The Borodin-Kostochka conjecture for graphs containing a doubly critical edge

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Submitted: Jul 23, 2007; Accepted: Oct 14, 2007; Published: Oct 24, 2007
Mathematics Subject Classification: 05C15

Abstract

We prove that if G is a graph containing a doubly-critical edge and satisfying $\chi \geq \Delta \geq 6$, then G contains a K_Δ.

1 Introduction

Way back in 1977, Borodin and Kostochka made the following conjecture (see [1]).

Conjecture. *Every graph satisfying $\chi \geq \Delta \geq 9$ contains a K_Δ.*

Examples exist showing that the $\Delta \geq 9$ condition is necessary (e.g. for the $\Delta = 8$ case, take a 5-cycle and expand each vertex to a triangle). In 1999, Reed proved the conjecture for $\Delta \geq 10^{14}$ (see [3]).

Definition 1. Let G be a graph. An edge $ab \in G$ is *doubly critical* just in case $\chi(G \setminus \{a, b\}) = \chi(G) - 2$.

We prove the following.

Theorem A. *Let G be a graph containing a doubly critical edge. If G satisfies $\chi \geq \Delta \geq 6$, then G contains a K_Δ.*

To see that this result is tight, consider the following graph. Put $A = \{1, 2\}$, $B = \{3, 4, 5\}$ and $C = \{6, 7, 8, 9\}$. Let G be the graph having $V(G) = A \cup B \cup C$ with A and C complete, B empty, and the additional edges $13, 14, 15, 23, 24, 25, 64, 65, 73, 75, 83, 84, 93, 94$. It is easily checked that G satisfies $\chi = \Delta = 5$ and $\omega = 4$. Also, G contains a doubly critical edge since removing both vertices 8 and 9 leaves a 3-chromatic graph. A counterexample with $\chi = \Delta = 4$ can be made by removing vertices 1 and 9 from G. The theorem holds trivially for $\Delta \leq 3$ since the only triangle-free graph containing a doubly critical edge is K_2.

We briefly mention a related conjecture of Lovàsz. He conjectures that the stronger condition that *every* edge of a connected graph G is doubly critical implies that G is complete (see [1]). Stiebitz has shown that this conjecture holds for graphs with chromatic number at most 5 (see [4]).

2 The Lonely Path Lemma

We reproduce the relevant definitions and lemmas from [2].

Definition 2. Let $C = \{I_1, \ldots, I_m\}$ be a coloring of a graph G. If there exists $j \neq k$ such that $v \in I_j$, $w \in I_k$ and $N(v) \cap I_k = \{w\}$, then the (directed) edge (v, w) is called *C-lonely*. If the coloring is clear from context we drop the C and just call the edge plain *lonely*.

The following lemma is clear from the definition of C-lonely.

Lemma 2.1. *Let C be a coloring of a graph G. If both (v, w) and (w, v) are C-lonely, then swapping v and w yields a new coloring C' with $|C| = |C'|$.*

Definition 3. Let C be a coloring of a graph G. The *C-lonely graph of G* (denoted $L_C(G)$) is the directed graph with vertex set $V(G)$ and edge set $\{(v, w) \mid (v, w) \text{ is } C\text{-lonely in } G\}$.

Definition 4. Let C be a coloring of a graph G. For any vertex $v \in G$, set

$$L_C(v) = \{w \in G \mid (v, w) \text{ is C-lonely}\}.$$

The following is the main lemma from [2]. We reproduce the proof here for completeness.

Lonely Path Lemma. *Let G be a graph. If C is an optimal coloring of G, $\{a\}, \{b\} \in C$ are distinct singleton color classes and p_a, p_b are vertex disjoint (directed) paths in $L_C(G)$ (starting at a, b respectively) both having at most one vertex in any given color class, then the vertices of p_a are completely joined to the vertices of p_b in G.*

Proof. Assume (to reach a contradiction) that the lemma is false. Of all counterexamples, pick an optimal coloring C of G, $\{a\}, \{b\} \in C$ distinct singleton color classes and p_a, p_b vertex disjoint (directed) paths in $L_C(G)$ (starting at a, b respectively) both having at most one vertex in any given color class where the sum of the lengths of p_a and p_b is minimized. Then, by the minimality condition, all but the ends of p_a and p_b must be joined in G. If p_a contains more than one vertex (say $p_a = a, a_2, a_3, \ldots, a_n$), then (a, a_2) is lonely since p_a is a path in $L_C(G)$. But {a} is a singleton color class, so (a_2, a) is also lonely. Hence, by Lemma 2.1, swapping a and a_2 yields another optimal coloring C' of G.

To apply the minimality condition, we need to show that $p'_a = a_2, a_3, \ldots, a_n$ and p_b are paths in $L_{C'}(G)$. Let I_j, I'_j be the color classes containing a_j in C, C' respectively. Assume that $p'_a \notin L_{C'}(G)$. Then we have $2 \leq k \leq n-1$ such that $|N(a_k) \cap I'_{k+1}| \neq 1$. Hence $I'_{k+1} \neq I_{k+1}$. Since swapping a and a_2 only changes $\{a\}$ and I_2, we must have $I_{k+1} = \{a\}$

or $I_{k+1} = I_2$. In the latter case, $a_{k+1} = a_2$ since p_a has at most one vertex in each color class. Thus $a_{k+1} = a$ or $a_{k+1} = a_2$. If $a_{k+1} = a_2$, then $I'_{k+1} = \{a_{k+1}\}$ contradicting the fact that $|N(a_k) \cap I'_{k+1}| \neq 1$. Whence $a_{k+1} = a$. Since p_a is a path, it has no repeated internal vertices; hence, $k+1 = n$. This is a contradiction since a_n is not joined to the end of p_b but a is. Whence $p'_a \in L_{C'}(G)$.

Now assume that $p_b \notin L_{C'}(G)$ (say $p_b = b, b_2, \ldots, b_m$). Let Q_j, Q'_j be the color classes containing b_j in C, C' respectively. Then we have $2 \leq e \leq m-1$ such that $|N(b_e) \cap Q'_{e+1}| \neq 1$. Hence $Q'_{e+1} \neq Q_{e+1}$. Since swapping a and a_2 only changes $\{a\}$ and I_2, we must have $Q_{e+1} = \{a\}$ or $Q_{e+1} = I_2$. The former is impossible since p_a and p_b are disjoint. Hence $Q_{e+1} = I_2$. Since $e < m$, b_e is adjacent to a_2. Since $|N(b_e) \cap I_2| = |N(b_e) \cap Q_{e+1}| = 1$, we must have $b_{e+1} = a_2$ contradicting the disjointness of p_a and p_b. Whence $p_b \in L_{C'}(G)$.

Hence p'_a and p_b are vertex disjoint paths in $L_{C'}(G)$ with the end of p'_a not joined to the end of p_b and p'_a shorter than p_a, contradicting the minimality condition. Hence p_a is the single vertex $\{a\}$. Similarly, p_b is the single vertex $\{b\}$. Since p_a is not joined to p_b, the color classes $\{a\}$ and $\{b\}$ can be merged, contradicting the fact that C is an optimal coloring. □

Lemma 2.2. *Let G be a graph and $C = \{I_1, \ldots, I_m\}$ an optimal coloring of G. Then, for each $1 \leq j \leq m$, there exists $v_j \in I_j$ such that $N(v_j) \cap I_k \neq \emptyset$ for each $k \neq j$.*

Proof. Otherwise C would not be optimal. □

3 Proof of The Main Result

Lemma 3.1. *Let G be a graph and $C = \{\{a\}, \{b\}, I_3, \ldots, I_m\}$ be an optimal coloring of G. Then $N(a) \cap N(b) \cap I_j \neq \emptyset$ for $3 \leq j \leq m$.*

Proof. Let $3 \leq j \leq m$. By Lemma 2.2, we have $v_j \in I_j$ such that $a, b \in N(v_j)$. □

The following is a simple application of the Lonely Path Lemma to paths of length one.

Lemma 3.2. *Let G be a graph and $C = \{\{a\}, \{b\}, I_3, \ldots, I_m\}$ be an optimal coloring of G. Then for any $X \subseteq L_C(a) \smallsetminus L_C(b)$ and $Y \subseteq L_C(b) \smallsetminus L_C(a)$ with $|X| \leq 1$ and $|Y| \leq 1$, $X \cup Y \cup L_C(a) \cap L_C(b)$ induces a clique in G.*

Lemma 3.3. *Let X be a set and $d \geq 3$. If $N_1, \ldots, N_d \subseteq X$ with $|N_i| = 2$ for all $1 \leq i \leq d$, $N_i \cap N_j \neq \emptyset$ for all $1 \leq i \leq j \leq d$ and $\bigcap_{i=1}^{d} N_i = \emptyset$, then $\left|\bigcup_{i=1}^{d} N_i\right| = 3$.*

Proof. Assume (to reach a contradiction) that this is not the case and let $N_1, \ldots, N_d$ be a counterexample with d minimal. Plainly, $d \geq 4$. By the minimality of d, the N_i are distinct. If $\{x_1, y_1\} = N_1 \not\subseteq \bigcup_{i=2}^{d} N_i$, then, without loss of generality, $x_1 \notin \bigcup_{i=2}^{d} N_i$. Hence

$x_1 \notin N_i$ for $2 \leq i \leq d$. But N_1 has non-trivial intersection with each of $N_2, \ldots, N_d$, so we must have $x_2 \in N_i$ for $2 \leq i \leq d$. Thus $x_2 \in \bigcap_{i=1}^{d} N_i$, giving a contradiction. Whence $N_1 \subseteq \bigcup_{i=2}^{d} N_i$. By the minimality of d, the lemma holds for $N_2, \ldots, , N_d$. If $\bigcap_{i=2}^{d} N_i = \emptyset$, then $\left|\bigcup_{i=2}^{d} N_i\right| = 3$. But $N_1 \subseteq \bigcup_{i=2}^{d} N_i$ giving $\left|\bigcup_{i=1}^{d} N_i\right| = 3$, a contradiction. Hence we have $z_1 \in \bigcap_{i=2}^{d} N_i$. Similarly, we have $z_2 \in N_1 \cap \bigcap_{i=3}^{d} N_i$ and $z_3 \in N_1 \cap N_2 \cap \bigcap_{i=4}^{d} N_i$. Since $\{z_1, z_2, z_3\} \subseteq N_4$ and $|N_4| = 2$, two of the z's coincide. Without loss of generality assume $z_1 = z_2$. Then $z_1 \in \bigcap_{i=1}^{d} N_i$ giving a final contradiction. $\square$

Proof of Theorem A. Assume (to reach a contradiction) that G satisfies $\chi \geq \Delta \geq 6$ and does not contain a K_Δ. Without loss of generality, we may assume that G is connected. By Brooks' theorem we must have $\chi(G) = \Delta(G)$. Set $m = \chi(G)$ and let $C = \{\{a\}, \{b\}, I_3, \ldots, I_m\}$ be an optimal coloring of G. By Lemma 2.2, a is adjacent to at least one vertex in each element of $C \smallsetminus \{a\}$. Hence $m - 1 \leq d(a) \leq \Delta(G) = m$ and thus $m - 2 \leq |L_C(a)| \leq m - 1$. Similarly, $m - 2 \leq |L_C(b)| \leq m - 1$. If $|L_C(a) \cup L_C(b)| = m$, then a straightforward application of Lemma 3.2 produces a K_m in G. Thus we have $|L_C(a) \cup L_C(b)| \leq m - 1$. Since $b \in L_C(a)$ and $a \in L_C(b)$, we must have $|L_C(a) \cup L_C(b)| = m - 1$. Let K be the unique color class that $L_C(a) \cup L_C(b)$ does not hit. Then $|N(a) \cap K| = 2$ and $|N(b) \cap K| = 2$.

Given $x \in L_C(a) \cup L_C(b) \smallsetminus \{a, b\}$, both (a, x) and (x, a) are C-lonely. Hence, by Lemma 2.1, we may swap x and a to yield a new optimal coloring C'. By an argument similar to above we conclude that $|L_{C'}(x) \cup L_{C'}(b)| = m - 1$. Since K is still a color class in C' and b hits two elements of K, we conclude that $|N(x) \cap K| = 2$.

Let $S = L_C(a) \cup L_C(b)$. By Lemma 3.2, S induces a clique of order $m - 1$. For $y \in S$, put $P_y = N(y) \cap K$. From the above we know that for each $y \in S$ we have $|P_y| = 2$. If there exists $z \in \bigcap_{y \in S} P_y$, then $S \cup \{z\}$ induces a K_m, giving a contradiction. Hence $\bigcap_{y \in S} P_y = \emptyset$. Given distinct $y_1, y_2 \in S$, we may swap y_1 with a and y_2 with b and apply Lemma 3.1, to conclude that $P_{y_1} \cap P_{y_2} = N(y_1) \cap N(y_2) \cap K \neq \emptyset$. Now applying Lemma 3.3 with $X = K$ and $\{N_1, \ldots, N_d\} = \{P_y \mid y \in S\}$ gives $\left|\bigcup_{y \in S} P_y\right| = 3$.

Put $T = \bigcup_{y \in S} P_y$. Let $A = G \smallsetminus K \smallsetminus S$. Since S induces a clique of order $m - 1$ and $|P_y| = 2$ for all $y \in S$, there are m edges from each $y \in S$ to $S \cup K$ and hence there are no edges between A and S. Plainly, A is $m-3$ colorable. Put $H = G[V(A) \cup K]$. We show that H has an $m-1$ coloring in which each element of T receives a different color. There are at most $3\Delta(G)$ edges from T to $G \setminus K$ and exactly $2|S| = 2(\Delta(G)-1)$ edges from T to S. Hence there are at most $\Delta(G)+2$ edges from T to H. Let $\{c_1, \ldots, c_{m-3}\}$ be a coloring of A. If each element of T hit all the c_i, then the number of edges from T to S would be at least $3(m-3) = 3(\Delta(G)-3)$. Hence we would have $3(\Delta(G)-3) \leq \Delta(G)+2$ and thus $\Delta(G) \leq \frac{11}{2}$. Whence, since $\Delta(G) \geq 6$, we have $z \in T$ and $1 \leq i \leq m-3$ such that z misses color c_i. Let c_{m-2} and c_{m-1} be two new colors. Coloring z with c_i, the other two elements of T with c_{m-2} and c_{m-1} and the rest of K with c_{m-2} gives an $m-1$ coloring D of H in which each element of T receives a different color.

We now extend D to S using Hall's theorem. Note from above that $|S| = m-1$. For each $y \in S$, let l_y be the elements of $\{c_1, \ldots, c_{m-1}\}$ not appearing on an element of P_y. Then for $y \in S$ we have $|l_y| = m-3$ since $|P_y| = 2$. Hence all we need to check is that the union of any $m-2$ of lists has at least $m-2$ elements and that the union of all of the lists has $m-1$ elements. If the former were false, then since T receives three distinct colors under D, we would have $y_1, \ldots, y_{m-2} \in S$ with $P_{y_i} = P_{y_j}$ for all $1 \leq i < j \leq m-2$. But the remaining element of S must be adjacent to at least one of the elements of P_{y_1} giving a K_m in G. If the union of all the lists had fewer than $m-1$ elements then we would have $P_w = P_y$ for all $w, y \in S$ giving a K_m once again. Hence Hall's theorem gives distinct $c_y \in l_y$ for $y \in S$. Since there are no edges between A and S, coloring y with c_y extends D to an $m-1$ coloring of G. This final contradiction completes the proof.

□

Acknowledgments

Thanks to the anonymous referee for pointing out that the original $\Delta \geq 9$ condition in Theorem A could be weakened and for suggesting the use of Hall's theorem to finish off the proof.

References

[1] Tommy R. Jenson and Bjarne Toft. Graph Coloring Problems *Wiley.*, 1995.

[2] Landon Rabern. Coloring and the Lonely Graph. *arXiv:0707.1069*, 2007.

[3] Bruce Reed. A strengthening of Brooks' Theorem. *J. Combinatorial Th. (B),* **76**, 1999, 136 - 149.

[4] Michael Stiebitz. K_5 is the only double-critical 5-chromatic graph. *Discrete Math.,* **64**, 1987, 91-93.