Fractional biclique covers and partitions of graphs

Valerie L. Watts
Department of Mathematics
Malaspina University-College
900 Fifth Street
Nanaimo, British Columbia, Canada V9R 5S5
wattsv@mala.bc.ca

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Abstract

A biclique is a complete bipartite subgraph of a graph. This paper investigates the fractional biclique cover number, $bc^*(G)$, and the fractional biclique partition number, $bp^*(G)$, of a graph G. It is observed that $bc^*(G)$ and $bp^*(G)$ provide lower bounds on the biclique cover and partition numbers respectively, and conditions for equality are given. It is also shown that $bc^*(G)$ is a better lower bound on the Boolean rank of a binary matrix than the maximum number of isolated ones of the matrix. In addition, it is noted that $bc^*(G) \leq bp^*(G) \leq \beta^*(G)$, the fractional vertex cover number. Finally, the application of $bc^*(G)$ and $bp^*(G)$ to two different weak products is discussed.

Keywords: fractional biclique covers, fractional biclique partitions, Boolean rank, weak products

1 Introduction

Fractional graph theory is the modification of integer-valued graph parameters to allow them to take on non-integer values. This article investigates the fractional analogues of the minimum number of complete bipartite subgraphs (bicliques) needed to cover or partition the edges of a graph. For more on fractional graph theory and other fractional graph parameters, see Berge [1] or Scheinerman and Ullman [17].

To begin, some definitions are given which are used throughout. A simple graph is denoted by G with vertex set $V(G) = \{1, 2, \cdots, n\}$ and edge set $E(G)$. An edge of G is an unordered pair of vertices $\{u, v\}$, usually written uv. For general graph theory terminology used throughout, see [2]. A subgraph of G whose edge set forms a complete bipartite graph is called a *biclique* of G. Let $K(R, S)$ denote the biclique of G with edge set $\{ij : i \in R, j \in S\}$ where R and S are disjoint non-empty subsets of vertices of G.

The sets of vertices R and S are called the *bipartition* of $K(R, S)$. If $|R| = r$ and $|S| = s$, then $K(R, S)$ is said to be a $K_{r,s}$. A biclique $K(R, S)$ is a *star* centered at vertex v if one of R or S contains only a single vertex v. The set of all bicliques of G is denoted $\mathcal{B}(G)$.

A *biclique cover* of a graph G is a collection $\mathcal{B}$ of bicliques $K(X_i, Y_i)$, $1 \leq i \leq k$, of G whose edge sets cover the edge set of G. That is, each edge of G is in at least one of the bicliques in $\mathcal{B}$. The *biclique cover number*, $bc(G)$, of a graph G is the minimum number of bicliques in a biclique cover of G. A *biclique partition* of a graph G is a collection $\mathcal{B}$ of bicliques $K(X_i, Y_i)$, $1 \leq i \leq k$, of G whose edge sets partition the edge set of G. That is, each edge of G is in exactly one of the bicliques in $\mathcal{B}$. The *biclique partition number*, $bp(G)$, is the minimum number of bicliques in a biclique partition of G. For more on biclique covers and partitions, see [3, 8, 11, 14, 15, 16].

Sections 2 and 3 introduce the fractional biclique cover and partition numbers, respectively, via linear programs that assign weights to either the bicliques or the edges of a graph. It is observed that the fractional biclique cover (resp. partition) number is a lower bound on $bc(G)$ (resp. $bp(G)$). In addition, it is shown that the fractional biclique cover number is a better lower bound on Boolean rank than a well known lower bound given by Gregory and Pullman [10]. An example is given which shows that the fractional biclique cover number may be smaller than the fractional biclique partition number. It is well known that for any graph G, $bc(G) \leq bp(G) \leq \beta(G)$, the vertex cover number, and section 4 shows that the fractional analogues of these three numbers share the same relationship. Finally, section 5 discusses the application of the fractional biclique cover and partition numbers to weak product and weak bipartite product. In particular, it is observed that the fractional biclique cover number of a weak bipartite product of bipartite graphs equals the product of their fractional biclique cover numbers. A similar result is true for weak product.

2 Fractional Biclique Covers

Another way to view a biclique cover is as a function w that assigns to each biclique B of G either 0 or 1 so that, for each edge $e \in E(G)$, $\sum w(B) \geq 1$ where the sum is taken over all bicliques that contain e. Then, $bc(G)$ is the minimum of $\sum_{B \in \mathcal{B}(G)} w(B)$ over all biclique covers. Thus, $bc(G)$ is the value of a $(0, 1)$-integer program and its linear relaxation defines the fractional biclique cover number.

A *fractional biclique cover* is a function w that assigns to each biclique B of a graph G a number so that $w(B) \geq 0$ and, for each edge $e \in E(G)$, $\sum w(B) \geq 1$ where the sum is taken over all bicliques that contain e. Note that every biclique cover is in fact a fractional biclique cover. To compare biclique covers and fractional biclique covers, consider the complete graph on four vertices, K_4, shown in Figure 1. A biclique cover of K_4 is given by bicliques $B_1 = K(\{1, 3\}, \{2, 4\})$ and $B_2 = K(\{1, 2\}, \{3, 4\})$. The function w associated with this cover assigns the values $w(B_1) = w(B_2) = 1$ and 0 for all other bicliques of K_4. As observed above, w is also a fractional biclique cover. Now, consider the function w_1 where $w_1(K(\{1, 3\}, \{2, 4\})) = w_1(K(\{1, 2\}, \{3, 4\})) =$

$w_1(K(\{1,4\},\{2,3\})) = \frac{1}{2}$ and $w_1(B) = 0$ for all other bicliques B of K_4. Note that w_1 is a fractional biclique cover of K_4 since each edge e of K_4 is in exactly two of the bicliques $K(\{1,3\},\{2,4\}), K(\{1,2\},\{3,4\}), K(\{1,4\},\{2,3\})$ and so $\sum_{\{B:e\in B\}} w_1(B) \geq 1$.

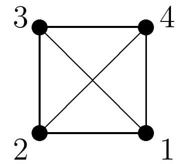

Figure 1

The *fractional biclique cover number*, $bc^*(G)$, is the infimum of $\sum_{B\in\mathcal{B}(G)} w(B)$ over all fractional biclique covers. That is, $bc^*(G)$ is the value of the linear program

$$\begin{array}{ll} \text{minimize} & \sum_{B\in\mathcal{B}(G)} w(B) \\ \text{subject to} & \sum_{\{B:e\in B\}} w(B) \geq 1 \text{ for each edge } e \text{ of } G \\ & w(B) \geq 0 \text{ for each biclique } B \text{ of } G \end{array} \tag{1}$$

where each biclique B of G is assigned a weight $w(B)$. Since the value of the fractional biclique cover number is determined by a linear program, $bc^*(G)$ may also be found using the dual program. Thus, $bc^*(G)$ is the value of the linear program

$$\begin{array}{ll} \text{maximize} & \sum_{e\in E(G)} v(e) \\ \text{subject to} & \sum_{e\in B} v(e) \leq 1 \text{ for each biclique } B \text{ of } G \\ & v(e) \geq 0 \text{ for each edge } e \text{ of } G \end{array} \tag{2}$$

where each edge e of G is assigned a weight $v(e)$. Note that $v(e) \leq 1$ since each edge is a biclique. For results regarding linear programming mentioned throughout, see Chvátal [4].

An *automorphism* of G is a permutation of the vertices of G which maps edges to edges and non-edges to non-edges. Let $\operatorname{Aut} G$ denote the *automorphism group* of a graph G. The orbits of $\operatorname{Aut} G$ partition the edge set of G into equivalence classes. It is straightforward to check that if v is an optimal weighting of the edges of G in (2) with $bc^*(G) = \sum_{e\in E(G)} v(e)$ and $\hat{v}(e) = \frac{1}{|\operatorname{Aut} G|}\sum_{\sigma\in\operatorname{Aut} G} v(\sigma(e))$ then $\hat{v}(e)$ also satisfies the constraints of (2) and $bc^*(G) = \sum_{e\in E(G)} \hat{v}(e)$. Thus, in finding $bc^*(G)$ using (2), edges in the same orbit of $\operatorname{Aut} G$ may be assumed to have the same weight. In particular, if G is edge-transitive (that is, if for each pair of edges e and f of G there exists $\sigma \in \operatorname{Aut} G$ with $\sigma(e) = f$) then each edge of G may be assumed to have the same weight. Consequently, $bc^*(G) = \frac{|E(G)|}{ab}$ where $K_{a,b}$ is the largest biclique of an edge-transitive graph G in terms of the number of edges. For the proofs of these statements, see Watts [19].

The cycle C_n on n vertices is an edge-transitive graph and for $n \neq 4$ its largest biclique is a $K_{1,2}$. Thus, $bc^*(C_n) = \frac{n}{2}$ for $n \neq 4$ and $bc^*(C_4) = 1$. Similarly, the complete graph K_n on n vertices is edge-transitive and its largest biclique is a $K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$. Consequently,

$$bc^*(K_n) = \begin{cases} \frac{2(n-1)}{n} & \text{if } n \text{ is even} \\ \frac{2n}{n+1} & \text{if } n \text{ is odd} \end{cases}$$

Harary, Hsu and Miller [12] showed that $bc(K_n) = \lceil \log_2 n \rceil$ and so as $n \to \infty$, $bc(K_n) \to \infty$ while $bc^*(K_n) \to 2$.

Consider the complement of the cycle on six vertices, $\overline{C}_6$, shown in Figure 2 below. The orbits of $\mathrm{Aut}\,\overline{C}_6$ yield the following equivalence classes: $A = \{13, 15, 24, 26, 35, 46\}$ and $B = \{14, 25, 36\}$. According to the statement given above, the edges in A may all receive the same weight, a, and the edges in B may all receive the same weight, b. It follows that the objective function of (2) for $\overline{C}_6$ is $6a + 3b$ since there are six elements in A and three elements in B. The seven different stars centered at vertex 1 produce the following constraints:

Biclique	Constraint
$K(\{1\},\{3\})$	$a \le 1$
$K(\{1\},\{4\})$	$b \le 1$
$K(\{1\},\{5\})$	$a \le 1$
$K(\{1\},\{3,5\})$	$2a \le 1$
$K(\{1\},\{3,4\})$	$a + b \le 1$
$K(\{1\},\{4,5\})$	$a + b \le 1$
$K(\{1\},\{3,4,5\})$	$2a + b \le 1$

Similar constraints occur for the stars centered at each vertex i. The only other bicliques in $\overline{C}_6$ are $K_{2,2}$'s, such as $K(\{1,2\},\{4,5\})$. Each $K_{2,2}$ yields the same constraint: $2a+2b \le 1$. Then from (2), $bc^*(\overline{C}_6)$ is the value of the linear program

$$\begin{array}{ll} \text{maximize} & 6a + 3b \\ \text{subject to} & 2a \le 1 \\ & a + b \le 1 \\ & 2a + b \le 1 \\ & 2a + 2b \le 1 \\ & a, b \le 1 \end{array}$$

Since $2a + b \le 1$, it follows that $6a + 3b \le 3$. Taking $a = \frac{1}{2}$ and $b = 0$, the maximum of 3 may be attained. Therefore, $bc^*(\overline{C}_6) = 3$.

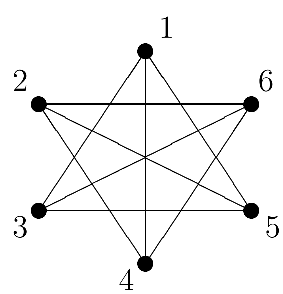

Figure 2

Theorem 2.1 shows that the fractional biclique cover number is a lower bound for the biclique cover number and conditions for equality are given. The proof of Theorem 2.1 first appeared in the thesis of the author [19].

Theorem 2.1 *For a graph G, $bc^*(G) \leq bc(G)$ with equality holding if and only if for some (and every) minimum biclique cover $\mathcal{B}$ and optimal weighting v of the edges of G for (2), $\sum_{e \in B} v(e) = 1$ for all $B \in \mathcal{B}$, $\sum_{e \in B} v(e) \leq 1$ for all bicliques B of G, $v(e) = 0$ for each edge e of G that is covered more than once by the bicliques of $\mathcal{B}$ and $v(e) \geq 0$ for each edge e of G.*

Proof. Let $\mathcal{B}$ be a minimum biclique cover of G. Let v be an optimal weighting of the edges of G for (2). Then $bc^*(G) = \sum_{e \in E(G)} v(e) \leq \sum_{B \in \mathcal{B}} \left(\sum_{e \in B} v(e)\right) \leq \sum_{B \in \mathcal{B}} 1 = |\mathcal{B}| = bc(G)$. Thus, $bc^*(G) = bc(G)$ if and only if $\sum_{e \in B} v(e) = 1$ for all $B \in \mathcal{B}$ and $v(e) = 0$ for each edge e of G which is covered more than once by the bicliques of $\mathcal{B}$. □

The case of bipartite graphs is of special interest since $bc^*(G)$ provides a lower bound on Boolean rank. The *Boolean rank*, $r_B(A)$, of an $m \times n$ Boolean matrix A is the smallest integer k such that $A = XY^T$ for some $m \times k$ binary matrix X and $n \times k$ binary matrix Y. If A is an $m \times n$ $(0, 1)$-matrix and G is the bipartite graph with bipartite adjacency matrix A, then $bc(G) = r_B(A)$, an observation provided by Orlin [16]. Consequently, $bc^*(G)$ is the fractional analogue of $r_B(A)$ and $r_B(A) \geq bc^*(G)$. For more on Boolean rank, see [6, 7, 8, 10, 13, 15]

As in [10], a set of ones of a binary matrix A is *isolated* if no pair of ones are in an all-ones submatrix of A together. Let $i(A)$ be the maximum number of ones in an isolated set of A. It follows that $r_B(A) \geq i(A)$. However, Theorem 2.2 below implies that $bc^*(G) \geq i(A)$ where G is the bipartite graph with bipartite adjacency matrix A. Consequently, fractional Boolean rank is a better lower bound for $r_B(A)$ than $i(A)$. For example, the 5×5 matrix $\overline{I}_5$ with zeros down the main diagonal and ones everywhere else has $r_B(\overline{I}_5) = 4$, $r_B^*(\overline{I}_5) = \frac{10}{3}$ and $i(\overline{I}_5) = 3$.

Theorem 2.2 *For a bipartite graph G, $bc^*(G) \geq i(A)$, where A is the bipartite adjacency matrix of G.*

Proof. A set of isolated ones in A corresponds to a matching in G with the property that the subgraph induced by the matching is $K_{2,2}$-free. This implies that the edges of the matching cannot be in any biclique together. Let M be the matching corresponding to a maximum isolated set of A. Define a weighting v on the edges of G as follows:

$$v(e) = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{otherwise} \end{cases}$$

This weighting of the edges of G satisfies the constraints of (2) since, for any biclique B of G, $\sum_{e \in B} v(e) \leq 1$. Thus, $bc^*(G) \geq \sum_{e \in E(G)} v(e) = |M| = i(A)$. □

3 Fractional Biclique Partitions

A biclique partition is a function w that assigns each biclique B of G either 0 or 1 so that, for each edge $e \in E(G)$, $\sum w(B) = 1$ where the sum is taken over all bicliques that contain e. Then, $bp(G)$ is the minimum of $\sum_{B\in\mathcal{B}(G)} w(B)$ over all biclique partitions. Thus, $bp(G)$ is the value of a $(0,1)$-integer program and its linear relaxation defines the fractional biclique partition number.

A *fractional biclique partition* is a function w that assigns to each biclique B of a graph G a number so that $w(B) \geq 0$ and, for each edge $e \in E(G)$, $\sum w(B) = 1$ where the sum is taken over all bicliques that contain e. The *fractional biclique partition number*, $bp^*(G)$, is the infimum of $\sum_{B\in\mathcal{B}(G)} w(B)$ over all fractional biclique partitions. That is, $bp^*(G)$ is the value of the linear program

$$\begin{array}{ll} \text{minimize} & \sum_{B\in\mathcal{B}(G)} w(B) \\ \text{subject to} & \sum_{\{B:e\in B\}} w(B) = 1 \text{ for each edge } e \text{ of } G \\ & w(B) \geq 0 \text{ for each biclique } B \text{ of } G \end{array} \tag{3}$$

where each biclique B of G is assigned a weight $w(B)$. As before, the dual program may also be used. Thus, $bp^*(G)$ is the value of the linear program

$$\begin{array}{ll} \text{maximize} & \sum_{e\in E(G)} v(e) \\ \text{subject to} & \sum_{e\in B} v(e) \leq 1 \text{ for each biclique } B \text{ of } G \end{array} \tag{4}$$

where each edge e of G is assigned a weight $v(e)$. Note that in (4) edges may receive negative weights, whereas in (2) non-negative weights are required. Similar to the case with $bc^*(G)$, edges in the same orbit of $\operatorname{Aut} G$ may receive the same weight when (4) is used to determine $bp^*(G)$.

In the examples immediately before Theorem 2.1, each optimal weighting of the edges for (2) is also an optimal weighting of the edges for (4). Consequently, $bc^*(C_n) = bp^*(C_n)$, $bc^*(K_n) = bp^*(K_n)$ and $bc^*(\overline{C}_6) = bp^*(\overline{C}_6)$. As with the fractional biclique cover number, the fractional biclique partition number is a lower bound on the biclique partition number. The proof of Theorem 3.1 is similar to the proof of Theorem 2.1 and is omitted. Alternatively, the proof may be found in Watts [19].

Theorem 3.1 *For a graph G, $bp^*(G) \leq bp(G)$ with equality holding if and only if for some (and every) minimum biclique partition $\mathcal{B}$ and optimal weighting v of the edges of G for (4), $\sum_{e\in B} v(e) = 1$ for all $B \in \mathcal{B}$ and $\sum_{e\in B} v(e) \leq 1$ for all bicliques B of G.*

A well-known lower bound on $bp(G)$ is the eigenvalue bound, attributed to H.S. Witsenhausen by Graham and Pollak [9], which states that $bp(G) \geq \max\{n_+(G), n_-(G)\}$, where $n_+(G)$ (resp. $n_-(G)$) is the number of positive (resp. negative) eigenvalues of the adjacency matrix of G. Although Theorem 3.1 provides a lower bound for $bp(G)$, in general $bp^*(G)$ is not as good a lower bound on $bp(G)$ as $\max\{n_+(G), n_-(G)\}$. Theorem 3.2 below shows that $bp^*(G) \leq \frac{n}{2}$, the proof of which first appeared in the thesis of the author [19]. Consequently, the only instances when $bp^*(G)$ may be of interest as a lower bound on $bp(G)$ is when $\max\{n_+(G), n_-(G)\} \leq \frac{n}{2}$. An example of this is $\overline{C}_6$ since it was observed above that $bp^*(\overline{C}_6) = bp(\overline{C}_6) = 3$ but $\max\{n_+(\overline{C}_6), n_-(\overline{C}_6)\} = 2$. Other graphs which have $\max\{n_+(G), n_-(G)\} \leq \frac{n}{2}$ include all the bipartite graphs.

Theorem 3.2 *For a graph G with n vertices, $bp^*(G) \leq \frac{n}{2}$.*

Proof. Let v be an optimal weighting of the edges of G for (4). For each vertex x in G, let S_x be the star centered at x, containing all the edges incident to x. Then $bp^*(G) = \sum_{e \in E(G)} v(e) = \frac{1}{2} \sum_{x \in V(G)} \left(\sum_{e \in S_x} v(e) \right) \leq \frac{1}{2} \sum_{x \in V(G)} 1 = \frac{n}{2}$. □

As with the fractional biclique cover number, the bipartite case is of special interest. Not only are bipartite graphs of interest because of the eigenvalue bound mentioned above, but also because the fractional biclique partition number gives a lower bound on the non-negative integer rank of a binary matrix . The *non-negative integer rank*, $r_{\mathbb{Z}^+}(A)$, of an $m \times n$ matrix A with non-negative integer entries is the smallest integer k such that $A = XY^T$ for some $m \times k$ matrix X and $n \times k$ matrix Y, both with non-negative integer entries. If A is an $m \times n$ $(0, 1)$-matrix and G is the bipartite graph with bipartite adjacency matrix A, then $bp(G) = r_{\mathbb{Z}^+}(A)$, an observation provided by Orlin [16]. Consequently, $bp^*(G)$ is the fractional analogue of $r_{\mathbb{Z}^+}(A)$ and $r_{\mathbb{Z}^+}(A) \geq bp^*(G)$ when G is bipartite. For more on non-negative integer rank, see [5, 8, 15].

It is always interesting to see if known integer results have corresponding fractional analogues. Since every biclique partition is a biclique cover, it follows that $bc(G) \leq bp(G)$. The fractional analogue of this statement is given in Theorem 3.3, together with conditions for equality. The proof of Theorem 3.3 first appeared in Watts [19].

Theorem 3.3 *For a graph G, $bc^*(G) \leq bp^*(G)$ with equality holding if and only if for some optimal weighting v of the edges of G for (4), $v(e) \geq 0$ for all edges e of G.*

Proof. It follows immediately from (2) and (4) that $bc^*(G) \leq bp^*(G)$ since a weighting v of the edges of G that satisfies the constraints of (2) also satisfies the constraints of (4). To prove the characterization for equality, suppose v is an optimal weighting of the edges of G for (4) with $v(e) \geq 0$ for all edges e of G. Then v satisfies the constraints of (2) and so $bc^*(G) \geq \sum_{e \in E(G)} v(e) = bp^*(G)$. It follows that $bc^*(G) = bp^*(G)$.

Conversely, suppose that $bc^*(G) = bp^*(G)$ and v is an optimal weighting of the edges of G for (2). Then v satisfies the constraints of (4) and $bc^*(G) = \sum_{e \in E(G)} v(e) = bp^*(G)$. Thus v must be an optimal weighting of the edges of G for (4) with $v(e) \geq 0$ for each edge e of G. □

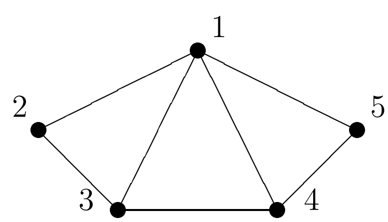

Figure 3

Theorem 3.3 implies that to have a graph with $bc^*(G) \neq bp^*(G)$, an optimal weighting of the edges of G for (4) must have some edges that receive negative weights. The graph

G shown in Figure 3 above is an example of a graph with $bc^*(G) \neq bp^*(G)$. Under the automorphism group, the orbits of $\operatorname{Aut} G$ yield the following equivalence classes: $X_1 = \{12, 15\}$, $X_2 = \{13, 14\}$, $X_3 = \{23, 45\}$, $X_4 = \{34\}$. Assign each edge in X_i weight x_i for $1 \leq i \leq 4$. Then, according to (4), $bp^*(G)$ is the value of the linear program

$$\begin{array}{ll} \text{maximize} & 2x_1 + 2x_2 + 2x_3 + x_4 \\ \text{subject to} & 2x_1 + 2x_2 \leq 1 \\ & 2x_1 + x_2 \leq 1 \\ & x_1 + x_2 \leq 1 \\ & 2x_1 \leq 1 \\ & 2x_2 \leq 1 \\ & x_1 + 2x_2 \leq 1 \\ & x_1 + x_3 \leq 1 \\ & x_2 + x_3 + x_4 \leq 1 \\ & x_2 + x_3 \leq 1 \\ & x_3 + x_4 \leq 1 \\ & x_2 + x_4 \leq 1 \\ & x_1 + x_2 + x_3 + x_4 \leq 1 \\ & x_1, x_2, x_3, x_4 \leq 1 \end{array} \tag{5}$$

Solving the linear program (5) yields $bp^*(G) = \frac{9}{4}$ with $x_1 = x_2 = \frac{1}{4}$, $x_3 = \frac{3}{4}$ and $x_4 = -\frac{1}{4}$. Adding the additional constraints $x_1, x_2, x_3, x_4 \geq 0$ to (5) gives the linear program for $bc^*(G)$ in (2). Solving this new linear program gives $bc^*(G) = 2$ with $x_1 = x_3 = \frac{1}{2}$ and $x_2 = x_4 = 0$.

4 Fractional Vertex Covers

The *vertex cover number*, $\beta(G)$, of a graph G is the minimum number of vertices in a vertex cover of G. As described in Berge [1], a *fractional vertex cover* is a function g that assigns to each vertex v of G a number so that $0 \leq g(v) \leq 1$ and for each edge $e \in E(G)$, $\sum g(v) \geq 1$ where the sum is taken over all vertices incident to e. The *fractional vertex cover number*, $\beta^*(G)$, is the infimum of $\sum_{v \in V(G)} g(v)$ over all fractional vertex covers. That is, $\beta^*(G)$ is the value of the linear program

$$\begin{array}{ll} \text{minimize} & \sum_{v \in V(G)} g(v) \\ \text{subject to} & \sum_{e \in v} g(v) \geq 1 \text{ for each edge } e \text{ of } G \\ & 0 \leq g(v) \leq 1 \text{ for each vertex } v \text{ of } G \end{array} \tag{6}$$

where each vertex v of G is assigned a weight $g(v)$. Dually, $\beta^*(G)$ is the value of the linear program

$$\begin{array}{ll} \text{maximize} & \sum_{e \in E(G)} f(e) \\ \text{subject to} & \sum_{\{e : v \in e\}} f(e) \leq 1 \text{ for each vertex } v \text{ of } G \\ & 0 \leq f(e) \leq 1 \text{ for each edge } e \text{ of } G \end{array} \tag{7}$$

where each edge e of G is assigned a weight $f(e)$. In fact, the dual program given in (7) is the linear program for the fractional matching number of G. For more on fractional vertex covers and fractional matchings, see Berge [1] or Scheinerman and Ullman [17].

A biclique partition may always be obtained by successively deleting the edge sets of stars centered at the vertices in a vertex cover of G. This gives a biclique partition of G consisting entirely of stars and consequently $bc(G) \leq bp(G) \leq \beta(G)$. It follows immediately from (2) and (7) that $bc^*(G) \leq \beta^*(G)$. In fact, Theorem 4.1 shows that $bc^*(G) \leq bp^*(G) \leq \beta^*(G)$.

Theorem 4.1 *For any graph G, $bc^*(G) \leq bp^*(G) \leq \beta^*(G)$.*

Proof. It was observed in Theorem 3.3 that $bc^*(G) \leq bp^*(G)$ and so it remains to prove $bp^*(G) \leq \beta^*(G)$. Let v be an optimal weighting of the edges of G for (4). Construct a new weighting f for the edges of G as follows:

$$f(e) = \begin{cases} v(e) & \text{if } v(e) \geq 0 \\ 0 & \text{if } v(e) < 0 \end{cases}$$

Note that for any vertex $x \in V(G)$, the edges incident to x with $v(e) \geq 0$ form a star centered at x. Since v satisfies the constraints of (4), $\sum_{v(e)\geq 0} v(e) \leq 1$. Thus, for each $x \in V(G)$, $\sum_{\{e: x\in e\}} f(e) = \sum_{v(e)\geq 0} v(e) \leq 1$. Hence, f satisfies the constraints of (7) and $bp^*(G) = \sum_{e\in E(G)} v(e) \leq \sum_{v(e)\geq 0} v(e) = \sum_{e\in E(G)} f(e) \leq \beta^*(G)$. □

5 Weak Bipartite Products and Weak Products

Let (X_G, Y_G) be the *(ordered) bipartition* of a bipartite graph G. The *weak bipartite product* of bipartite graphs G and H is the bipartite graph $G\tilde{\times}H$ with ordered bipartition $(X_G \times X_H, Y_G \times Y_H)$. Two vertices (g_i, h_k) and (g_j, h_ℓ) in $G\tilde{\times}H$ are adjacent if and only if g_i is adjacent to g_j in G and h_k is adjacent to h_ℓ in H. In fact, $G\tilde{\times}H$ is one of the components of the weak product $G \times H$ described below. Note that each edge g_ig_j of G and each edge h_kh_ℓ of H yield only one edge of $G\tilde{\times}H$. Further, every edge of $G\tilde{\times}H$ is the result of a unique pair of edges of G and H.

For a bipartite graph G with ordered bipartition (X_G, Y_G) each biclique of G may be written as $K(R, S)$ with $R \subseteq X_G$ and $S \subseteq Y_G$. Then, the biclique $K(R, S)$ is called an *ordered biclique* of G with ordered bipartition (R, S). All of the bicliques of G may assumed to be ordered. Hence, the bicliques of G, H and $G\tilde{\times}H$ may all be assumed to be ordered bicliques. Note that the weak bipartite product of an ordered biclique $K(R_G, S_G)$ of G and an ordered biclique $K(R_H, S_H)$ of H yields an ordered biclique $K(R_G, S_G)\tilde{\times}K(R_H, S_H) = K(R_G \times R_H, S_G \times S_H)$ of $G\tilde{\times}H$ with ordered bipartition $(R_G \times R_H, S_G \times S_H)$. A biclique of $G\tilde{\times}H$ which can be written as the weak bipartite product of an ordered biclique of G and an ordered biclique of H is called an *ordered product biclique* of $G\tilde{\times}H$. Every ordered product biclique of $G\tilde{\times}H$ can be expressed uniquely as the weak bipartite product of an ordered biclique from $\mathcal{B}(G)$ and an ordered biclique from $\mathcal{B}(H)$. Let $\mathcal{B}(G)\tilde{\times}\mathcal{B}(H)$ denote the set of ordered product bicliques of $G\tilde{\times}H$. Then $\mathcal{B}(G)\tilde{\times}\mathcal{B}(H)$ is a subset of all the ordered bicliques of $G\tilde{\times}H$ and $|\mathcal{B}(G)\tilde{\times}\mathcal{B}(H)| = |\mathcal{B}(G)||\mathcal{B}(H)|$.

The *weak product* of two graphs G and H, not necessarily bipartite, is the graph $G \times H$ with vertex set $V(G) \times V(H)$. Two vertices (g_i, h_k) and (g_j, h_ℓ) are adjacent in $G \times H$ if and only if g_i is adjacent to g_j in G and h_k is adjacent to h_ℓ in H. Note that each edge g_ig_j of G and each edge h_kh_ℓ of H yields two edges of $G \times H$: the edges $(g_i, h_k)(g_j, h_\ell)$ and $(g_i, h_\ell)(g_j, h_k)$. Further, every edge of $G \times H$ belongs to a unique pair of this type.

The weak product, $B_G \times B_H$, of a biclique $B_G = K(R_G, S_G)$ of G and a biclique $B_H = K(R_H, S_H)$ of H yields two disjoint bicliques of $G \times H$: one is $B_G\tilde{\times}B_H = K(R_G \times R_H, S_G \times S_H)$ defined above and the other is $K(R_G \times S_H, S_G \times R_H)$. A biclique of $G \times H$ which is one of the two bicliques produced from the weak product of a biclique of G and a biclique of H is called a *product biclique* of $G \times H$. Each product biclique of $G \times H$ belongs to a unique pair of bicliques $B_G \times B_H$. Let $\mathcal{B}(G) \times \mathcal{B}(H)$ denote the set of all product bicliques of $G \times H$. Then $\mathcal{B}(G) \times \mathcal{B}(H)$ is a subset of all the bicliques of $G \times H$ and $|\mathcal{B}(G) \times \mathcal{B}(H)| = 2|\mathcal{B}(G)||\mathcal{B}(H)|$.

Let A_G and A_H be the bipartite adjacency matrices of bipartite graphs G and H respectively. Then $A_G \otimes A_H$ is the bipartite adjacency matrix of $G\tilde{\times}H$, where $\otimes$ denotes the Kronecker product of matrices A_G and A_H. A result of de Caen, Gregory and Pullman [7] showed that $r_{\mathbb{Z}^+}(A_G \otimes A_H) \leq r_{\mathbb{Z}^+}(A_G)r_{\mathbb{Z}^+}(A_H)$. It follows that $bp(G\tilde{\times}H) \leq bp(G)bp(H)$. Also, Kratzke, Reznick and West [14] observed that $bp(G \times H) \leq 2bp(G)bp(H)$. Theorem 5.1 gives the corresponding fractional analogues, the proof of which first appeared in the thesis of the author [19]. Note that Theorem 5.1 gives an inequality for the fractional analogue of the non-negative integer rank of the Kronecker product of binary matrices.

Theorem 5.1 *1. Let G_i, $1 \leq i \leq k$, be bipartite graphs with ordered bipartitions (X_{G_i}, Y_{G_i}) respectively. Then $bp^*(G_1\tilde{\times} \cdots \tilde{\times}G_k) \leq \prod_{i=1}^k bp^*(G_i)$.*

2. Let G_i, $1 \leq i \leq k$, be graphs. Then $bp^*(G_1 \times \cdots \times G_k) \leq 2^{k-1} \prod_{i=1}^k bp^*(G_i)$.

Proof. The proof of 1 follows below. The proof of 2 is similar and is omitted. It suffices to prove the result for $k = 2$; the general case follows directly by induction on k. Let $G = G_1$ and $H = G_2$. Let w_G and w_H be optimal weightings of the ordered bicliques of G and H, respectively, for (3). Construct a weighting w of the ordered bicliques of $G\tilde{\times}H$ from the ordered bicliques of G and H. If B is an ordered product biclique of $G\tilde{\times}H$ and $B = B_G\tilde{\times}B_H$ where $B_G \in \mathcal{B}(G)$ and $B_H \in \mathcal{B}(H)$, let $w(B) = w_G(B_G)w_H(B_H)$. For all other ordered bicliques B of $G\tilde{\times}H$, let $w(B) = 0$.

Note that by construction $w(B) \geq 0$ for each ordered biclique B of $G\tilde{\times}H$. Let $e = (g_i, h_k)(g_j, h_\ell)$ be an edge of $G\tilde{\times}H$ with $(g_i, h_k) \in X_G \times X_H$ and $(g_j, h_\ell) \in Y_G \times Y_H$. Then g_ig_j and h_kh_ℓ are edges of G and H respectively. Note that e is an edge of an ordered product biclique $B = B_G\tilde{\times}B_H$ if and only if g_ig_j and h_kh_ℓ are edges of the ordered bicliques B_G of G and B_H of H respectively. Then, remembering that at each stage the bicliques are ordered accordingly, for each such edge e,

$$
\begin{aligned}
\sum_{\{B:e\in B\}} w(B) &= \sum_{\substack{\{B:e\in B\}\\ B\in\mathcal{B}(G)\tilde{\times}\mathcal{B}(H)}} w(B)\\
&= \sum_{\substack{\{B_G\tilde{\times}B_H: e\in B_G\tilde{\times}B_H\}\\ B_G\in\mathcal{B}(G), B_H\in\mathcal{B}(H)}} w_G(B_G)w_H(B_H)\\
&= \sum_{\substack{\{B_G:g_ig_j\in B_G\}\\ \{B_H:h_kh_\ell\in B_H\}}} w_G(B_G)w_H(B_H)\\
&= \Bigg(\sum_{\{B_G:g_ig_j\in B_G\}} w_G(B_G)\Bigg)\Bigg(\sum_{\{B_H:h_kh_\ell\in B_H\}} w_H(B_H)\Bigg)\\
&= 1.
\end{aligned}
$$

Thus, w satisfies the constraints of (3) and it follows that

$$
\begin{aligned}
bp^*(G\tilde{\times}H) &\le \sum_{B\in\mathcal{B}(G)\tilde{\times}\mathcal{B}(H)} w(B)\\
&= \sum_{\substack{B=B_G\tilde{\times}B_H\\ B\in\mathcal{B}(G)\tilde{\times}\mathcal{B}(H)}} w_G(B_G)w_H(B_H)\\
&= \sum_{\substack{B_G\in\mathcal{B}(G)\\ B_H\in\mathcal{B}(H)}} w_G(B_G)w_H(B_H)\\
&= \Bigg(\sum_{B_G\in\mathcal{B}(G)} w_G(B_G)\Bigg)\Bigg(\sum_{B_H\in\mathcal{B}(H)} w_H(B_H)\Bigg)\\
&= bp^*(G)bp^*(H). \qquad \square
\end{aligned}
$$

A result of de Caen, Gregory and Pullman [7] showed $r_B(A_G \otimes A_H) \le r_B(A_G)r_B(A_H)$ where A_G and A_H are the bipartite adjacency matrices of bipartite graphs G and H respectively. Consequently, $bc(G\tilde{\times}H) \le bc(G)bc(H)$. Watts [18] observed that this inequality can be strict. However, equality always holds for the fractional analogue, given in Theorem 5.3. That is, $bc^*(G\tilde{\times}H) = bc^*(G)bc^*(H)$. The same can be said for the weak product. Watts [19] observed that $bc(G \times H) \le 2bc(G)bc(H)$ and that equality need not hold, while equality holds in the corresponding fractional analogue, also given in Theorem 5.3. The proofs of Lemma 5.2 and Theorem 5.3 first appeared in [19].

Lemma 5.2 *1. Let G and H be bipartite graphs with ordered bipartitions (X_G, Y_G) and (X_H, Y_H) respectively. Each ordered biclique of $G\tilde{\times}H$ is a subgraph of an ordered product biclique of $G\tilde{\times}H$.*

2. Let G and H be graphs. Each biclique of $G \times H$ is a subgraph of a product biclique of $G \times H$.

Proof. The proof of 1 follows below. The proof of 2 is similar and is omitted. Let $K(R,S)$ be a biclique of $G\tilde{\times}H$ with ordered bipartition $R \subseteq X_G \times X_H$ and $S \subseteq Y_G \times Y_H$. Let

$$\begin{aligned} R_G &= \{g_i \in X_G : (g_i, h_k) \in R \text{ for some } h_k \in X_H\} \\ R_H &= \{h_k \in X_H : (g_i, h_k) \in R \text{ for some } g_i \in X_G\} \\ S_G &= \{g_j \in Y_G : (g_j, h_\ell) \in S \text{ for some } h_\ell \in Y_H\} \\ S_H &= \{h_\ell \in Y_H : (g_j, h_\ell) \in S \text{ for some } g_j \in Y_G\}. \end{aligned}$$

Then $R \subseteq R_G \times R_H \subseteq X_G \times X_H$ and $S \subseteq S_G \times S_H \subseteq Y_G \times Y_H$. Each vertex of R_G is adjacent to each vertex of S_G, so $K(R_G, S_G)$ is an ordered biclique of G. Similarly, $K(R_H, S_H)$ is an ordered biclique of H. Consequently, $K(R,S)$ is a subgraph of the ordered product biclique $K(R_G, S_G)\tilde{\times}K(R_H, S_H) = K(R_G \times R_H, S_G \times S_H)$. □

Theorem 5.3 *1. Let G_i, $1 \leq i \leq k$, be bipartite graphs with ordered bipartitions (X_{G_i}, Y_{G_i}) respectively. Then $bc^*(G_1\tilde{\times} \cdots \tilde{\times}G_k) = \prod_{i=1}^k bc^*(G_i)$.*

2. Let G_i, $1 \leq i \leq k$, be graphs. Then $bc^*(G_1 \times \cdots \times G_k) = 2^{k-1} \prod_{i=1}^k bc^*(G_i)$.

Proof. The proof of 1 follows below. The proof of 2 is similar and is omitted. It suffices to prove the theorem for $k = 2$; the general case follows directly by induction on k. Let $G = G_1$ and $H = G_2$. The proof that $bc^*(G\tilde{\times}H) \leq bc^*(G)bc^*(H)$ is similar to the proof of Theorem 5.1 and is omitted.

To show the reverse inequality, let v_G and v_H be optimal weightings of the edges of G and H, respectively, for (2). For each edge $e = (g_i, h_k)(g_j, h_\ell)$ of $G\tilde{\times}H$ with $(g_i, h_k) \in X_G \times X_H$ and $(g_j, h_\ell) \in Y_G \times Y_H$, construct a weighting v of the edges of $G\tilde{\times}H$ with $v(e) = v_G(g_i g_j)v_H(h_k h_\ell)$.

Note that by construction $v(e) \geq 0$ for each edge e of $G\tilde{\times}H$. Let $B = K(R,S)$ be an ordered biclique of $G\tilde{\times}H$ with $R \subseteq X_G \times X_H$ and $S \subseteq Y_G \times Y_H$. By Lemma 5.2 B is a subgraph of an ordered product biclique of $G\tilde{\times}H$. That is, B is a subgraph of an ordered product biclique $B_G\tilde{\times}B_H$ for some $B_G \in \mathcal{B}(G)$ and $B_H \in \mathcal{B}(H)$. Note that $e = (g_i, h_k)(g_j, h_\ell)$ is an edge of $B_G\tilde{\times}B_H$ if and only if $g_i g_j$ and $h_k h_\ell$ are edges of B_G and B_H, respectively. Then for each such ordered biclique B,

$$\begin{aligned} \sum_{e \in B} v(e) &\leq \sum_{e \in B_G\tilde{\times}B_H} v(e) \\ &= \sum_{(g_i,h_k)(g_j,h_\ell) \in B_G\tilde{\times}B_H} v_G(g_i g_j)v_H(h_k h_\ell) \\ &= \sum_{\substack{g_i g_j \in B_G \\ h_k h_\ell \in B_H}} v_G(g_i g_j)v_G(h_k h_\ell) \\ &= \left(\sum_{e \in B_G} v_G(e)\right)\left(\sum_{e \in B_H} v_H(e)\right) \\ &\leq 1. \end{aligned}$$

Therefore, v satisfies the constraints of (2) and it follows that

$$\begin{aligned} bc^*(G\tilde{\times}H) &\geq \sum_{e\in E(G\tilde{\times}H)} v(e) \\ &= \sum_{(g_i,h_k)(g_j,h_\ell)\in E(G\tilde{\times}H)} v_G(g_ig_j)v_H(h_kh_\ell) \\ &= \sum_{\substack{g_ig_j\in E(G)\\ h_kh_\ell\in E(H)}} v_G(g_ig_j)v_H(h_kh_\ell) \\ &= \Big(\sum_{e\in E(G)} v_G(e)\Big)\Big(\sum_{e\in E(H)} v_H(e)\Big) \\ &= bc^*(G)bc^*(H). \end{aligned}$$

Consequently, $bc^*(G\tilde{\times}H) = bc^*(G)bc^*(H)$. □

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