A Bound for Size Ramsey Numbers of Multi-partite Graphs

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Abstract

It is shown that the (diagonal) size Ramsey numbers of complete m-partite graphs $K_m(n)$ can be bounded from below by $cn^2 2^{(m-1)n}$, where c is a positive constant.

Key words: Size Ramsey number, Complete multi-partite graph

1 Introduction

Let G, G_1 and G_2 be simple graphs with at least two vertices, and let

$$G \to (G_1, G_2)$$

signify that in any edge-coloring of edge set $E(G)$ of G in red and blue, there is either a monochromatic red G_1 or a monochromatic blue G_2. With this notation, the *Ramsey number* $r(G_1, G_2)$ can be defined as

$$\begin{aligned} r(G_1, G_2) &= \min\{N : K_N \to (G_1, G_2)\} \\ &= \min\{|V(G)| : G \to (G_1, G_2)\}. \end{aligned}$$

As the number of edges of a graph is often called the size of the graph, Erdős, Faudree, Rousseau and Schelp [2] introduced an idea of measuring minimality with respect to size rather than order of the graphs G with $G \to (G_1, G_2)$. Let $e(G)$ be the number of edges of G. Then the *size Ramsey number* $\hat{r}(G_1, G_2)$ is defined as

$$\hat{r}(G_1, G_2) = min\{e(G) : G \to (G_1, G_2)\}.$$

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As usual, we write $\hat{r}(G,G)$ as $\hat{r}(G)$. Erdős and Rousseau in [3] showed

$$\hat{r}(K_{n,n}) > \frac{1}{60}n^2 2^n. \tag{1}$$

Gorgol [4] gave

$$\hat{r}(K_m(n)) > cn^2 2^{mn/2}, \tag{2}$$

where and henceforth $K_m(n)$ is a complete m-partite graph with n vertices in each part, and $c > 0$ is a constant. Bielak [1] gave

$$\hat{r}(K_{n,n,n}) > c_n\, n^2\, 2^{2n}, \tag{3}$$

where $c_n \to \frac{3^{1/3}}{4e^{8/3}}$ as $n \to \infty$. We shall generalize (1) and (3) by improving (2) as

$$\hat{r}(K_m(n)) > cn^2 2^{(m-1)n},$$

where $c = c_m > 0$ that has a positive limit as $n \to \infty$.

2 Main results

We need an upper bound for the number of subgraphs isomorphic to $K_m(n)$ in a graph of given size. The following counting lemma generalizes a result of Erdős and Rousseau [3] and we made a minor improvement for the case $m = 2$.

Lemma 1 *Let $n \geq 2$ be an integer. A graph with q edges contains at most $A(m,n,q)$ copies of complete m-partite graph $K_m(n)$, where*

$$A(m,n,q) = \frac{2eq}{(m-1)m!n}\left(\frac{2e^2 q}{n^2}\right)^{mn/2}\left(\frac{2m-2}{m}\right)^{(m-2)n/2}.$$

Proof. Let F denote $K_m(n)$ and let G be a graph of q edges on vertex set V. Set

$$s = \left\lceil \frac{e(F)}{2}\log\frac{2q}{e(F)} \right\rceil,$$

where $\log x$ is the natural logarithmic function. Set $d_{s+1} = \infty$ and

$$d_k = (m-1)ne^{k/e(F)}, \quad k = 0, 1, 2, \cdots, s,$$

and

$$X_k = \{x \in V : d_k \leq \deg(x) < d_{k+1}\}.$$

Then $X_0, X_1, \ldots, X_s$ form a partition of the set $W_0 = \{x \in V : \deg(x) \geq (m-1)n\}$. Let

$$W_k = \cup_{j=k}^{s} X_j = \{x \in V : \deg(x) \geq d_k\}.$$

Let us say that a subgraph F in G is of *type* k if k is the smallest index such that $X_k \cap V(F) \neq \phi$. Then

- every vertex of $V(F)$ belongs to W_k;
- at least one vertex of $V(F)$ belongs to X_k.

Let M_k be the number of type k copies of F in G. Then $M = \sum_{k=0}^{s} M_k$ is the total number of copies of F. Notice that in a type k copy of F at least one vertex, say x, belongs to X_k and every vertex belongs to W_k. Thus all $F-$neighbors of x belong to an $(m-1)n$-element subset Y of the $G-$neighborhood of x in W_k. Moreover all other $(n-1)$ vertices of F belong to an $(n-1)$-element subset of $W_k - Y - \{x\}$. Since the neighborhood of x in F is a complete $(m-1)$-partite graph, say H, then we get at most

$$t(m,n) \quad = \quad \frac{1}{(m-1)!}\binom{(m-1)n}{n}\binom{(m-2)n}{n}\cdots\binom{2n}{n}\binom{n}{n}$$

subgraphs isomorphic to H in the graph induced by the set Y. Furthermore, the m parts in $K_m(n)$ can be interchanged arbitrarily. Note that a vertex $x \in X_k$ has degree at most d_{k+1}, so

$$M_k \le |X_k| \frac{t(m,n)}{m} \binom{\lfloor d_{k+1} \rfloor}{(m-1)n} \binom{|W_k|}{n}.$$

The elementary formulas

$$\binom{D}{t}\binom{t}{n} = \binom{D}{n}\binom{D-n}{t-n}$$

and

$$\binom{D}{n} \le \frac{D^n}{n!} < \left(\frac{eD}{n}\right)^n$$

give

$$\begin{aligned}
& \binom{\lfloor d_{k+1} \rfloor}{(m-1)n}\binom{(m-1)n}{n}\binom{(m-2)n}{n}\cdots\binom{2n}{n} \\
\le \quad & \binom{\lfloor d_{k+1} \rfloor}{n}\binom{\lfloor d_{k+1} \rfloor - (m-1)n}{(m-2)n}\binom{(m-2)n}{n}\cdots\binom{2n}{n} \\
\le \quad & \binom{\lfloor d_{k+1} \rfloor}{n}\binom{\lfloor d_{k+1} \rfloor}{(m-2)n}\binom{(m-2)n}{n}\cdots\binom{2n}{n} \\
\le \quad & \binom{\lfloor d_{k+1} \rfloor}{n}^{m-1} \le \left(\frac{ed_{k+1}}{n}\right)^{(m-1)n}.
\end{aligned}$$

It implies that for $k = 0, 1, 2, \dots, s-1$,

$$\begin{aligned}
M_k \quad & \le \quad \frac{|X_k|}{m!}\left(\left(\frac{ed_{k+1}}{n}\right)^{m-1}\frac{e|W_k|}{n}\right)^n \\
& \le \quad \frac{|X_k|}{m!}\left(\frac{e^m}{n^2}\left(\frac{d_{k+1}}{n}\right)^{m-2} d_{k+1}|W_k|\right)^n.
\end{aligned}$$

From the definition of W_k, we have $d_k|W_k| \le 2q$. Hence

$$d_{k+1}|W_k| = d_k|W_k|e^{1/e(F)} \le 2qe^{1/e(F)},$$

and $d_{k+1}/n = (m-1)e^{(k+1)/e(F)}$, so

$$M_k \le \frac{|X_k|}{m!}\left(\frac{2qe^m(m-1)^{m-2}}{n^2}\exp\left(\frac{(m-2)k+m-1}{e(F)}\right)\right)^n.$$

As $k \le s-1 \le \frac{e(F)}{2}\log\frac{2q}{e(F)}$ and $e(F) = m(m-1)n^2/2$,

$$\exp\left(\frac{(m-2)k+m-1}{e(F)}\right) \le e^{2/(mn^2)}\left(\frac{4q}{m(m-1)n^2}\right)^{(m-2)/2},$$

and hence

$$M_k \le \frac{e|X_k|}{m!}\cdot\left(\frac{2e^2q}{n^2}\right)^{mn/2}\left(\frac{2m-2}{m}\right)^{(m-2)n/2}.$$

Since $d_s \ge 2e^{-1/e(F)}\sqrt{(m-1)q/m}$, so $|X_s| = |W_s| \le e^{1/e(F)}\sqrt{mq/(m-1)}$, and if the subgraph F is of type s, then each vertex of $V(F)$ must belong to X_s. Thus we have

$$\begin{aligned} M_s &\le \frac{t(m,n)}{m}\binom{|X_s|}{(m-1)n}\binom{|X_s|}{n} \\ &< \frac{1}{m!}\binom{|X_s|}{n}^m \le \frac{e}{m!}\left(\frac{2e^2q}{n^2}\right)^{mn/2}\left(\frac{m}{2m-2}\right)^{mn/2}. \end{aligned}$$

If $|X_s| = 0$ then $|M_s| = 0$; thus we can write

$$M_s \le \frac{e|X_s|}{m!}\left(\frac{2e^2q}{n^2}\right)^{mn/2}\left(\frac{m}{2m-2}\right)^{mn/2}.$$

Hence for all $k = 0, 1, \ldots, s$, we have

$$M_k \le \frac{e|X_k|}{m!}\left(\frac{2e^2q}{n^2}\right)^{mn/2}\left(\frac{2m-2}{m}\right)^{(m-2)n/2}.$$

Finally, we obtain

$$\begin{aligned} M &= \sum_{k=0}^{s} M_k \le |W_0|\cdot\frac{e}{m!}\left(\frac{2e^2q}{n^2}\right)^{mn/2}\left(\frac{2m-2}{m}\right)^{(m-2)n/2} \\ &\le \frac{2eq}{n(m-1)m!}\left(\frac{2e^2q}{n^2}\right)^{mn/2}\left(\frac{2m-2}{m}\right)^{(m-2)n/2}. \end{aligned}$$

The assertion follows.

Theorem 1 *Let $m \geq 2$ be fixed and $n \to \infty$, then*

$$\hat{r}(K_m(n)) > (c - o(1))n^2 2^{(m-1)n},$$

where $c = \frac{m}{16e^2(m-1)} \left(\frac{4m-4}{m}\right)^{2/m}$.

Proof. We shall prove that

$$\hat{r}(K_m(n)) > c(m,n)n^2 2^{(m-1)n},$$

where

$$c(m,n) = \frac{m}{16e^2(m-1)} \left(\frac{4m-4}{m}\right)^{2/m} \left(\frac{(m-1)m!}{4en}\right)^{2/(mn)}.$$

Let G be arbitrary graph with q edges, where $q \leq c(m,n)\, n^2 2^{(m-1)n}$. Let us consider a random red-blue edge-coloring of G, in which each edge is red with probability $1/2$ and the edges are colored independently. Then the probability P that such a random coloring yields a monochromatic copy of $K_m(n)$ satisfies

$$\begin{aligned} P &\leq \frac{4eq}{n(m-1)m!} \left(\frac{2e^2 q}{n^2}\right)^{mn/2} \left(\frac{2m-2}{m}\right)^{(m-2)n/2} \left(\frac{1}{2}\right)^{m(m-1)n^2/2} \\ &< \frac{4en^2 2^{(m-1)n}}{n(m-1)m!} (2e^2)^{mn/2} \left(\frac{2m-2}{m}\right)^{(m-2)n/2} c^{mn/2} = 1. \end{aligned}$$

Thus $G \not\to (K_m(n), K_m(n))$, and the desired lower bound follows from the fact that $c(m,n) \to c$ as $n \to \infty$.

References

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