Perfect dominating sets in the Cartesian products of prime cycles

Hamed Hatami[a] **and Pooya Hatami**[b]

[a] *Department of Computer Science*
University of Toronto
e-mail:hamed@cs.toronto.edu

[b] *Department of Mathematical Science*
Sharif University of Technology
e-mail:p_hatami@ce.sharif.edu

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Abstract

We study the structure of a minimum dominating set of C^n_{2n+1}, the Cartesian product of n copies of the cycle of size $2n+1$, where $2n+1$ is a prime.

Keywords: Perfect Lee codes; dominating sets; defining sets.

1 Introduction

Let G and H be two graphs. The *Cartesian product* of G and H is a graph with vertices $\{(x,y) : x \in G, y \in H\}$ where $(x,y) \sim (x',y')$ if and only if $x = x'$ and $y \sim y'$, or $x \sim x'$ and $y = y'$. Let G^n denote the Cartesian product of n copies of G. This article deals with C^n_{2n+1} where C_{2n+1} is the cycle of size $p := 2n+1$ and p is a prime.

For our purpose, it is more convenient to view the vertices of C^n_{2n+1} as the elements of the group $G := \mathbb{Z}^n_{2n+1}$. Then $x \sim y$ if and only if $x - y = \pm e_i$ for some $i \in [n]$, where $e_i = (0,\dots,1,\dots,0)$ is the unit vector with 1 at the ith coordinate. In other words, C^n_{2n+1} is the Cayley graph $\Gamma(\mathbb{Z}^n_{2n+1}, \mathcal{U})$ over the group $\mathbb{Z}^n_{2n+1}$ with the set of generators $\mathcal{U} = \{\pm e_1, \dots, \pm e_n\}$. From this point on, to emphasis the group structure of the graph we will use the Cayley graph notation $\Gamma(\mathbb{Z}^n_{2n+1}, \mathcal{U})$ instead of the Cartesian product notation of C^n_{2n+1}.

Let u and v be two vertices of a graph G. We say that u dominates v if $u = v$ or $u \sim v$. A subset S of the vertices of G is called a *dominating set*, if every vertex of G is dominated by at least one vertex of S. A dominating set is *perfect*, if no vertex is dominated by more than one vertex.

Remark 1 Let G be a graph with m vertices. Every function $f : V(G) \to \mathbb{C}$ can be viewed as a vector $\vec{f} \in \mathbb{C}^m$. Let A denote the adjacency matrix of G. Note that $f : V(G) \to \{0,1\}$ is the characteristic function of a perfect dominating set if and only if $(A+I)\vec{f} = \vec{1}$.

We are interested in perfect dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. Note that for an r-regular graph $G = (V, E)$ a dominating set is perfect if and only if it is of size $|V|/(r+1)$. Since $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ is $2n$-regular and has $(2n+1)^n$ vertices, a dominating set is perfect if and only if it is of size $(2n+1)^{n-1}$.

Fix an arbitrary $(\epsilon_1, \ldots, \epsilon_{n-1}) \in \{-1,1\}^{n-1}$, and a $k \in \{0, \ldots, 2n\}$. The set

$$\{(x_1, \ldots, x_{n-1}, k + \sum_{i=1}^{n-1} \epsilon_i (i+1) x_i) : x_i \in \mathbb{Z}_{2n+1} \ \forall i \in [n-1]\} \tag{1}$$

forms a perfect dominating set in $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$, where the additions are in $\mathbb{Z}_{2n+1}$. To see this, consider $y = (y_1, \ldots, y_n) \in \mathbb{Z}_{2n+1}^n$. Let $t = k + \sum_{i=1}^{n-1} \epsilon_i (i+1) y_i$, and $\Delta = (t - y_n) \bmod 2n+1$ so that $|\Delta| \le n$. If $\Delta \in \{-1,0,1\}$ then y is dominated by $(y_1, \ldots, y_{n-1}, t)$. If $\Delta \notin \{-1,0,1\}$, then with the notation $j := |\Delta| - 1$, y is adjacent to $(y_1, \ldots, y_{j-1}, y_j - \epsilon_j \times \mathrm{sgn}(\Delta), y_{j+1}, \ldots, y_n)$ which can be easily seen that it is in the considered set.

There are many results in the direction of constructing perfect dominating sets in the Cartesian product of cycles (see [5] and its references). However the authors are unaware of any result in the direction of characterizing the structure of perfect dominating sets. We consider the simplest case $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ where $2n+1$ is a prime. Even in this simple case we are unable to characterize all the perfect dominating sets. However we prove the following theorem in this direction.

Theorem 1 *Let $2n+1$ be a prime and $S \subseteq \Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ be a perfect dominating set. Then for every $(x_1, \ldots, x_n) \in \mathbb{Z}_{2n+1}^n$ and every $i \in [n]$,*

$$|S \cap \{(y_1, \ldots, y_n) : y_j = x_j \ \forall j \neq i\}| = 1.$$

Theorem 1 says that when $2n+1$ is a prime, every parallel-axis line contains exactly one point from every perfect dominating set of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. It is easy to construct examples to show that the condition of $2n+1$ being a prime is necessary [4].

Let $\mathcal{F}$ be a family of sets. For $S \in \mathcal{F}$, a set $D \subseteq S$ is called a *defining set* for $(S, \mathcal{F})$ (or for S when there is no ambiguity), if and only if S is the only superset of D in $\mathcal{F}$. The size of the minimum defining set for $(S, \mathcal{F})$ is called its defining number. Defining sets are studied for various families of $\mathcal{F}$ (See [3] for a survey on the topic). Let $\mathcal{F}$ be the family of all minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. Note that since $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ is regular and contains at least one perfect dominating set, a set $S \subseteq V(G)$ is a minimum dominating set if and only if it is a perfect dominating set. In [2] Chartrand et al. studied the size of defining sets of $\mathcal{F}$ for $n = 2$. Based on this case they conjectured that the smallest defining set over all minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ is of size exactly n. As it is noticed by Richard Bean [private communication], the conjecture fails for $n = 3$, as in this case there are perfect dominating sets with defining number 2 (See Remark 3). So far there is no nontrivial bound known for the defining numbers of minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. We prove the following theorem.

Theorem 2 *Let $2n+1$ be a prime and $\mathcal{F}$ be the family of all minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. Every $S \in \mathcal{F}$ has a defining set of size at most $n!2^n$.*

The proof of Theorem 1 uses Fourier analysis on finite Abelian groups. In Section 2 we review Fourier analysis on $\mathbb{Z}_p^n$. Section 3 is devoted to the proof of Theorem 2. Section 4 contains further discussions about the defining sets of minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$.

2 Background

In this section we introduce some notations and review Fourier analysis on $G = \mathbb{Z}_p^n$. For a nice and more detailed, but yet brief introduction we refer the reader to [1]. See also [6] for a more comprehensive reference.

Aside from its group structure we will also think of G as a measure space with the uniform (product) measure, which we denote by μ. For any function $f : G \to \mathbb{C}$, let

$$\int_G f(x)dx = \frac{1}{|G|}\sum_{x \in G} f(x).$$

The inner product between two functions f and g is $\langle f, g\rangle = \int_G f(x)\overline{g(x)}dx$. Let

$$\omega = e^{2\pi i/p},$$

where i is the imaginary number. For any $x \in G$, let $\chi_x : G \to \mathbb{C}$ be defined as

$$\chi_x(y) = \omega^{\sum_{i=1}^n x_i y_i}.$$

It is easy to see that these functions form an orthonormal basis. So every function $f : G \to \mathbb{C}$ has a unique expansion of the form $f = \sum \widehat{f}(x)\chi_x$, where $\widehat{f}(x) = \langle f, \chi_x\rangle$ is a complex number.

3 Proof of Theorem 1

Let $\vec{0} = (0, \ldots, 0)$, $\vec{1} = (1, \ldots, 1)$, and $e_i = (0, \ldots, 1, \ldots, 0)$, the unit vector with 1 at the i-th coordinate. Let $p = 2n+1$ be a prime and S be a perfect dominating set in G, and let f be the characteristic function of S, i.e. $f(x) = 1$ if $x \in S$ and $f(x) = 0$ otherwise. Let

$$D = \{\pm e_1, \pm e_2, \ldots, \pm e_n\},$$

be the set of unit vectors and their negations. For every $\tau \in D$ define $f_\tau(x) = f(x+\tau)$. Note that

$$\widehat{f_\tau}(y) = \int f(x+\tau)\overline{\chi_y(x)}dx = \int f(x)\overline{\chi_y(x-\tau)}dx = \int f(x)\overline{\chi_y(x)}\chi_y(\tau)dx = \widehat{f}(y)\chi_y(\tau).$$

Let

$$g = f + \sum_{\tau\in D} f_\tau.$$

We have

$$g = \left(\sum_{y\in G} \widehat{f}(y)\chi_y\right) + \sum_{\tau\in D}\sum_{y\in G} \widehat{f_\tau}(y)\chi_y = \sum_{y\in G} \widehat{f}(y)\left(\sum_{\tau\in D\cup\{\vec{0}\}} \chi_y(\tau)\right)\chi_y. \tag{2}$$

Since f is the characteristic function of a perfect dominating set, we have $g(x) = 1$, for every $x \in G$. So $g = \chi_{\vec{0}}$. By uniqueness of Fourier expansion, for every $y \neq \vec{0}$,

$$0 = \widehat{g}(y) = \widehat{f}(y) \sum_{\tau\in D\cup\{\vec{0}\}} \chi_y(\tau) = \widehat{f}(y)\left(1 + \sum_{i=1}^{n} \omega^{y_i} + \sum_{i=1}^{n} \omega^{-y_i}\right). \tag{3}$$

Now we turn to the key step of the proof. Since $2n+1$ is a prime, (3) implies that whenever $\widehat{f}(y) \neq 0$, we have

$$\{y_1, \ldots, y_n\} \cup \{-y_1, \ldots, -y_n\} = \{1, \ldots, 2n\}. \tag{4}$$

Denote the set of all y satisfying (4) by $\mathcal{Y}$. For $1 \le i \le n$, let

$$D_i = \{ke_i : 0 \le k \le 2n\}.$$

Define $g_i = \sum_{\tau\in D_i} f_\tau$. Similar to (2), we get

$$g_i = \sum \widehat{f}(y)\left(\sum_{\tau\in D_i} \chi_y(\tau)\right)\chi_y.$$

When $y \in \mathcal{Y}$, since $y_i \neq 0$, we have

$$\sum_{\tau\in D_i} \chi_y(\tau) = \sum_{k=0}^{2n} \omega^{ky_i} = 0.$$

When $y \notin \mathcal{Y}$ and $y \neq \vec{0}$, $\widehat{f}(y) = 0$. So

$$g_i = \left(\widehat{f}(0) \sum_{\tau\in D_i} \chi_{\vec{0}}(\tau)\right)\chi_{\vec{0}} = \chi_{\vec{0}} = 1. \tag{5}$$

Note that $g_i(x)$ counts the number of elements in $S \cap \{(y_1, \ldots, y_n) : y_j = x_j \ \forall j \neq i\}$. This completes the proof.

Remark 2 The above proof can be translated to the language of linear algebra (However in the linear algebra language the key observation (4) becomes less obvious). Indeed, let $m=(2n+1)^n$ denote the number of vertices. From Remark 1 we know that $f:\mathbb{Z}_{2n+1}^n \to \mathbb{C}$ is the characteristic function of a perfect dominating set if and only if $(A+I)\vec{f}=\vec{1}$, where A is the adjacency matrix of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. The reader may notice that in the proof of Theorem 1, $\vec{g}=(A+I)\vec{f}$, and thus (2) shows that $\vec{\chi_y}$ is a family of orthonormal eigenvectors of $A+I$. Moreover, among these eigenvectors, the ones that correspond to the 0 eigenvalue are exactly $\vec{\chi_y}$ with $y \in \mathcal{Y}$. Hence the rank of $A+I$ is $m-|\mathcal{Y}|=(2n+1)^n-n!2^n$. We will use this fact in the proof of Theorem 2.

4 Proof of Theorem 2

As it is observed in Remark 1, every perfect dominating set of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ corresponds to a zero-one vector $\vec{f} \in \mathbb{C}^m$ that satisfies $(A+I)\vec{f}=\vec{1}$. Let

$$V := \mathrm{span}\{\vec{f} : f \in \mathcal{F}\}.$$

Trivially

$$\dim V \leq 1 + (m - \mathrm{rank}(A+I)) = 1 + n!2^n.$$

Also for a subset D of vertices of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$, define

$$V_D := \mathrm{span}\{\vec{f} : f \in \mathcal{F} \text{ and } \forall x \in D,\ f(x)=1\},$$

Note that $V = V_\emptyset$.

To prove Theorem 2 we start from $D=\emptyset$. At every step, if D does not extend uniquely to S, then there exists a vertex $v \in S$ such that $\dim V_{D\cup\{v\}} < \dim V_D$; we add v to D. Since $\dim V_\emptyset \leq 1+n!2^n$, we can obtain a set D of size at most $n!2^n$ such that the dimension of V_D is at most 1. This completes the proof as there is at most one non-zero, zero-one vector in a vector space of dimension 1.

5 Future directions

We ask the following question:

Question 1 *For prime $2n+1$, are there examples of perfect dominating sets in $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ that are not of the form (1)?*

If the answer to Question 1 turns out to be negative, then we can improve the bound of Theorem 2:

Proposition 1 *Let $p=2n+1$ be a prime, and let $\mathcal{T}$ denote the set of perfect dominating sets of the form (1). Every $(S,\mathcal{T})$ where $S \in \mathcal{T}$ has a defining set of size $1+\lceil \frac{n-1}{\lfloor \log_2 p \rfloor} \rceil$.*

Proof. Suppose that $S \in \mathcal{T}$. Then S is of the form:

$$\{(x_1, \ldots, x_{n-1}, k + \sum_{i=1}^{n-1} \epsilon_i (i+1) x_i) : x_i \in \mathbb{Z}_p \ \forall i \in [n-1]\}.$$

Let $m = \lfloor \log_2 p \rfloor$. We will use the easy fact that for any $c \in \mathbb{Z}_p$, the equation $\sum_{i=0}^{m-1} \epsilon_i 2^i =_p c$ has at most one solution $(\epsilon_0, \epsilon_1, ..., \epsilon_{m-1}) \in \{-1, +1\}^m$. For $i, j \geq 0$, define $\alpha_{i,j} \in \mathbb{Z}_p$ to be the solution to $(i+j+1)\alpha_{i,j} =_p 2^j$.

Let $u = (0, 0, ..., 0, b)$ be the unique vertex in S with the first $n-1$ coordinates equal to 0, and for every $1 \leq i \leq n-1$ consider the unique vector

$$u_i = (0, ..., 0, \alpha_{i,0}, \alpha_{i,1}, ..., \alpha_{i,k_i}, 0, ..., 0, b_i) \in S,$$

where $\alpha_{i,0}$ is in the ith coordinate and $k_i = \min(m-1, n-i-1)$. We claim that the set $D = \{u, u_0, u_m, ..., u_{m(\lceil \frac{n-1}{m} \rceil - 1)}\}$ is a defining set for $(S, \mathcal{T})$. Since S is of form (1), clearly $k = b$, and for every $0 \leq i \leq \lceil \frac{n-1}{m} \rceil - 1$, we have:

$$b_{mi} - b = \sum_{j=0}^{k_{mi}} \epsilon_{mi+j} (mi + j + 1) \alpha_{mi,j} = \sum_{j=0}^{k_{mi}} \epsilon_{mi+j} 2^j.$$

The above equation has only one solution $(\epsilon_{mi}, \epsilon_{mi+1}, ..., \epsilon_{mi+k_{mi}}) \in \{-1, +1\}^{k_{mi}+1}$. Considering this for all $u_{mi} \in D$ determines $(\epsilon_1, \epsilon_2, ..., \epsilon_{n-1})$. Thus the set D is a defining set for $(S, \mathcal{T})$. ■

Remark 3 For $n = 2, 3$ the answer to Question 1 is negative. Thus when $n = 3$, Proposition 1 implies that there is a defining set of size 2 for a perfect dominating set. This disproves the conjecture of [2] which is already observed by Richard Bean [private communication].

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