On oriented arc-coloring of subcubic graphs

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Abstract

A homomorphism from an oriented graph G to an oriented graph H is a mapping φ from the set of vertices of G to the set of vertices of H such that $\overrightarrow{\varphi(u)\varphi(v)}$ is an arc in H whenever $\overrightarrow{uv}$ is an arc in G. The oriented chromatic index of an oriented graph G is the minimum number of vertices in an oriented graph H such that there exists a homomorphism from the line digraph $LD(G)$ of G to H (Recall that $LD(G)$ is given by $V(LD(G)) = A(G)$ and $\overrightarrow{ab} \in A(LD(G))$ whenever $a = \overrightarrow{uv}$ and $b = \overrightarrow{vw}$). We prove that every oriented subcubic graph has oriented chromatic index at most 7 and construct a subcubic graph with oriented chromatic index 6.

Keywords: Graph coloring, oriented graph coloring, arc-coloring, subcubic graphs.

1 Introduction

We consider finite simple *oriented graphs*, that is digraphs with no opposite arcs. For an oriented graph G, we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs.

In [2], Courcelle introduced the notion of vertex-coloring of oriented graphs as follows: an *oriented k-vertex-coloring* of an oriented graph G is a mapping φ from $V(G)$ to a set of k colors such that (i) $\varphi(u) \neq \varphi(v)$ whenever $\overrightarrow{uv}$ is an arc in G, and (ii) $\varphi(u) \neq \varphi(x)$ whenever $\overrightarrow{uv}$ and $\overrightarrow{wx}$ are two arcs in G with $\varphi(v) = \varphi(w)$. The *oriented chromatic number* of an oriented graph G, denoted by $\chi_o(G)$, is defined as the smallest k such that G admits an oriented k-vertex-coloring.

Let H and H' be two oriented graphs. A *homomorphism* from H to H' is a mapping φ from $V(H)$ to $V(H')$ that preserves the arcs: $\overrightarrow{\varphi(u)\varphi(v)} \in A(H')$ whenever $\overrightarrow{uv} \in A(H)$. An oriented k-vertex-coloring of G can be equivalently defined as a homomorphism φ from

G to H, where H is an oriented graph of order k. The existence of such a homomorphism from G to H is denoted by $G \to H$. The graph H will be called *color-graph* and its vertices will be called *colors*, and we will say that G is H-colorable. The oriented chromatic number can be then equivalently defined as the smallest order of an oriented graph H such that $G \to H$.

Oriented vertex-colorings have been studied by several authors in the last past years (see e.g. [1, 3, 5] or [7] for an overview).

One can define *oriented arc-colorings* of oriented graphs in a natural way by saying that, as in the undirected case, an oriented arc-coloring of an oriented graph G is an oriented vertex-coloring of the line digraph $LD(G)$ of G (Recall that $LD(G)$ is given by $V(LD(G)) = A(G)$ and $\overrightarrow{ab} \in A(LD(G))$ whenever $a = \overrightarrow{uv}$ and $b = \overrightarrow{vw}$). We will say that an oriented graph G is *H-arc-colorable* if there exists a homomorphism φ from $LD(G)$ to H and φ is then an *H-arc-coloring* or simply an *arc-coloring* of G. Therefore, an oriented arc-coloring φ of G must satisfy (i) $\varphi(\overrightarrow{uv}) \neq \varphi(\overrightarrow{vw})$ whenever $\overrightarrow{uv}$ and $\overrightarrow{vw}$ are two consecutive arcs in G, and (ii) $\varphi(\overrightarrow{vw}) \neq \varphi(\overrightarrow{xy})$ whenever $\overrightarrow{uv}, \overrightarrow{vw}, \overrightarrow{xy}, \overrightarrow{yz} \in A(G)$ with $\varphi(\overrightarrow{uv}) = \varphi(\overrightarrow{yz})$. The *oriented chromatic index* of G, denoted by $\chi'_o(G)$, is defined as the smallest order of an oriented graph H such that $LD(G) \to H$.

The notion of oriented chromatic index can be extended to undirected graphs as follows. The oriented chromatic index $\chi'_o(G)$ of an undirected graph G is the maximum of the oriented chromatic indexes taken over all the orientations of G (an orientation of an undirected graph G is obtained by giving one of the two possible orientations to every edge of G).

In this paper, we are interested in oriented arc-coloring of subcubic graphs, that is graphs with maximum degree at most 3.

Oriented vertex-coloring of subcubic graphs has been first studied in [4] where it was proved that every oriented subcubic graph admits an oriented 16-vertex-coloring. In 1996, Sopena and Vignal improved this result:

Theorem 1 *[6] Every oriented subcubic graph admits an oriented 11-vertex-coloring.*

It is not difficult to see that every oriented graph having an oriented k-vertex-coloring admits a k-arc-coloring (from a k-vertex-coloring f, we obtain a k-arc-coloring g by setting $g(\overrightarrow{uv}) = f(u)$ for every arc $\overrightarrow{uv}$). Therefore, every oriented subcubic graph admits an oriented 11-arc-coloring.

We improve this bound and prove the following

Theorem 2 *Every oriented subcubic graph admits an oriented 7-arc-coloring.*

More precisely, we shall show that every oriented subcubic graph admits a homomorphism to QR_7, a tournament on 7 vertices described in section 3.

Note that Sopena conjectured that every oriented connected subcubic graph admits an oriented 7-vertex-coloring [4].

This paper is organized as follows. In the next section, we introduce the main definitions and notation. In section 3, we described the tournament QR_7 and give some properties of this graph. Finally, Section 4 is dedicated to the proof of Theorem 2.

2 Definitions and notation

In the rest of the paper, oriented graphs will be simply called *graphs*. For a graph G and a vertex v of G, we denote by $d_G^-(v)$ the indegree of v, by $d_G^+(v)$ its outdegree and by $d_G(v)$ its degree. A vertex of degree k (resp. at most k, at least k) will be called a k-vertex (resp. $^{\leq}k$-vertex, $^{\geq}k$-vertex). A *source vertex* (or simply *a source*) is a vertex v with $d^-(v) = 0$ and a *sink vertex* (or simply a *sink*) is a vertex v with $d^+(v) = 0$. A source (resp. sink) of degree k will be called a k-source (resp. a k-sink).

We denote by $N_G^+(v)$, $N_G^-(v)$ and $N_G(v)$ respectively the set of successors of v, the set of predecessors of v and the set of neighbors of v in G. The *maximum degree* and *minimum degree* of a graph G are respectively denoted by $\Delta(G)$ and $\delta(G)$.

We denote by $\overrightarrow{uv}$ the arc from u to v or simply uv whenever its orientation is not relevant (therefore $uv = \overrightarrow{uv}$ or $uv = \overrightarrow{vu}$).

For a graph G and a vertex v of $V(G)$, we denote by $G \setminus v$ the graph obtained from G by removing v together with the set of its incident arcs; similarly, for an arc a of $A(G)$, $G \setminus a$ denotes the graph obtained from G by removing a. These two notions are extended to sets in a standard way: for a set of vertices V', $G \setminus V'$ denotes the graph obtained from G by successively removing all vertices of V' and their incident arcs, and for a set of arcs A', $G \setminus A'$ denotes the graph obtained from G by removing all arcs of A'.

Let G be an oriented graph and f be an oriented arc-coloring of G. For a given vertex v of G, we denote by $C_f^+(v)$ and $C_f^-(v)$ the outgoing color set of v (i.e. the set of colors of the arcs outgoing from v) and the incoming color set of v (i.e. the set of colors of the arcs incoming to v), respectively.

The drawing conventions for a configuration are the following: a vertex whose neighbors are totally specified will be black (i.e. vertex of fixed degree), whereas a vertex whose neighbors are partially specified will be white. Moreover, an edge will represent an arc with any of its two possible orientations.

3 Some properties of the tournament QR_7

For a prime $p \equiv 3 \pmod 4$, the Paley tournament QR_p is defined as the oriented graph whose vertices are the integers modulo p and such that $\overrightarrow{uv}$ is an arc if and only if $v - u$ is a non-zero quadratic residue of p.

For instance, let us consider the tournament QR_7 with $V(QR_7) = \{0, 1, \ldots, 6\}$ and $\overrightarrow{uv} \in A(QR_7)$ whenever $v - u \equiv r \pmod 7$ for $r \in \{1, 2, 4\}$.

This graph has the two following useful properties [1]:

(P_1) Every vertex of QR_7 has three successors and three predecessors.

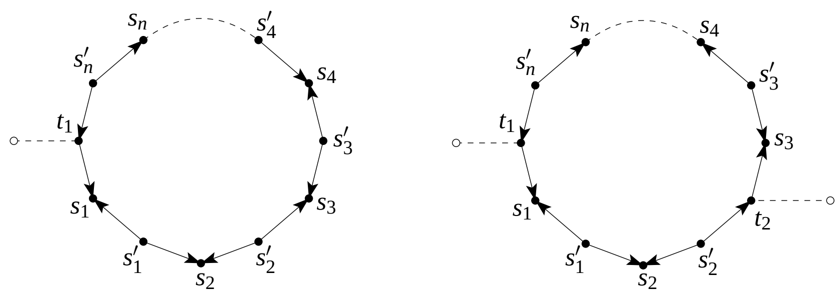


Figure 1: Two special cycles

(P_2) For every two distinct vertices u and v, there exists four vertices w_1, w_2, w_3 and w_4 such that:

- $\overrightarrow{uw_1} \in A(QR_7)$ and $\overrightarrow{vw_1} \in A(QR_7)$;
- $\overrightarrow{uw_2} \in A(QR_7)$ and $\overrightarrow{w_2v} \in A(QR_7)$;
- $\overrightarrow{w_3u} \in A(QR_7)$ and $\overrightarrow{w_3v} \in A(QR_7)$;
- $\overrightarrow{w_4u} \in A(QR_7)$ and $\overrightarrow{vw_4} \in A(QR_7)$.

4 Proof of Theorem 2

Let G be an oriented subcubic graph and C be a cycle in G (C is a subgraph of G). A vertex u of C is a *transitive vertex of* C if $d_C^+(u) = d_C^-(u) = 1$ (therefore $2 \le d_G(u) \le 3$).

A cycle C in G is a *special cycle* if and only if:

(1) every non-transitive vertex of C is a 2-source or a 2-sink in G;

(2) C has either exactly 1 transitive vertex or exactly 2 transitive vertices, and in this case, both transitive vertices have the same orientation on C.

Figure 1 shows two special cycles; the first one has exactly 1 transitive vertex while the second has exactly 2 transitive vertices oriented in the same direction. Vertices s_i, s'_j and t_k are respectively the sinks, sources, and transitive vertices of the special cycles.

Remark 3 Every 2-source (resp. 2-sink) in a special cycle C is necessarily adjacent to a 2-sink (resp. 2-source). This directly follows from the fact that C does not contain two transitive vertices oriented in opposite direction.

We shall denote by $SS_G(C)$ the set of 2-sources and 2-sinks of the cycle C in G.

Remark 4 Note that a special cycle may only be connected to the rest of the graph by its transitive vertices (see Figure 2 for an example).

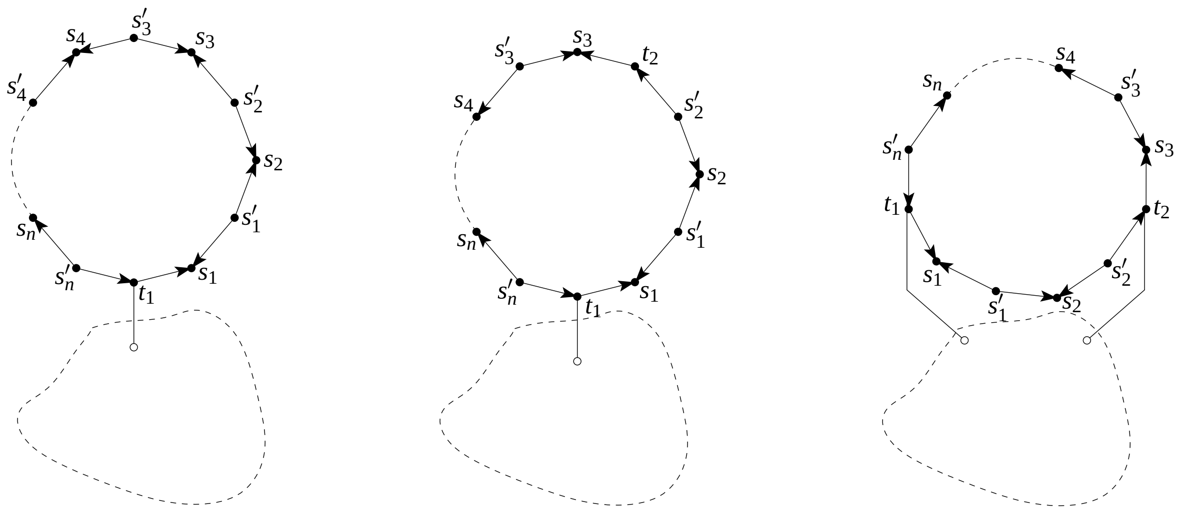


Figure 2: Graphs with a special cycle

A QR_7-arc-coloring f of an oriented subcubic graph G is *good* if and only if :

- for every 2-source u, $|C_f^+(u)| = 1$,
- for every 2-sink v, $|C_f^-(v)| = 1$.

Note that if a subcubic graph G admits a good QR_7-arc-coloring, then for every 2-vertex v of G, $|C_f^+(v)| \leq 1$ and $|C_f^-(v)| \leq 1$.

We first prove the following:

Theorem 5 *Every oriented subcubic graph with no special cycle admits a good QR_7-arc-coloring.*

We define a partial order $\prec$ on the set of all graphs. Let $n_2(G)$ be the number of ${}^{\geq}2$-vertices of G. For any two graphs G_1 and G_2, $G_1 \prec G_2$ if and only if at least one of the following conditions holds:

- G_1 is a proper subgraph of G_2;
- $n_2(G_1) < n_2(G_2)$.

Note that this partial order is well-defined, since if G_1 is a proper subgraph of G_2, then $n_2(G_1) \leq n_2(G_2)$. The partial order $\prec$ is thus a partial linear extension of the subgraph poset.

In the rest of this section, let H a be counter-example to Theorem 5 which is minimal with respect to $\prec$.

We shall show in the following lemmas that H does not contain some configurations.

In all the proofs which follow, we shall proceed similarly. We suppose that H contains some configurations and, for each of them, we consider a reduction H' of H with no special cycle such that $H' \prec H$. Therefore, due to the minimality of H, there exists a good QR_7-arc-coloring f of H'. The coloring f is a partial good QR_7-arc-coloring of H, that is an arc-coloring of some subset S of $A(H)$ and we show how to extend it to a good QR_7-arc-coloring of H. This proves that H cannot contain such configurations.

We will extensively use the following proposition:

Proposition 6 *Let $\overrightarrow{G}$ be an oriented graph which admits a good QR_7-arc-coloring. Let $\overleftarrow{G}$ be the graph obtained from $\overrightarrow{G}$ by giving to every arc its opposite direction. Then, $\overleftarrow{G}$ admits a good QR_7-arc-coloring.*

Proof: Let f be a good QR_7-arc-coloring of $\overrightarrow{G}$. Consider the coloring $f' : V(QR_7) \rightarrow A(\overleftarrow{G})$ defined by $f'(\overrightarrow{uv}) = 6 - f(\overrightarrow{vu})$.

It is easy to see that for every arc $\overrightarrow{uv} \in A(QR_7)$, we have $\overrightarrow{xy} \in A(QR_7)$ for $x = 6 - v$ and $y = 6 - u$. Moreover, the two incident arcs to a 2-source (or a 2-sink) will get the same color by f' since they got the same color by f. □

Therefore, when considering good QR_7-arc-coloring of an oriented graph G, we may assume that one arc in G has a given orientation.

The following remark will be extensively used in the following lemmas :

Remark 7 Let G be a graph with no special cycle and $A \subseteq A(G)$ be an arc set. If the graph $G' = G \setminus A$ contains a special cycle C, then at least one of the vertices incident to A is a 2-source or a 2-sink in G' and belongs to $V(C)$, since otherwise C would be a special cycle in G.

Lemma 8 *The graph H is connected.*

Proof: Suppose that $H = H_1 \uplus H_2$ (disjoint union). We have $H_1 \prec H$ and $H_2 \prec H$. The graphs H_1 and H_2 contain no special cycle and then, by minimality of H, H_1 and H_2 admits good QR_7-arc-colorings f_1 and f_2 respectively that can easily be extended to a good QR_7-arc-coloring $f = f_1 \cup f_2$ of H. □

Lemma 9 *The graph H contains no 3-source and no 3-sink.*

Proof: By Proposition 6, we just have to consider the 3-source case. Let u be a 3-source in H and H' be the graph obtained from H by splitting u into three 1-vertices u_1, u_2, u_3. We have $H' \prec H$ since $n_2(H') = n_2(H) - 1$. Any good QR_7-arc-coloring of H' is clearly a good QR_7-arc-coloring of H. □

Lemma 10 *The graph H contains no 1-vertex.*

Proof: Let u_1 be a 1-vertex in H, v be its neighbor and $N_H(v) = \{u_i, 1 \leq i \leq d_H(v)\}$. By Proposition 6, we may assume $\overrightarrow{u_1v} \in A(H)$. We consider three subcases.

1. $d_H(v) = 1$.

   By Lemma 8, $H = \overrightarrow{u_1v}$ and obviously, H admits a good QR_7-arc-coloring.

2. $d_H(v) = 2$.

   Let $H' = H \setminus u_1$; we have $H' \prec H$ and H' contains no special cycle by remark 7. By minimality of H, H' admits a good QR_7-arc-coloring f that can easily be extended to H: if v is a 2-sink, we set $f(\overrightarrow{u_1v}) = f(\overrightarrow{u_2v})$; otherwise, we have three available colors for $f(\overrightarrow{u_1v})$ by Property (P_1).

3. $d_H(v) = 3$.

   Let $H' = H \setminus u_1$; we have $H' \prec H$.

   If H' contains no special cycle then, by minimality of H, H' admits a good QR_7-arc-coloring f such that $|C_f^+(v)| \leq 1$. The coloring f can then be extended to H since we have three available colors to set $f(\overrightarrow{u_1v})$ by property (P_1).

   If H' contains a special cycle C, $v \in C$ and v is a 2-source in H' by Remark 7 and Lemma 9. We may assume w.l.o.g. that u_2 is a 2-sink by Remark 3. Let $N_H(u_2) = \{v, x\}$ and $H'' = H \setminus \{\overrightarrow{vu_2}, \overrightarrow{u_1v}\}$. We have $H'' \prec H$ and H'' contains no special cycle by Remark 7. By minimality of H, H'' admits a good QR_7-arc-coloring f that can be extended to H: we set $f(\overrightarrow{vu_2}) = f(\overrightarrow{xu_2})$, and we have at least one available color for $f(\overrightarrow{u_1v})$ by Property (P_2).

□

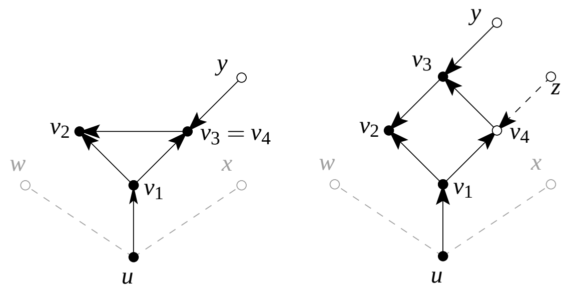
Recall that a *bridge* in a graph G is an edge whose removal increases the number of components of G.

Lemma 11 *The graph H contains no bridge.*

Proof: Suppose that H contains a bridge uv. Let $H \setminus uv = H_1 \uplus H_2$. For $i = 1, 2$, consider $H'_i = H_i + uv$. By Lemma 10, uv is not a dangling arc in H. Moreover $H'_i \prec H$ for $i = 1, 2$. Clearly, the graphs H'_1 and H'_2 have no special cycle and therefore, by minimality of H, they admit good QR_7-arc-colorings f_1 and f_2 respectively. By cyclically permuting the colors of f_2 if necessary, we may assume that $f_1(uv) = f_2(uv)$. The mapping $f = f_1 \cup f_2$ is then clearly a good QR_7-arc-coloring of H. □

Lemma 12 *The graph H contains no 2-sink adjacent to a 2-source.*

Proof: Suppose that H contains a 2-sink v adjacent to a 2-source w. Let $N(v) = \{u, w\}$ and $N(w) = \{v, x\}$. Since H contains no special cycle, u and x are distinct vertices and $\overrightarrow{xu} \notin A(H)$.



(a) (b)

Figure 3: Configurations of Lemma 14

Let H' be the graph obtained from $H \setminus \{v, w\}$ by adding $\overrightarrow{ux}$ (if it did not already belong to $A(H)$). We have $H' \prec H$ since $n_2(H') \leq n_2(H) - 2$. Since the vertices u and x are neither 3-sources nor 3-sinks in H by Lemma 9, they are neither 2-sources nor 2-sinks in H' and therefore, by Remark 7, H' contains no special cycle. Hence, by minimality of H, H' admits a good QR_7-arc-coloring f' that can be extended to H by setting $f(\overrightarrow{uv}) = f(\overrightarrow{wv}) = f(\overrightarrow{wx}) = f(\overrightarrow{ux})$. □

Lemma 13 *Every 2-source (resp. 2-sink) of H is adjacent to a vertex v with $d^+(v) = 2$ (resp. $d^-(v) = 2$).*

Proof: Suppose that H contains a 2-source u adjacent to two vertices v and w such that $d^+(v) \neq 2$ and $d^+(w) \neq 2$ (by Proposition 6, it is enough to consider this case). Let $H' = H \setminus u$; by hypothesis and by Lemmas 9 and 12, the vertices v and w are such that $d^+_{H'}(v) = d^-_{H'}(v) = d^+_{H'}(w) = d^-_{H'}(w) = 1$. Therefore, the graph H' contains no special cycle by Remark 7. By minimality of H, H' admits a good QR_7-arc-coloring f that can be extended to H in such a way that $f(\overrightarrow{uu_1}) = f(\overrightarrow{uu_2})$ thanks to Property (P_2). □

Recall that we denote by $SS_G(C)$ the set of 2-sources and 2-sinks of the cycle C in G.

Lemma 14 *Let u be a vertex of H and $H' = H \setminus u$. Then H' does not contain a special cycle C with $|N_H(u) \cap SS_{H'}(C)| = 1$.*

Proof: Let $v_1 \in N(u)$ and w.l.o.g., suppose that $H' = H \setminus u$ contains a special cycle C such that $N_H(u) \cap SS_{H'}(C) = \{v_1\}$; by Remark 7, v_1 is a 2-source or a 2-sink in H' and by Proposition 6 we may assume w.l.o.g. that v_1 is a 2-source.

By Remark 3, v_1 is adjacent to a 2-sink v_2. By Lemma 12, the only pair of adjacent 2-source and 2-sink in H' is v_1, v_2. Therefore, we have $3 \leq |C| \leq 4$. Let $V(C) = \{v_1, v_2, v_3, v_4\}$ and $v_3 = v_4$ if $|C| = 3$. Moreover v_3 and v_4 are necessarily two transitive vertices of C. Furthermore, we have $\overrightarrow{yv_3} \in A(H)$ by Lemma 13 and $\overrightarrow{uv_1} \in A(H)$ by Lemma 9. Then, we have only two possible configurations, depicted in Figure 3.

- If $|C| = 3$ (see Figure 3(a)), consider $H'_1 = H \setminus \overrightarrow{v_1v_2}$. This graph contains no special cycle by Remark 7 and we have $H'_1 \prec H$. By minimality of H, H'_1 admits a good QR_7-arc-coloring f that can be extended to H: we first erase $f(\overrightarrow{v_1v_3})$; then, we can set $f(\overrightarrow{v_1v_2}) = f(\overrightarrow{v_3v_2})$ thanks to Property (P_2) and then we have one available color for $f(\overrightarrow{v_1v_3})$ by Property (P_2) since $f(\overrightarrow{uv_1}) \neq f(\overrightarrow{v_3v_2})$.

- If $|C| = 4$ (see Figure 3(b)), consider the graph $H'_2 = H \setminus v_2$. We have $H'_2 \prec H$.

  - If H'_2 contains no special cycle, by minimality of H, H'_2 admits a good QR_7-arc-coloring f that can be extended to H in such a way that $f(\overrightarrow{v_3v_2}) = f(\overrightarrow{v_1v_2})$ thanks to Property (P_2) since $f(\overrightarrow{v_4v_3}) = f(\overrightarrow{yv_3})$.

  - Suppose now that H'_2 contains a special cycle C'. By Remark 7, v_3 belong to C' and by Remark 3, y is a 2-sink. By Lemma 12, the only pair of adjacent 2-source and 2-sink in H' is v_3, y, and therefore $|C'|$ is a special cycle of length 3 or 4. Suppose first that $\{u, v_1, v_4, v_3, y\} \subseteq V(C')$; we thus have $u = y$, that is a contradiction since by hypothesis $N_H(u) \cap SS_{H'}(C) = \{v_1\} \neq \{v_1, v_3\}$.

    Therefore $V(C') = \{y, v_3, v_4, z\}$, and then $\overrightarrow{zv_4} \in A(H)$. If $|C'| = 3$, we have $y = z$ and in this case, the graph H contains a bridge $\overrightarrow{uv_1}$ that is forbidden by Lemma 11. Therefore, we have $|C'| = 4$ and z is a transitive vertex of C'.

    Consider in this case the graph $H'_3 = H \setminus v_4$. This graph contains no special cycle since the vertices v_1 and v_3 are two transitive 2-vertices oriented in opposite directions. We have $H'_3 \prec H$ and therefore, by minimality of H, there exists a good QR_7-arc-coloring f of H'_3 such that $C^-_f(v_1) = \{c_1\}, C^-_f(v_2) = \{c_2\}$ and $C^+_f(y) = C^-_f(z) = \{c_3\}$. The mapping f can be extended to H as follows: we can set $f(\overrightarrow{v_4v_3}) = c_4 \notin \{c_1, c_3\}$ thanks to Property (P_1). Then, by Property (P_2), we have one available color for $f(\overrightarrow{v_1v_4})$ since $c_1 \neq c_4$ and one available color for $f(\overrightarrow{zv_4})$ since $c_3 \neq c_4$.

□

Lemma 15 *The graph H does not contain two adjacent 2-vertices.*

Proof: Suppose that H contains two adjacent 2-vertices v and w. Let $N(v) = \{u, w\}$ and $N(w) = \{v, x\}$ and $H' = H \setminus v$. By Lemma Remark 7 and 14, H contains no special cycle. We have $H' \prec H$ and by minimality of H, H' admits a good QR_7-arc-coloring f.

We shall consider two cases depending on the orientation of the arcs incident to v and w (by Proposition 6, we may assume that $\overrightarrow{uv} \in A(H)$).

1. v is a 2-sink and w is a transitive vertex.
   By Lemma 12, u is not a 2-source in H. We have $|C^-_f(u)| \leq 1$ and then, we can set $f(\overrightarrow{uv}) = f(\overrightarrow{wv})$ thanks to Property (P_2).

2. v and w are transitive vertices.
   By the previous case, u is not a 2-source. We have $|C^-_f(u)| \leq 1$. Thanks to

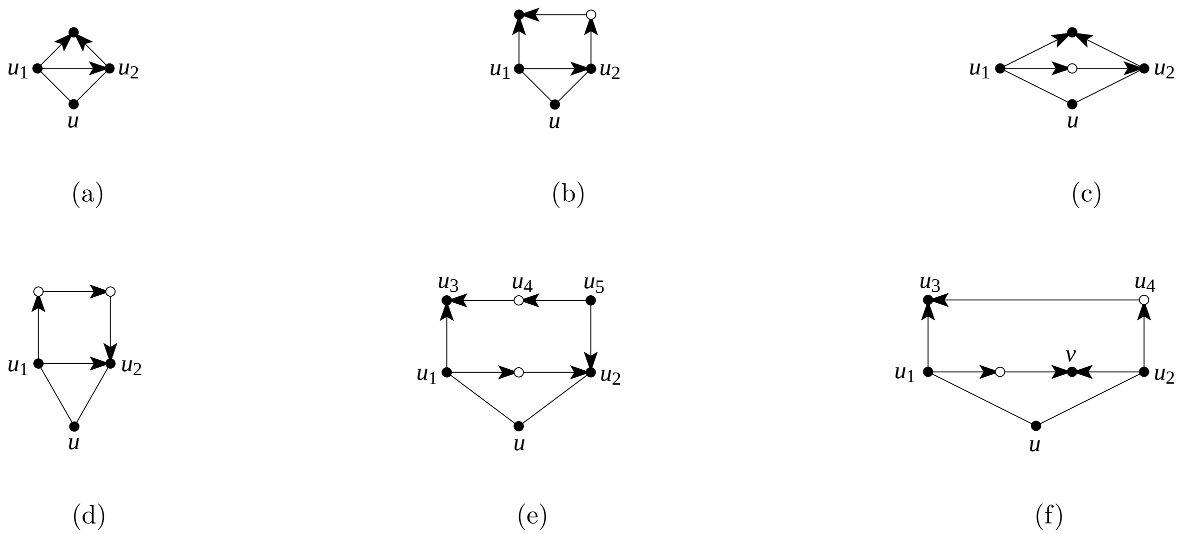


Figure 4: Configurations of Lemma 16

Property (P_1), we can set $f(\overrightarrow{uv}) \neq f(\overrightarrow{wx})$ and finally, we have one available color for $f(\overrightarrow{vw})$ by Property (P_2) since $f(\overrightarrow{uv}) \neq f(\overrightarrow{wx})$.

□

Lemma 16 *The graph H contains no 2-vertex.*

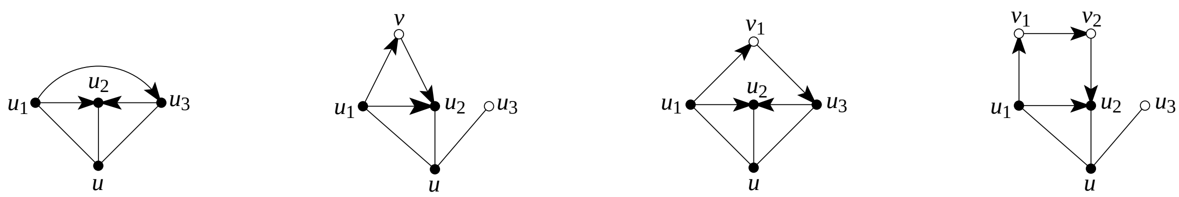
Proof: Suppose that H contains a 2-vertex u and let $N(u) = \{u_1, u_2\}$. The vertices u_1 and u_2 are 3-vertices by Lemma 15. By Proposition 6, we may assume w.l.o.g. that $\overrightarrow{uu_1} \in A(H)$. Let $H'_1 = H \setminus u$; we have $H'_1 \prec H$.

If H'_1 contains no special cycle, then by minimality of H, H'_1 admits a good QR_7-arc-coloring f of H'_1 that can be extended to H as follows. If u is a 2-source , we can set $f(\overrightarrow{uu_1}) = f(\overrightarrow{uu_2})$ thanks to Property (P_2) since $|C^+_f(u_1)| \leq 1$ and $|C^+_f(u_2)| \leq 1$. If u is a transitive vertex, we can set $f(\overrightarrow{uu_1}) \notin C^-_f(u_2)$ thanks to Property (P_1) and then we have one available color for $f(\overrightarrow{u_2u})$ by Property (P_2).

Suppose now that H'_1 contains a special cycle C. By Lemma 14, u_1 and u_2 belongs to C and at least one of them is a 2-source or a 2-sink.

Suppose first that u_1 is a 2-source in H'_1 and u_2 is neither a 2-source nor a 2-sink in H'_1. Then, since H contains no adjacent 2-vertices by Lemma 15, we have only three possible configurations depicted in Figures 4(a), 4(b) and 4(c).

Clearly, the configuration of Figure 4(a) admits a good QR_7-arc-coloring. The white vertex of the configuration of Figure 4(b) is a 3-vertex by Lemma 15, but in this case, the graph contains a bridge, that is forbidden by Lemma 11. The white vertex of the configuration of Figure 4(c) is of degree two by Lemma 11 and this configuration clearly admits a good QR_7-arc-coloring.



(a) (b) (c) (d)

Figure 5: Configurations of Lemma 17

Therefore, u_1 and u_2 are either 2-sources or 2-sinks in H'_1. In this case, since H contains no adjacent 2-vertices by Lemma 15, we have only three possible configurations depicted in Figure 4(d), 4(e) and 4(f).

- Figure 4(d): by Lemma 9, we have $\overrightarrow{u_2u}, \overrightarrow{uu_1} \in A(H)$. Consider the graph $H'_2 = H\backslash\overrightarrow{u_1u_2}$; H'_2 contains no special cycle. Since $H'_2 \prec H$, by minimality of H, H'_2 admits a good QR_7-arc-coloring f that can be extended to H thanks to Property (P_2) since $f(\overrightarrow{u_2u}) \neq f(\overrightarrow{uu_1})$.
- Figure 4(e): by Lemma 9, we have $\overrightarrow{u_2u}, \overrightarrow{uu_1} \in A(H)$. By Lemma 15, u_4 is a 3-vertex. If $d^-(u_4) = 2$, this configuration is forbidden by Lemma 13. If $d^+(u_4) = 2$, this configuration is also forbidden by Lemma 13.
- Figure 4(f): by Lemma 9, we have $\overrightarrow{uu_1}, \overrightarrow{uu_2} \in A(H)$. Therefore, by Lemma 13, $d^-(u_4) = 2$. Consider $H'_4 = H \setminus \overrightarrow{u_1u_3}$; clearly, H'_4 contains no special cycle. By minimality of H, H'_4 admits a good QR_7-arc-coloring that can be extended to H as follows. We first erase $f(\overrightarrow{u_2u_4})$ and $f(\overrightarrow{u_4u_3})$; then, thanks to Property (P_2), we can set $f(\overrightarrow{u_1u_3}) = f(\overrightarrow{u_4u_3})$. Finally, since $f(\overrightarrow{uu_2}) \neq f(u_4u_3)$, we can extend f to a good QR_7-arc-coloring of H thanks to Property (P_2).

□

Lemma 17 *The graph H contains no 3-vertex.*

Proof: By Lemmas 10 and 16, H is a 3-regular graph. Let u be a vertex of H with neighbors u_1, u_2 and u_3. By Lemma 9, u is neither a 3-source nor a 3-sink and therefore, by Proposition 6, we may assume w.l.o.g. that $d^+(u) \geq d^-(u)$. Let $\overrightarrow{u_1u}, \overrightarrow{uu_2}, \overrightarrow{uu_3} \in A(H)$.

If $H'_1 = H\backslash u$ contains no special cycle, by minimality of H, H'_1 admits a good QR_7-arc-coloring f that can be extended to H as follows. We can set $f(\overrightarrow{u_1u}) \notin C^+_f(u_2) \cup C^+_f(u_3)$ thanks to Property (P_1). Then, thanks to Property (P_2), we can extend f to a good QR_7-arc-coloring of H.

Suppose now that H'_1 contains a special cycle C. The graph H'_1 contains three 2-vertices. Since a special cycle consists in k pairs of 2-sources and 2-sinks, C contains only one pair of adjacent 2-source and 2-sink (w.l.o.g. u_1 and u_2 respectively). Therefore, we have only four possible configurations depicted in Figure 5.

Clearly, the configuration of Figure 5(a) admits a good QR_7-arc-coloring. The white vertex of the configuration of Figure 5(c) is a 2-vertex by Lemma 11 and it is easy to check that there exits a good QR_7-arc-coloring of this graph. Consider now the configurations of Figures 5(b) and 5(d) and let $H_2' = H \setminus \overrightarrow{u_1u_2}$. We have $H_2' \prec H$ and clearly, H_2' contains no special cycle. Therefore, by minimality of H, H_2' admits a good QR_7-arc-coloring f that can be extended to H thanks to Property (P_2) since for any orientation of H, $C_f^-(u_1) \cap C_f^+(u_2) = \emptyset$. □

Proof of Theorem 2: By Lemmas 10, 16 and 17, a minimal counter-example to Theorem 5 does not exist.

We now say that a QR_7-arc-coloring f of an oriented subcubic graph G is *quasi-good* if and only if for every 2-source u, $|C_f^+(u)| = 1$.

Note that if a subcubic graph admits a quasi-good QR_7-arc-coloring f, we have $|C_f^+(v)| \leq 1$ for every ${}^{\leq}2$-vertex v of G.

We shall then prove Theorem 2 by showing that every subcubic graph admits a quasi-good $QR7$-arc-coloring.

Let H be a minimal counter-example to Theorem 2.

If H contains no special cycle, by Theorem 5, H admits a good QR_7-arc-coloring which is a quasi-good QR_7-arc-coloring.

Suppose now that H contains at least one special cycle. By definition, a special cycle contains at least one 2-source. We inductively define a sequence of graphs $H_0, H_1, \ldots, H_n$ for $n \geq 0$, and a sequence of vertices $u_0, u_1, \ldots, u_{n-1}$ such that:

- $H_0 = H$;
- H_i contains a special cycle, and thus a 2-source u_i for $0 \leq i < n$;
- $H_{i+1} = H_i \setminus u_i$ for $0 \leq i < n$;
- H_n has no special cycle.

By Theorem 5, H_n admits a good QR_7-arc-coloring, and therefore a quasi-good QR_7-arc-coloring. Suppose that H_{i+1} admits a quasi-good QR_7-arc-coloring f_{i+1} for $1 \leq i < n$; we claim that we can extend f_{i+1} to a quasi-good QR_7-arc-coloring f_i of H_i as follows. To see that, let v_i and w_i be the two neighbors of u_i which are ${}^{\leq}2$-vertices in H_{i+1}. Therefore, we have $|C^+_{f_{i+1}}(v_i)| \leq 1$ and $|C^+_{f_{i+1}}(w_i)| \leq 1$ and thanks to Property (P_2), we can set $f_i(\overrightarrow{u_iv_i}) = f_i(\overrightarrow{u_iw_i})$.

Therefore, any quasi-good QR_7-arc-coloring of H_n can be extended to $H_0 = H$, that is a contradiction. A minimal counter-example to Theorem 2 does not exist, that completes the proof. □

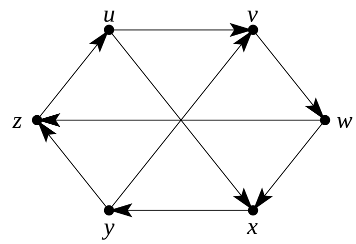


Figure 6: Cubic graph G with $\chi'_o(G) = 6$

Currently, we cannot provide an oriented subcubic graph with oriented chromatic index 7. However, the oriented cubic graph G depicted in Figure 6 has oriented chromatic index 6.

Suppose we want to color G with five colors $1, 2, 3, 4, 5$. Necessarily the colors of $\overrightarrow{vw}$, $\overrightarrow{xy}$ and $\overrightarrow{zu}$ are pairwise distinct and we may assume w.l.o.g. that $f(\overrightarrow{vw}) = 1$, $f(\overrightarrow{xy}) = 2$ and $f(\overrightarrow{zu}) = 3$. Clearly, each of the colors 4 and 5 will appear at most once on $\overrightarrow{uv}$, $\overrightarrow{wx}$ and $\overrightarrow{yz}$. Therefore, w.l.o.g. we may assume that $f(\overrightarrow{yz}) = 1$, which implies w.l.o.g. that we must set $f(\overrightarrow{ux}) = 4$. Thus, we must set $f(\overrightarrow{yv}) = 5$, and then we have no remaining color to color $f(\overrightarrow{wz})$.

Therefore, we have the following:

Proposition 18 *Let $\mathcal{C}$ be the class of subcubic graphs. Then $6 \leq \chi'_o(\mathcal{C}) \leq 7$.*

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