Nonexistence of permutation binomials of certain shapes

Ariane M. Masuda*

Department of Mathematics and Statistics
University of Ottawa, Ottawa, ON K1N 6N5, Canada
amasuda@uottawa.ca

Michael E. Zieve*

Center for Communications Research
805 Bunn Drive, Princeton, NJ 08540
zieve@math.rutgers.edu

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Abstract

Suppose $x^m + ax^n$ is a permutation polynomial over $\mathbb{F}_p$, where $p > 5$ is prime and $m > n > 0$ and $a \in \mathbb{F}_p^*$. We prove that $\gcd(m-n, p-1) \notin \{2, 4\}$. In the special case that either $(p-1)/2$ or $(p-1)/4$ is prime, this was conjectured in a recent paper by Masuda, Panario and Wang.

1 Introduction

A polynomial over a finite field is called a *permutation polynomial* if it permutes the elements of the field. These polynomials have been studied intensively in the past two centuries. Permutation monomials are completely understood: for $m > 0$, x^m permutes $\mathbb{F}_q$ if and only if $\gcd(m, q-1) = 1$. However, even though dozens of papers have been written about them, permutation binomials remain mysterious. In this note we prove the following result:

Theorem 1.1. *If $p > 5$ is prime and $f := x^m + ax^n$ permutes $\mathbb{F}_p$, where $m > n > 0$ and $a \in \mathbb{F}_p^*$, then* $\gcd(m-n, p-1) \notin \{2, 4\}$.

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In case $(p-1)/2$ or $(p-1)/4$ is prime, this was conjectured in the recent paper [2] by Panario, Wang and the first author. It is well-known that the gcd is not 1: for in that case, f has more than one root in $\mathbb{F}_p$, since x^{m-n} is a permutation polynomial. It is much more difficult to show that the gcd is not 2 or 4.

In Section 2 we prove some general results about permutation binomials, and in particular we show that it suffices to prove Theorem 1.1 when $m-n$ divides $p-1$. Then we prove Theorem 1.1 in Section 3.

Throughout this paper, we want to ignore permutation binomials that are really monomials in disguise. Here one can disguise a permutation monomial (over $\mathbb{F}_q$) by adding a constant plus a multiple of $x^q - x$; such addition does not affect the permutation property. Thus, we say a permutation binomial of $\mathbb{F}_q$ is *trivial* if it is congruent modulo $x^q - x$ to the sum of a constant and a monomial. In other words, the nontrivial permutation binomials are those whose terms have degrees being positive and incongruent modulo $q-1$.

2 Permutation binomials in general

Lemma 2.1. *If f is a permutation polynomial over $\mathbb{F}_q$, then the greatest common divisor of the degrees of the terms of f is coprime to $q-1$.*

Proof. Otherwise f is a polynomial in x^d, where $d > 1$ divides $q-1$, but x^d is not a permutation polynomial so f is not one either. □

Lemma 2.2. *Let $d \mid (q-1)$, and suppose there are no nontrivial permutation binomials over $\mathbb{F}_q$ of the form $x^e(x^d + a)$. Then there are no nontrivial permutation binomials over $\mathbb{F}_q$ of the form $x^n(x^k + a)$ with $\gcd(k, q-1) = d$.*

Proof. Suppose $f(x) := x^n(x^k + a)$ permutes $\mathbb{F}_q$, where $n, k, a \neq 0$. Let $d = \gcd(k, q-1)$. Pick $r > 0$ such that $kr \equiv d \pmod{q-1}$ and $\gcd(r, q-1) = 1$. Then $f(x^r)$ permutes $\mathbb{F}_q$ and $f(x^r) \equiv x^{nr}(x^d + a) \pmod{x^q - x}$. □

Lemma 2.2 immediately implies the following result from [2]:

Corollary 2.3. *If $q-1$ is a Mersenne prime, then there are no nontrivial permutation binomials over $\mathbb{F}_q$.*

We give one further reduction along the lines of Lemma 2.2:

Lemma 2.4. *Let $d, n, e > 0$ satisfy $d|(q-1)$, $\gcd(ne, d) = 1$ and $n \equiv e \pmod{(q-1)/d}$. Then $x^n(x^d + a)$ permutes $\mathbb{F}_q$ if and only if $x^e(x^d + a)$ does.*

Proof. Write $f := x^n(x^d + a)$ and $g := x^e(x^d + a)$. For any $z \in \mathbb{F}_q$ with $z^d = 1$, we have $f(zx) = z^n f(x)$; since $\gcd(n, d) = 1$, this implies that the values of f on $\mathbb{F}_q$ comprise all the d^{th} roots of the values of $f(x)^d$. Since $f(x)^d \equiv g(x)^d \pmod{x^q - x}$, the result follows. □

Finally, since we constantly use it, we give here a version of Hermite's criterion [1]:

Lemma 2.5. *A polynomial $f \in \mathbb{F}_q[x]$ is a permutation polynomial if and only if*

1. *for each i with $0 < i < q-1$, the reduction of f^i modulo $x^q - x$ has degree less than $q-1$; and*
2. *f has precisely one root in $\mathbb{F}_q$.*

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We treat the cases of gcd 2 and 4 separately.

Theorem 3.1. *If p is prime and $x^n(x^k + a)$ is a nontrivial permutation binomial over $\mathbb{F}_p$, then* $\gcd(k, p-1) > 2$.

Proof. There are no nontrivial permutation binomials over $\mathbb{F}_2$ or $\mathbb{F}_3$, so we may assume $p = 2\ell + 1$ with $\ell > 1$. By Lemma 2.2, it suffices to show there are no nontrivial permutation binomials of the form $f := x^n(x^d + a)$ with $d \in \{1, 2\}$. This is clear for $d = 1$ (since then $f(0) = f(-a)$), so we need only consider $d = 2$. Assume $f := x^n(x^2 + a)$ is a permutation binomial. Lemma 2.1 implies n is odd.

Suppose ℓ is odd. We will use Hermite's criterion with exponent $\ell - 1$; to this end, we compute

$$f^{\ell-1} = x^{n\ell-n}(x^2 + a)^{\ell-1} = x^{n\ell-n} \sum_{i=0}^{\ell-1} \binom{\ell-1}{i} a^{\ell-1-i} x^{2i}.$$

Write $f^{\ell-1} = \sum_{i=0}^{\ell-1} b_i x^{n\ell-n+2i}$, where $b_i = \binom{\ell-1}{i} a^{\ell-1-i}$. Since $\ell - 1 < p$ and p is prime, each b_i is nonzero. Thus, the degrees of the terms of $f^{\ell-1}$ are precisely the elements of

$$S = \{n\ell - n, n\ell - n + 2, n\ell - n + 4, \dots, n\ell - n + 2\ell - 2\}.$$

Since ℓ is odd, S consists of ℓ consecutive even numbers, so it contains a unique multiple of $p-1 = 2\ell$. Thus the reduction of $f^{\ell-1}$ modulo $x^p - x$ has degree $p-1$, which contradicts Hermite's criterion.

If ℓ is even then $f^\ell = \sum_{i=0}^{\ell} c_i x^{n\ell+2i}$, where each $c_i = \binom{\ell}{i} a^{\ell-i}$ is nonzero. The degrees of the terms of f^ℓ consist of the $\ell + 1$ consecutive even numbers $n\ell, n\ell + 2, \dots, n\ell + 2\ell$. Since n is odd, $n\ell$ is not a multiple of $p - 1 = 2\ell$. Thus f^ℓ has a unique term of degree divisible by $p - 1$, which again contradicts Hermite's criterion. □

Theorem 3.2. *If p is prime and $x^n(x^k + a)$ is a nontrivial permutation binomial over $\mathbb{F}_p$, then* $\gcd(k, p-1) \neq 4$.

Proof. Plainly we need only consider primes p with $p \equiv 1 \pmod 4$. By Lemma 2.2, it suffices to show there are no nontrivial permutation binomials of the form $x^n(x^4 + a)$. By Lemma 2.1, we may assume n is odd. By Lemma 2.4, it suffices to show nonexistence with $0 < n < (p-1)/4$ if $p \equiv 1 \pmod 8$, and with $0 < n < (p-1)/2$ if $p \equiv 5 \pmod 8$. Assume $f := x^n(x^4 + a)$ is a nontrivial permutation binomial with n satisfying these constraints.

First suppose $p \equiv 1 \pmod 8$, say $p = 8\ell + 1$; here our assumption is $0 < n < 2\ell$. The set of degrees of terms of $f^{2\ell}$ is

$$S = \{2\ell n, 2\ell n + 4, 2\ell n + 8, \ldots, 2\ell n + 8\ell\}.$$

When ℓ is even, S consists of $2\ell+1$ consecutive multiples of 4. Since n is odd, $2\ell n$ is not a multiple of 8ℓ, so S contains precisely one multiple of $p-1 = 8\ell$, contradicting Hermite's criterion. So assume ℓ is odd; since $8\ell+1$ is prime, we have $\ell \geq 5$. Now the set of degrees of terms of $f^{2\ell+2}$ is

$$S = \{2\ell n + 2n, 2\ell n + 2n + 4, 2\ell n + 2n + 8, \ldots, 2\ell n + 2n + 4(2\ell + 2)\}.$$

Here S consists of $2\ell+3$ consecutive multiples of 4, so it contains a multiple of $p-1 = 8\ell$. By Hermite's criterion, S must have at least two such multiples. Thus, 8ℓ divides either $2\ell n + 2n$, $2\ell n + 2n + 4$ or $2\ell n + 2n + 8$, so ℓ divides either n, $n+2$ or $n+4$. Since $\ell \geq 5$ and $0 < n < 2\ell$, we have $n + 4 < 3\ell$; since n is odd, it follows that ℓ equals either n, $n+2$ or $n+4$. But then f^8 has a unique term of degree divisible by $p - 1 = 8\ell$, contradicting Hermite's criterion.

Thus we have $p \equiv 5 \pmod 8$; write $p = 4\ell + 1$ with ℓ odd, where again $0 < n < 2\ell$. Suppose $\ell \equiv 1 \pmod 4$. If $\ell = 1$ then f is trivial, so assume $\ell > 1$. The set of degrees of terms of $f^{\ell-1}$ is

$$S = \{n\ell - n, n\ell - n + 4, n\ell - n + 8, \ldots, n\ell - n + 4\ell - 4\}.$$

Since $\ell \equiv 1 \pmod 4$, the set S consists of ℓ consecutive multiples of 4, so S contains precisely one multiple of $p - 1 = 4\ell$, contradicting Hermite's criterion.

Thus $\ell \equiv 3 \pmod 4$. The set of degrees of terms of $f^{\ell+1}$ is

$$S = \{n\ell + n, n\ell + n + 4, n\ell + n + 8, \ldots, n\ell + n + 4\ell + 4\}.$$

Since S consists of $\ell + 2$ consecutive multiples of 4, it certainly contains a multiple of 4ℓ, so (by Hermite's criterion) it must contain two such multiples. Thus either $n(\ell + 1)$ or $n(\ell + 1) + 4$ is a multiple of 4ℓ, so ℓ divides either n or $n + 4$. Since n is odd and $0 < n < 2\ell$, the only possibilities are $n = \ell$ or $n = \ell - 4$ or $(n, \ell) = (5, 3)$. If $n = \ell - 4$ then f^4 has degree $4\ell = p - 1$, contradicting Hermite's criterion. If $(n, \ell) = (5, 3)$, then $p = 13$ and $a^{-1}f(x^{11})$ permutes $\mathbb{F}_p$; since $a^{-1}f(x^{11}) \equiv x^3(x^4 + a^{-1}) \pmod{x^{13} - x}$, it suffices to treat the case $n = \ell$. Finally, suppose $n = \ell$, so $f = x^\ell(x^4 + a)$ permutes $\mathbb{F}_p$. The degrees of the terms of f^4 are

$$4\ell, 4\ell + 4, 4\ell + 8, 4\ell + 12, 4\ell + 16.$$

We have our usual contradiction if the degree 4ℓ term is the unique term of f^4 with degree divisible by 4ℓ, so the only remaining possibility is that 4ℓ divides either 4, 8, 12 or 16. Since $\ell \equiv 3 \pmod 4$, the only possibility is $\ell = 3$. Finally, when $\ell = 3$, the coefficient of x^{12} in the reduction of f^4 modulo $x^{13} - x$ is $a^4 + 4a$, which must be zero (by Hermite), so $a^3 = -4$; but the cubes in $\mathbb{F}_{13}^*$ are ± 1 and ± 8, contradiction. □

References

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[2] A. Masuda, D. Panario, and Q. Wang, *The number of permutation binomials over $\mathbb{F}_{4p+1}$ where p and $4p+1$ are primes*, Electronic J. Combin. **13** (2006), R65.