption which is not easy to verify since recognizing tough graphs is NP-Hard ([10]). Ignoring the hypothesis of 1-toughness but conserving the constraint on n, that is $n \geq 3\varphi - 1$, we obtained in ([4]) a characterization of graphs in $\mathcal{O}(n, \varphi \leq 4)$. Without any constraint on n, a characterization of graphs in $\mathcal{O}(n, \varphi \leq 2)$ is given in ([2] and [9]). The same characterization was given by Schiermeyer ([12]) in terms of the dual-closure of G.

In this paper we go a step further than Shiermeyer by giving a complete map of graphs in $\mathcal{O}(n, \varphi \leq 3)$ with respect to the hamiltonian property. The dual closure ([1, 2, 5]) of those graphs is completely determined. This is indeed useful since then finding a cycle in G of maximum length becomes a polynomial problem.

2 Preliminary results

A vertex of degree $n-1$ is a dominating vertex and Ω will denote the set of dominating vertices. The circumference $c(G)$ of G is the length of its longest cycle. For $u \in V(G)$, let $N_H(u)$ denote the set and $d_H(u)$ the number of neighbors of u in H, a subgraph of G. If $H = G$ we will write simply $N(u)$ for $N_G(u)$ and $d(u)$ for $d_G(u)$ respectively. For convenience, we extend this notation as follows. Given a subset $S \subset V$, we define the degree of a vertex x with respect to S as $d_S(x)$ to be the number of vertices of S adjacent to x. For $X \subset V$, put $N(X) = \cup_{u \in X} N(u)$. If $X, Y \subset V$, let $E(X, Y)$ denote the set of edges joining vertices of X to vertices of Y. As we need very often to refer to a presence or not of an edge, we write xy to mean that $xy \in E$ and $\overline{xy}$ to mean $xy \notin E$ For each pair (a, b) of nonadjacent vertices we associate

$$\begin{gathered} G_{ab} := G - N(a) \cup N(b),\ \gamma_{ab} := \left|N(a) \cup N(b)\right|,\ \lambda_{ab} := \left|N(a) \cap N(b)\right| \\ T_{ab} := V \backslash \left(N\left[a\right] \cup N\left[b\right]\right),\ t_{ab} := \left|T_{ab}\right|,\ \overline{\alpha}_{ab} := 2 + t_{ab} = \left|V(G_{ab})\right| \\ \delta_{ab} := \min\left\{d(x) \mid x \in T_{ab}\right\} \text{ if } T_{ab} \neq \varnothing \text{ and } \delta_{ab} := \delta(G) \text{ otherwise} \\ \alpha_{ab} = \alpha(G_{ab}),\ \nu_{ab} = \nu(G_{ab}),\ \omega(T_{ab}) = \omega(G\left[T_{ab}\right]). \end{gathered}$$

In this paper there is a specially chosen pair (a, b) of vertices. To remain simple, we omit the reference to a, b for all parameters defined above. Moreover we understand T as the set, the graph induced by its vertices and its edge set. Our proofs are all based on the concept of the hamiltonian closure ([11], [1], [2]). The two conditions of closure developed in [1], [2] are both generalizations of Bondy-Chvàtal's closure. To state the condition under which our closure is based we define a binary variable ε_{ab} associated with (a, b).

Definition 2.1 *Let $\varepsilon_{ab} \in \{0, 1\}$ be a binary variable, associated with a pair (a, b) of nonadjacent vertices. We set $\varepsilon_{ab} = 0$ if and only if*

1. $\varnothing \neq T$ and all vertices of T have the same degree $1 + t$. Moreover $\lambda_{ab} \leq 1$ if $N(T) \backslash T \subseteq N(a) \bigtriangleup N(b)$, (where $\bigtriangleup$ denotes the symmetric difference).

2. one of the following two local configurations holds

   (a) T is a clique (possibly with one element), $\lambda_{ab} \leq 2$ and there exist $u, v \notin T$ such that $T \subset N(u) \cap N(v)$.

   (b) T is an independent set (with at least two elements), $\lambda_{ab} \leq 1 + t$ and either $N(T) \subseteq D$ or there exists a vertex $u \in N(a) \bigtriangleup N(b)$ such that $|N_T(u)| \geq |T| - \max\left(\lambda_{ab} - 1, 0\right)$. Moreover T is a clique in G^2, the square of G.

Lemma 2.2 (the main closure condition) *Let G be a 2-connected graph and let (a, b) be a pair of nonadjacent vertices satisfying the condition*

$$\overline{\alpha}_{ab} \leq \max\{\lambda_{ab} + \nu_{ab}, \delta_{ab} + \varepsilon_{ab}\} \tag{ncc}$$

Then $c(G) = p$ if and only if $c(G + ab) = p$, $p \leq n$.

The first condition is a relaxation of the condition $\alpha_{ab} \leq \max\{\lambda_{ab}, 2\}$ given in [1]. Since by definition $\overline{\alpha}_{ab}$ is the order of G_{ab}, it follows that $\alpha_{ab} \leq \overline{\alpha}_{ab}$. As α_{ab} is not easy to compute we developed many upper bounds of α_{ab}, computable in polynomial time ([1], [6]) . One of these upper bounds is precisely ν_{ab}. It is known that for any graph H, $\alpha(H)+\nu(H) \leq n(H)$. We note that $\nu_{ab} = \nu(G_{ab}) = \nu(T)$ since a, b are isolated vertices in G_{ab}, we see that $\overline{\alpha}_{ab} - \nu_{ab} \leq \alpha_{ab}$ and hence $\alpha_{ab} \leq \max\{\lambda_{ab}, 2\}$ implies $\overline{\alpha}_{ab} \leq \lambda_{ab} + \nu_{ab}$. We note that $\overline{\alpha}_{ab} \leq \lambda_{ab} + \nu_{ab}$ is stronger than Bondy-Chvàtal's hamiltonian closure condition ([11]) since $d(a) + d(b) \geq n \Leftrightarrow \overline{\alpha}_{ab} \leq \lambda_{ab}$. The second part of the condition (ncc) is a relaxation of a strongest one given in [1], improved in ([5]) .The condition $\overline{\alpha}_{ab} \leq \delta_{ab} + \varepsilon_{ab}$, especially with the addition of the term ε_{ab} will prove to be a most useful tool in obtaining the main properties of the dual closure of any graph $G \in \mathcal{O}(n, 3)$. The condition $\overline{\alpha}_{ab} \leq \lambda_{ab}+\nu_{ab}$ is only used in very particular cases. Note that $\overline{\alpha}_{ab} \leq \delta_{ab}+\varepsilon_{ab} \Leftrightarrow \gamma_{ab}+\delta_{ab}+\varepsilon_{ab} \geq n$ and $\overline{\alpha}_{ab} \leq \lambda_{ab} + \nu_{ab} \Leftrightarrow d(a) + d(b) + \nu_{ab} \geq n$.

The 0-dual neighborhood closure $nc_0^*(G)$ (the 0−dual closure for short) is the graph obtained from G by successively joining (a, b) satisfying the condition (ncc) until no such pair remains. Throughout we denote $nc_0^*(G)$ by H. All closures based on the above conditions are well defined. Moreover, it is shown in ([6], [5]) that it takes a polynomial time to construct H and to exhibit a longest cycle in G whenever a longest cycle is known in H.

As a direct consequence of Lemma 2.2 we have.

Corollary 2.3 *Let G be a 2-connected graph. Then G is hamiltonian if and only if H is hamiltonian.*

3 Results

To state our results, we define first three nonhamiltonian graphs (H_7^1 to H_7^3) on the set $\{a, b, d, u, v, x, y\}$ of 7 vertices. For all the three graphs, d is dominating and au, bv, ux are edges. We refer to H as H_7^1 if vx, xy and uv. We refer to H as H_7^2 by removing uv from H_7^1. In H_7^3, uv and vy are edges. These three graphs are all in $\mathcal{O}(7, 3)$ and only H_7^1 is 1-tough. Next we define a family $\mathcal{K}_n^\varphi$ of nonhamiltonian graphs. A graph G of order n is in $\mathcal{K}_n^\varphi$ for $\varphi \geq 1$ if its dual closure H satisfies the condition $|\Omega| + 1 \leq \omega(H - \Omega) \leq |\Omega| + \varphi$ and each component of $H - \Omega$ is any graph on maximum φ vertices.

Theorem 3.1 *Let $G \in \mathcal{O}(n, \varphi)$, $0 \leq \varphi \leq 3$, and let $H := nc_0^*(G)$. Then (i) G is hamiltonian if and only if either $H = C_7$, in which case $\varphi = 3$ or $H = K_n$ and (ii) G is nonhamiltonian if and only if either $\varphi = 3$, $n = 7$ and $H = H_7^i$, $i = 1, 2, 3$ or $H \in \mathcal{K}_n^\varphi$.*

Proof. Follows directely from Lemmas 5.1 to 5.5 in section 5. ∎

Corollary 3.2 *Let $G \in \mathcal{O}(n, \varphi)$, $0 \leq \varphi \leq 3$. If $n \geq 3\varphi - 1$ then G is hamiltonian if and only if $H = K_n$ and nonhamiltonian if and only if $H \in \mathcal{K}_n^{\varphi}$.*

Corollary 3.3 *Let $G \in \mathcal{O}(n, \varphi)$, $0 \leq \varphi \leq 3$. Then $H \in \{K_n, C_7, H_7^1\}$ if G is 1-tough.*

Corollary 3.4 *Let $G \in \mathcal{O}(n, \varphi)$, $0 \leq \varphi \leq 3$. If G is not hamiltonian then $c(G) = c(H) \geq n - \varphi$. Moreover $c(G) = c(H) = n - 1$ if $n \geq 3(\varphi + 1)$.*

4 General Lemmas

In this section we assume $G \in \mathcal{O}(n, \varphi)$, $\varphi \geq 0$ and all neighborhood sets and degrees are understood under H, unless otherwise stated. With each pair (a, b) we adopt the following decomposition of V by setting $A := N(a)\backslash N(b)$, $A^+ := A \cup \{a\}$,$B := N(b)\backslash N(a)$, $B^+ := B \cup \{b\}$, $D := N(a) \cap N(b)$, $T := T_{ab}$ where $t = |T|$. Also we set $T_i := \{x \in T \mid d_T(x) = i\}$, $i \geq 0$. We point out that $T \neq \varnothing$ by (ncc) whenever $H \neq K_n$ since H is 2-connected. For an ordered pair (x, y) of nonadjacent vertices we set $N(x, y) := N(x)\backslash N(y)$ and $n(x, y) := |N(x, y)|$. With this notation, we have $A = N(a, b)$ and $B = N(b, a)$. We shall say that $H \neq K_n$ is (a, b)-*well-shaped* if $E(A \cup B, T) \cup E(A, B) = \varnothing$ and $\Omega = D$.

Throughout, a, b are chosen as follows:

(i) $\overline{ab}$ and $d(a) + d(b) = \sigma_2 = n - \varphi$,
(ii) subject to (i), λ_{ab} is minimum.
(iii) subject to (i) and (ii) and if possible H is (a, b)-well-shaped.

Moreover we always assume $d(a) \leq d(b) \leq d(x)$ for any $x \in T$. This choice implies immediately.

Lemma 4.1 *If $H \neq K_n$ and $\varphi \geq 1$ then*

$(L1)$ $2 + t = \lambda_{ab} + \varphi$.

$(L2)$ $\forall p, q \in V$, $\overline{pq} \Rightarrow \max\{n(p, q), n(q, p)\} + \varepsilon_{pq} < \varphi$.

$(L3)$ $|A| \leq |B| < \varphi - \varepsilon_{ab}$.

$(L4)$ $T = \cup_{j=0}^{\varphi-1} T_j$. *Furthermore either $T_{\varphi-1} = \varnothing$ or $E(A \cup B, T) \cup E(A, B) = \varnothing$.*

$(L5)$ *if $u \in A$ then $d_{A \cup T}(u) + \varepsilon_{bu} \leq \varphi - 2 + d(a) - \delta_{bu}$. Similarly if $v \in B$ then $d_{B \cup T}(v) + \varepsilon_{av} \leq \varphi - 2 + d(b) - \delta_{av}$.*

$(L6)$ *if $A \cup B = \varnothing$ then $\overline{xy} = \varnothing \Rightarrow d_T(x) + d_T(y) + \nu_{xy} < \varphi$ for all $x, y \in T$*

Proof. $(L1)$. By choice of a, b we have $d(a) + d(b) = n - \varphi$. This is equivalent to $2 + t = \lambda_{ab} + \varphi$.

$(L2)$ If $\overline{pq}$ then $\gamma_{pq} + \delta_{pq} + \varepsilon_{pq} < n$. Let us choose $r \in T_{pq}$ such that $d(r) = \delta_{pq}$. This vertex exists since $T_{pq} \neq \varnothing$ by (ncc). Since clearly $\gamma_{pq} = d(q) + n(p,q)$, we have $n(p,q) + d(q) + d(r) + \varepsilon_{pq} < n$. By hypothesis $d(q) + d(r) \geq n - \varphi$ since $\overline{qr}$. It follows that $(L2)$ holds. In particular if $\overline{pq}$ and $n(p,q) = \varphi - 1$ then $\varepsilon_{pq} = 0$.

$(L3)$ This is a consequence of $(L2)$ since $B = N(b,a)$.

$(L4)$ Clearly $T = \cup_{j=0}^{t-1} T_j$. If $T_j \neq \varnothing$ for $j \geq \varphi$ then $n(x,a) = |N_T(x)| \geq \varphi$, a contradiction to $(L2)$. Suppose next vy for some $(v,y) \in B \times T$ and choose $z \in T_{\varphi-1}$. Clearly $z \neq y$ for otherwise $n(y,a) \geq \varphi$, a contradiction to $(L2)$. By $(L2)$, $\varepsilon_{az} = 0$ and hence T_{az} must be a clique since $bv \in T_{az}$. But then by since vy. Thus $E(B,T) = \varnothing$. Similarly $E(A,T) = \varnothing$. Next suppose vu for some $(v,u) \in B \times A$. Again T_{az} is a clique and $vu \Rightarrow bu$. Therefore $E(A,B) = \varnothing$.

$(L5)$ Because $\overline{ub}$, $u \in A$ and by (ncc) we have $\overline{\alpha}_{ub} > \delta_{ub} + \varepsilon_{ub}$. Obviously $\overline{\alpha}_{ub} = 1 + |A| - d_A(u) + t - d_T(u) = 1 + d(a) - \lambda_{ab} - d_A(u) + t - d_T(u)$. By $(L1)$ we get $\overline{\alpha}_{ub} = 1 + d(a) + \varphi - 2 - (d_A(u) + d_T(u))$. On the other hand $\delta_{ub} \geq d(b) \geq d(a)$.From these inequalities we obtain $d_A(u) + d_T(u) + \varepsilon_{ub} \leq \varphi - 2$. Similarly $d_A(u) + d_T(u) + \varepsilon_{ub} \leq \varphi - 2$.

$(L6)$ We observe that $d(x) + d(y) = 2\lambda_{ab} + d_T(x) + d_T(y)$. Then $2\lambda_{ab} + d_T(x) + d_T(y) + \nu_{ab} < n$ by (ncc). On the other hand $n = d(a) + d(b) + \varphi = 2\lambda_{ab} + \varphi$. Statement $(L6)$ follows easily. ■

5 Application to graphs in $O(n, \varphi)$, $\varphi \leq 3$

Throughout, we assume $H := nc_0^*(G) \neq K_n$.

Lemma 5.1 *If $G \in \mathcal{O}(n,1)$ then $H \in \mathcal{K}_n^1$.*

Proof. By hypothesis, $d(a) + d(b) \geq n - 1$ or equivalently $\overline{\alpha}_{ab} \leq \lambda_{ab} + 1$. By (ncc) $\overline{\alpha}_{ab} > \max\{\lambda_{ab} + \nu_{ab}, \delta_{ab} + \varepsilon_{ab}\}$ since $\overline{ab}$. It follows that $\nu_{ab} = \varepsilon_{ab} = 0$. Moreover T is independent and $d(x) = \delta_{ab} = \lambda_{ab} = d(a) = d(b)$ holds for any $x \in T$. This means in particular that $A \cup B = \varnothing$.and $N_D(v) = D$ is true for each vertex $v \in V \backslash D$. Furthermore D must be a clique for if $\overline{ef}$ for some $(e,f) \in D^2$ then $\overline{\alpha}_{ef} \leq |D| = \lambda_{ab} \leq \lambda_{ef} = |V \backslash D|$, a contradiction to (ncc). Therefore $\Omega = D$ and $\omega(H - \Omega) = |V \backslash D|$. Clearly $|D| = \frac{n-1}{2}$ and $|V \backslash D| = \frac{n+1}{2}$ since $d(a) + d(b) = n - 1 = 2\lambda_{ab}$. It follows that $\omega(H - \Omega) = \frac{n+1}{2}$ and $H \in \mathcal{K}_n^1$. ■

Lemma 5.2 *If $G \in \mathcal{O}(n,2)$ then $H \in \mathcal{K}_n^2$.*

Proof. Now $\nu_{ab} \leq 1$. As a first step, we prove that $N_D(v) = D$ is true for each vertex $v \in V \backslash D$. Choose $(x,e) \in T \times D$. If $\overline{ex}$ then $n(e,x) \geq |\{a,b\}| = 2$, a contradiction to $(L2)$. Moreover $E(A \cup B, T) = \varnothing$ for if there exists an edge ux with $(u,x) \in A \times T$ then

$n(u,b) \geq 2$, a contradiction to $(L2)$. Similarly $E(A,B) = \varnothing$ for if there exists uv for some $(u,v) \in A \times B$ then $n(u,x) \geq 2$. It follows that $N_D(u) = D$ is also true for each vertex $u \in A \cup B$ since by the choice of a, b, $d(u) \geq d(a)$ if $u \in A$ and $d(u) \geq d(b)$ if $u \in B$. As for the proof of the above Lemma we get $\Omega = D$. If $\nu_{ab} = 0$ then clearly $H = (m+2)K_1 + K_m$ with $m = \lambda_{ab}$. If $\nu_{ab} = 1$ and $t = 2$ then $(K_r \cup K_s \cup K_2) + K_2$, $1 \leq r \leq s \leq 2$. If $\nu_{ab} = 1$ and $t > 2$ then $((m+1)K_1 \cup K_2) + K_m$ with $m \geq 3$. In all cases, one can easily check that ch that $H \in \mathcal{K}_n^2$. ■

Lemma 5.3 *If $G \in \mathcal{O}(n,3)$ and $\lambda_{ab} = 0$ then $H = C_7$.*

Proof. By $(L1)$ we obtain $t = 1$. Assuming $T = \{x\}$, we get $d(x) = 2$ by (ncc). It follows that $d(a) = d(b) = 2$ by the choice of a, b. As $d(a) + d(b) = 4 = n - 3$, we have $n = 7$. Set $N(a) = \{a_1, a_2\}$ and $N(b) = \{b_1, b_2\}$. If $N(x) = B$ then $T_{ab_2} = \{b_1\}$ and hence $d(b_1) = 2$ by (ncc). But now $H - b_2$ is disconnected. With this contradiction, we deduce that $N(x) \neq B$. Similarly $N(x) \neq A$. Assume then xa_1 and xb_1. Now $T_{ab_1} = \{b_2\}$ and hence $d(b_2) = 2$ by (ncc). Similarly $d(a_2) = 2$. As H is 2-connected, we must have $N_B(a_2) \neq \varnothing$ and $N_A(b_2) \neq \varnothing$. Suppose first a_2b_1 and a_1b_2. This would contradict (ncc) since $T_{a_1b_1} = \varnothing$. It remains to admit that a_2b_2, in which case $H = C_7$, as claimed. ■

Lemma 5.4 *If $G \in \mathcal{O}(n,3)$ and $\lambda_{ab} = 1$ then $H = H_7^i$, $i = 1, 2, 3$.*

Proof. By $(L1)$, $t = 2$ and we may assume $T := \{x, y\}$, $d(y) \leq d(x)$ and $D := \{d\}$. Moreover $T \subseteq T_0 \cup T_1$.

Claim 1. $\varepsilon_{ab} = 1$

By contradiction, suppose $\varepsilon_{ab} = 0$. Then $d(x) = d(y) = 3$, T is either a clique or a stable and $d(a) \leq d(b) \leq 3$. If xy then by Definition 2.1(2.a), there exist $r, s \in N(a) \cup N(b)$ and $N(r) \cap N(s) \supset \{x, y\}$. Assuming $r \neq d$ then $r \in A \cup B$. It follows that $d_T(r) = 2$, a contradiction to $(L5)$. Suppose next $\overline{xy}$. In one hand we obviously have $|N_{A\cup B}(x)| \geq 2$, $|N_{A\cup B}(y)| \geq 2$ and $|N_{A\cup B}(x) \cap N_{A\cup B}(y)| \leq 1$ by $(L5)$. As $|A \cup B| \leq 2$, we deduce that $|N_{A\cup B}(x)| = |N_{A\cup B}(y)| = 2$, $|A| = |B| = 2$ and $N(d) \supset \{x, y\}$. Set $A = \{u, u'\}$ and $B = \{v, v'\}$. Only two configurations are possible : $N_{A\cup B}(x) = \{u, v\}$, $N_{A\cup B}(y) = \{u', v'\}$ or $N_B(x) = B$, $N_A(y) = A$. For the first case $T_{av} = \{y, v'\}$ and $\varepsilon_{av} = 0$ by $(L2)$ since $n(v,a) = 2 = \varphi - 1$. Since T_{av} is a clique and $|N(T_{av}) \backslash T_{av}| \geq 3$, we get a contradiction to the definition of ε_{av}. For the second case we may assume uv since H is 2-connected. We note that $u'v$ for otherwise $n(v,u') = 3$ since $\overline{u'u}$ by $(L5)$. Now $T_{av'} = \{y, v\}$ and $d(v) \geq 4$. By (ncc), av and we get the required contradiction for the proof of Claim 1. As a consequence of this claim, we must have $d(y) = 2$. This in turn implies $A = \{u\}$ and $B = \{v\}$.

Claim 2. $H = H_7^i$, $i = 1, 2, 3$

By the choice of a, b, $d(y) = d(a) = d(b) = 2 \Rightarrow N(y) \cap \{u, d\} \neq \varnothing$ and $N(y) \cap \{v, d\} \neq \varnothing$. We claim that yd for otherwise $N(y) = \{u, v\}$ and $N(x) = \{d\}$ since $N(x) \cap N(y) \cap \{u, v\} = \varnothing$ by $(L5)$. This is obviously a contradiction. Next we show that dx. If $d(x) = 2$, this is true by symmetry with y. Otherwise $N(x) = \{y, u, v\}$ and hence dx since $T_{dx} = \varnothing$. Moreover ud and vd, that is $d \in \Omega$. To see this, suppose $\overline{ud}$ for instance. Then $T_{ud} = \{v\}$ and hence $d(v) = 2$. But now we have $\lambda_{av} = 0$ and choosing (a, v) instead of (a, b) we get a contradiction. Let us consider two cases.

Case 1: xy.

Set $F := \{ux, ux, vx\}$. Since $H - d$ must be connected, H must contain at least two edges of F. If H contains all edges of F then $H = H_7^1$. This graph is 1-tough (in fact it is the smallest 1-tough, non-hamiltonian graph). If H contains 2 edges of F then $H = H_7^2$ (we have three isomorphic graphs).

Case 2: $\overline{xy}$.

Since $H - d$ must be connected, we must have uv. Since $N(x) \cap N(y) \{u, v\} = \varnothing$, we may assume ux and vy. We have now the third nonhamiltonian graph $H = H_7^3$ and the proof is complete ■

Lemma 5.5 *If $G \in \mathcal{O}(n, 3)$ and $\lambda_{ab} \geq 2$ then $H \in \mathcal{K}_n^3$.*

Proof. By $(L2)$, $t \geq 3$ and we recall that $\nu_{ab} \leq 2$. The proof is split into three claims.

Claim 1: $E(A \cup B, T) \cup E(A, B) = \varnothing$.

By contradiction suppose first $A \cup B \neq \varnothing$ and $E(A \cup B, T) \neq \varnothing$. If $\varepsilon_{ab} = 0$ then by Definition 2.1 (2.b), $d_T(v) \geq t - \lambda_{ab} + 1$. By $(L1)$, $d_T(v) \geq \varphi - 1 \geq 2$, a contradiction to $(L5)$. With this contradiction, we assume $\varepsilon_{ab} = 1$, in which case $B = \{v\}$ and $A \subseteq \{u\}$ by $(L3)$. Without loss of generality, assume vx for some $(v, x) \in B \times T$. Consider now $T_{av} = A^+ \cup (T_{ab} \backslash \{x\})$. As $n(v, a) = 2$,we deduce that $\varepsilon_{av} = 0$ by $(L2)$ and consequently T_{av} is either a clique or a stable. Clearly T_{av} cannot be a clique and hence it is a stable and in particular $A = \varnothing$. Moreover $d(a) = d(w)$ for any vertex of $T_{ab} \backslash \{x\}$, a contradiction to the choice of a, b since now $\delta_{ab} = d(a) < d(b)$. We have just proved that $E(A \cup B, T) = \varnothing$. Next suppose uv, $(u, v) \in A \times B$. By $(L4)$, $T = T_0 \cup T_1$. Furthermore $T_0 = \varnothing$ for if $x \in T_0$ then $N(x) \subseteq D$ and hence $d(x) < d(b)$. Therefore $T = pK_2$ with $p \geq 2$ since $t = \lambda_{ab} + \varphi - 2 \geq 3$. As $N(u, x) = \{a, v\}$ we have $\varepsilon_{ux} = 0$ for any $x \in T$. This is a contradiction since T_{ux} is neither a clique nor a stable since it contains at least one edge and the single vertex b.

Claim 2: H is (a, b)-well-shaped.

By Claim 1, it suffices to prove that $\Omega = D$. We first note, by (ncc), that $\Omega = D$ if $N_{V \backslash D}(e) = V \backslash D$ holds for any $e \in D$. Choose $(e, x) \in D \times T$ such that $\overline{ex}$. If $A \cup B \neq \varnothing$ then $B = \{v\}$ and $N_D(v) = D$ since $d(v) \geq d(b)$ by the choice of a, b. Therefore $N(e, x) \supseteq \{a, b, v\}$, a contradiction to $(L2)$. For the remaining we assume $A \cup B = \varnothing$. Clearly $T_0 = \varnothing$ since obviously $x \notin T_0$ for if there exists $y \in T_0$ then necessarily $\overline{xy}$ and $N_D(y) = D$ and hence $n(e, y) \geq 3$, a contradiction to $(L2)$. Therefore $T = T_1 \cup T_2$. Suppose first $x \in T_2$. As $n(x, a) = 2$ then $\varepsilon_{ax} = 0$ and hence $d(z) = d(b)$ for any vertex $z \in T_{ax} \cap T$. By the choice of (a, b) we must have $\lambda_{az} = \lambda_{ab}$. So if $z \in T_{ax} \cap T$ then $z \in T_0$. Since $T_0 = \varnothing$ we conclude that $T_{ax} = \{b\}$. It follows that $d(b) = 2$ and $t = \varphi = 3$ by $(L1)$. Therefore $N[x] = T$ and

$T_{ex} \subseteq \{f\}$, where $D = \{e, f\}$. By (ncc), we have $d(f) = 2$. This is a contradiction since $H - e$ cannot be disconnected. Finally suppose $T = T_1$. Now $d(x) = d(b)$ since $\overline{ex}$, while $\lambda_{ax} < \lambda_{ab}$, a contradiction. The proof of Claim 2 is now complete.

Claim 3: $H \in \mathcal{K}_n^3$.

We set $m = \lambda_{ab}$ and we recall that $\Omega = D$, $E(A \cup B, T) = E(A, B) = \varnothing$. By $(L4)$, $T \subseteq T_0 \cup T_1 \cup T_2$.

Case 1: $A \cup B \neq \varnothing$ and $T_2 \neq \varnothing$.

In this case we necessarily have $T = T_1 \cup T_2$. Choose a vertex $z \in T_2$. By $(L2)$, applied to (a, z) and (b, z) we get $\varepsilon_{az} = \varepsilon_{bz} = 0$. It follows that, for instance $T_{az} \supset B^+$ must be either a clique or a stable.

If T_{az} is a clique then necessarily $T_{az} = B^+$ and hence $t = 3$ and $\lambda_{ab} = 2$ by $(L1)$. It is not diffficult to check that $H = (K_r \cup K_s \cup M_3) + K_2$, $1 \leq r \leq s \leq 3$ where $r = |A^+|$, $s = |B^+|$ and $M_3 \in \{P_3, K_3\}$. If $s = 3$ then$M_3 = K_3$ by the choice of a, b. For this sub-case we have $H \in \mathcal{K}_n^3$, as claimed.

Case 2: $A \cup B \neq \varnothing$ and $T_2 = \varnothing$.

In this case we necessarily have $T = T_1$. Set $T = pK_2$. Thus $\nu_{ab} = p$ and by (ncc), $2t = s < 3$ for otherwise ab. It follows that $t = 4$ since t is even and greater than 3. Therefore $H = (K_r \cup K_s \cup 2K_2) + K_3$, $1 \leq r \leq s \leq 2$, ie $H \in \mathcal{K}_n^3$.

Case 3: $A \cup B = \varnothing$.

By $(L6)$ we have $\overline{xy} = \varnothing \Rightarrow d_T(x) + d_T(y) + \nu_{xy} \leq 2$ $(*)$. By $(L4)$, $T = T_2 \cup T_1 \cup T_0$. Suppose first $T_2 \neq \varnothing$ and let $z \in T_2$. By $(*)$, we necessarily have $T = N[z] \cup T_0$, that is $T = M_3 \cup (m-2) K_1$ where $M_3 \in \{P_3, K_3\}$ (recall that $t = \lambda_{ab} + 1 = m + 1$). Thus $H = (mK_1 \cup M_3) + K_m$, $m \geq 3$, that is $H \in \mathcal{K}_n^3$.

Next suppose $T_2 = \varnothing$ but $T_1 \neq \varnothing$. Set $T = pK_2 \cup qK_1$. Clearly $p \leq 2$ by $(*)$. If $p = 2$ and $T_0 = \varnothing$ then $H = (K_r \cup K_s \cup 2K_2) + K_3$, $r = s = 1$. If $p = 2$ and $T_0 \neq \varnothing$ then $H = ((m-1)K_1 \cup 2K_2) + K_m$, $m \geq 4$. If $p = 1$ and $T_0 = \varnothing$ then we have $\varphi = 2$.Finally, if $p = 1$ and $T_0 \neq \varnothing$ then $H = ((m+2)K_1 \cup K_2) + K_m$, $m \geq 3$.

Finally, suppose $T = T_0$. In this case $H = (2+t)K_1 + K_m$, $m = \lambda_{ab} \geq 2$. By $(L1)$, $2 + t = m + 3$. Therefore $H = (m+3)K_1 + K_m$. In all cases $H \in \mathcal{K}_n^3$. ■

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