The universal embedding of the near polygon $\mathbb{G}_n$

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Abstract

In [12], it was shown that the dual polar space $DH(2n-1,4)$, $n \geq 2$, has a sub near-$2n$-gon $\mathbb{G}_n$ with a large automorphism group. In this paper, we determine the absolutely universal embedding of this near polygon. We show that the generating and embedding ranks of $\mathbb{G}_n$ are equal to $\binom{2n}{n}$. We also show that the absolutely universal embedding of $\mathbb{G}_n$ is the unique full polarized embedding of this near polygon.

1 Introduction

1.1 Definitions

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point p and every line L, there exists a unique point on L nearest to p. Here, distances are measured in the point or collinearity graph Γ of $\mathcal{S}$. If d is the diameter of Γ, then the near polygon is called a *near* $2d$*-gon*. A near 0-gon is just a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles (Payne and Thas [23]). Near polygons were introduced by Shult and Yanushka in [26]. For a discussion on the basic theory of near polygons, we refer to the recent book [13] of the author.

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 from each other have at least two common neighbours. By Theorem 4 of Brouwer and Wilbrink [6], every two points of a dense near $2d$-gon at distance $\delta \in \{0, \ldots, d\}$ from each other are contained in a unique convex sub-2δ-gon. These convex sub-2δ-gons are called *quads* if $\delta = 2$, *hexes* if $\delta = 3$ and *maxes* if $\delta = d-1$.

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The existence of quads in a dense near polygon was already shown by Shult and Yanushka [26].

A proper convex subspace F of a dense near polygon $\mathcal{S}$ is called *big* in $\mathcal{S}$ if every point x of $\mathcal{S}$ not contained in F is collinear with a necessarily unique point of F. We will denote this point by $\pi_F(x)$. If moreover every line of $\mathcal{S}$ is incident with precisely 3 points, then a *reflection* $\mathcal{R}_F$ about F can be defined which is an automorphism of $\mathcal{S}$ (see [13, Theorem 1.11]). If $x \in F$, then we define $\mathcal{R}_F(x) := x$. If $x \notin F$, then $\mathcal{R}_F(x)$ denotes the unique point of the line $x\pi_F(x)$ different from x and $\pi_F(x)$.

With every polar space Π of rank $n \geq 2$ there is associated a near $2n$-gon Δ, which is called a *dual polar space* (see Shult and Yanushka [26]; Cameron [7]). The points and lines of Δ are the maximal and next-to-maximal singular subspaces of Π with reverse containment as incidence relation. If Π is the polar space associated with a nonsingular hermitian variety $H(2n-1, q^2)$ in $\mathrm{PG}(2n-1, q^2)$, then the corresponding dual polar space is denoted by $DH(2n-1, q^2)$.

Let $H(2n-1, 4)$, $n \geq 2$, denote the hermitian variety $X_0^3 + X_1^3 + \cdots + X_{2n-1}^3 = 0$ of $\mathrm{PG}(2n-1, 4)$ (with respect to a given reference system). The number of nonzero coordinates (with respect to the same reference system) of a point p of $\mathrm{PG}(2n-1, 4)$ is called the *weight* of p. The maximal and next-to-maximal subspaces of $H(2n-1, 4)$ define a dual polar space $DH(2n-1, 4)$. Let $\mathbb{G}_n = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the following substructure of $DH(2n-1, 4)$:

(i) $\mathcal{P}$ is the set of all maximal subspaces of $H(2n-1, 4)$ which are generated by n points of weight 2 whose sum has weight $2n$;

(ii) $\mathcal{L}$ is the set of all $(n-2)$-dimensional subspaces of $H(2n-1, 4)$ containing at least $n-2$ points of weight 2;

(iii) incidence is reverse containment.

By De Bruyn [12], see also De Bruyn [13, 6.3], $\mathbb{G}_n$ is a dense near $2n$-gon with 3 points on each line and its above-defined embedding in $DH(2n-1, 4)$ is isometric, i.e. preserves distances. The generalized quadrangle $\mathbb{G}_2$ is isomorphic to the classical generalized quadrangle $Q(5, 2)$. A construction of the near hexagon $\mathbb{G}_3$ was already given in Brouwer et al. [4]. The near $2n$-gon $\mathbb{G}_n$ contains $\frac{3^n \cdot (2n)!}{2^n \cdot n!}$ points. For every permutation ϕ of $\{0, \ldots, 2n-1\}$, every automorphism θ of $\mathrm{GF}(4)$ and all $\lambda_0, \lambda_1, \ldots, \lambda_{2n-1} \in \mathrm{GF}(4) \setminus \{0\}$, the map $(X_0, X_1, \ldots, X_{2n-1}) \mapsto (\lambda_0 (X_{\phi(0)})^\theta, \lambda_1 (X_{\phi(1)})^\theta, \ldots, \lambda_{2n-1}(X_{\phi(2n-1)})^\theta)$ induces an automorphism of $\mathbb{G}_n$. If $n \geq 3$, then every automorphism of $\mathbb{G}_n$ is obtained in this way, see Theorem 6.38 of De Bruyn [13]. This conclusion does not hold if $n = 2$. In that case, we have $\mathrm{Aut}(\mathbb{G}_2) \cong \mathrm{Aut}(Q(5, 2)) \cong P\Gamma U(4, 4)$.

If X is a non-empty set of points of a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, then $\langle X \rangle_{\mathcal{S}}$ denotes the smallest subspace of $\mathcal{S}$ containing the set X, i.e. $\langle X \rangle_{\mathcal{S}}$ is the intersection of all subspaces of $\mathcal{S}$ containing X. The minimal number $gr(\mathcal{S}) := \min\{|X| : X \subseteq \mathcal{P} \text{ and } \langle X \rangle_{\mathcal{S}} = \mathcal{P}\}$ of points which are necessary to generate the whole point-set $\mathcal{P}$ is called the *generating rank* of $\mathcal{S}$. A *hyperplane* of a partial linear space $\mathcal{S}$ is a proper subspace meeting every line (necessary in a unique point of the whole line). If $\mathcal{S}$ is a near polygon, then the set of points at non-maximal distance from a given point x is a hyperplane of $\mathcal{S}$ which is called the *singular hyperplane with deepest point* x.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial linear space. A *full embedding* e of $\mathcal{S}$ into a projective space $\Sigma = \mathrm{PG}(V)$ is an injective mapping e from $\mathcal{P}$ to the set of points of Σ satisfying: (i) $\langle e(P) \rangle_\Sigma = \Sigma$, (ii) $e(L) := \{e(x) \,|\, x \in L\}$ is a line of Σ for every line L of $\mathcal{S}$. The dimensions $\dim(\Sigma)$ and $\dim(V) = \dim(\Sigma) + 1$ are respectively called the *projective dimension* and the *vector dimension* of the embedding e. The maximal dimension $er(\mathcal{S})$ of a vector space V for which $\mathcal{S}$ has a full embedding in $\mathrm{PG}(V)$ is called the *embedding rank* of $\mathcal{S}$. Off course, $er(\mathcal{S})$ is only defined when $\mathcal{S}$ admits a full embedding, in which case it holds that $er(\mathcal{S}) \leq gr(\mathcal{S})$.

If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$ and if α is a hyperplane of Σ, then $e^{-1}(e(\mathcal{P}) \cap \alpha)$ is a hyperplane of $\mathcal{S}$. We will say that such a hyperplane *arises from the embedding* e. If $\mathcal{S}$ is a near polygon and if all singular hyperplanes arise from the embedding e, then e is called a *polarized embedding*. The dual polar space $DH(2n-1, q^2)$ has a nice full polarized embedding into the projective space $\mathrm{PG}(\binom{2n}{n} - 1, q)$, see Cooperstein [9] and De Bruyn [14]. We refer to this embedding as the Grassmann-embedding of $DH(2n-1, q^2)$. The Grassmann-embedding of $DH(2n-1, 4)$ induces a full polarized embedding of the near $2n$-gon $\mathbb{G}_n$. We refer to Section 2 for more details.

Two full embeddings $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ of $\mathcal{S}$ are called *isomorphic* ($e_1 \cong e_2$) if there exists an isomorphism $f : \Sigma_1 \to \Sigma_2$ such that $e_2 = f \circ e_1$. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$ and if U is a subspace of Σ satisfying (C1): $\langle U, e(p) \rangle_\Sigma \neq U$ for every point p of $\mathcal{S}$, (C2): $\langle U, e(p_1) \rangle_\Sigma \neq \langle U, e(p_2) \rangle_\Sigma$ for any two distinct points p_1 and p_2 of $\mathcal{S}$, then there exists a full embedding e/U of $\mathcal{S}$ into the quotient space Σ/U mapping each point p of $\mathcal{S}$ to $\langle U, e(p) \rangle_\Sigma$. If $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ are two full embeddings, then we say that $e_1 \geq e_2$ if there exists a subspace U in Σ_1 satisfying (C1), (C2) and $e_1/U \cong e_2$. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$, then by Ronan [24], there exists a unique (up to isomorphisms) full embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ satisfying (i) $\widetilde{e} \geq e$, (ii) if $e' \geq e$ for some embedding e' of $\mathcal{S}$, then $\widetilde{e} \geq e'$. We say that $\widetilde{e}$ is *universal relative to* e. If $\widetilde{e} \cong e$ for some full embedding e of $\mathcal{S}$, then we say that e is *relatively universal*. A full embedding e of $\mathcal{S}$ is called *absolutely universal* if it is universal relative to any full embedding of $\mathcal{S}$ defined over the same division ring as e. Kasikova and Shult [20] gave sufficient conditions for a relatively universal embedding to be absolutely universal.

By Ronan [24], every fully embeddable geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with three points per line admits the absolutely universal embedding which is obtained in the following way. Let V be a vector space over the field $\mathbb{F}_2$ with a basis whose vectors are indexed by the elements of $\mathcal{P}$, e.g. $B = \{\overline{v}_p \,|\, p \in \mathcal{P}\}$. Let W denote the subspace of V generated by all vectors $\overline{v}_{p_1} + \overline{v}_{p_2} + \overline{v}_{p_3}$ where $\{p_1, p_2, p_3\}$ is a line of $\mathcal{S}$. Then the map $p \in \mathcal{P} \mapsto \{\overline{v}_p + W, W\}$ defines a full embedding of $\mathcal{S}$ into the projective space $\mathrm{PG}(V/W)$ which is isomorphic to the absolutely universal embedding of $\mathcal{S}$.

1.2 Main results

Although every embeddable point-line geometry with three points per line admits the absolutely universal embedding, it is very often nontrivial to determine the embedding rank of such a geometry or to decide whether a given embedding of such a geometry is absolutely universal. We refer to Cooperstein [11, page 27] for an overview of what is known about the generating and embedding ranks of point-line geometries with three points per line. Regarding absolutely universal embeddings of near polygons with three points per line, we mention the following important results from the literature:

(1) The universal embedding dimension of the dual polar space $DW(2n-1,2)$, $n \geq 2$, was determined by Li [21] and Blokhuis and Brouwer [3].

(2) The universal embedding dimension of the dual polar space $DH(2n-1,4)$, $n \geq 2$, was determined by Li [22].

(3) The universal embedding dimensions of the ${}^3D_4(2)$-generalized hexagon and the J_2 near octagon were determined by Frohardt and Smith [18].

(4) The universal embedding dimensions of the two generalized hexagons of order 2 were determined in Frohardt and Johnson [17]. For an alternative proof, see also Thas and Van Maldeghem [27].

(5) The universal embedding dimension of the $U_4(3)$ near hexagon was determined by Yoshiara [28]. Alternative proofs were given by Bardoe [1] and De Bruyn [16].

(6) The universal embedding dimension of the near polygon $\mathbb{H}_n$, $n \geq 2$, on the 1-factors of the complete graph on $2n+2$ vertices was determined by Blokhuis and Brouwer [2].

In the present paper, we determine the absolutely universal embedding of the near polygon $\mathbb{G}_n$, $n \geq 2$. We prove the following in Section 2:

Theorem 1.1 *The Grassmann-embedding of $DH(2n-1,4)$, $n \geq 2$, induces a full polarized embedding of $\mathbb{G}_n$ of vector dimension $\binom{2n}{n}$.*

In Section 3, we prove the following:

Theorem 1.2 *The dual polar space $\mathbb{G}_n$, $n \geq 2$, can be generated by $\binom{2n}{n}$ points.*

Recall that $er(\mathbb{G}_n) \leq gr(\mathbb{G}_n)$. Now, $er(\mathbb{G}_n) \geq \binom{2n}{n}$ by Theorem 1.1 and $gr(\mathbb{G}_n) \leq \binom{2n}{n}$ by Theorem 1.2. Hence, we can say the following:

Corollary 1.3 (1) *The generating and embedding ranks of $\mathbb{G}_n$, $n \geq 2$, are equal to $\binom{2n}{n}$.*

(2) *The full embedding of $\mathbb{G}_n$, $n \geq 2$, induced by the Grassmann-embedding of $DH(2n-1,4)$ is isomorphic to the absolutely universal embedding of $\mathbb{G}_n$.*

Finally, in Section 4, we prove the following:

Theorem 1.4 *The absolutely universal embedding of $\mathbb{G}_n$, $n \geq 2$, is the unique (up to isomorphisms) full polarized embedding of $\mathbb{G}_n$.*

Remarks. (1) In Corollary 1.3 (1), we mentioned that the generating and embedding ranks of $\mathbb{G}_n$, $n \geq 2$, are equal. This is a property which holds for almost all embeddable point-line geometries with three points per line for which these two ranks are known, see Cooperstein [11, page 27]. A counterexample provided by Heiss [19] shows however that this is not always the case.

(2) The fact that the absolutely universal embedding of $\mathbb{G}_3$ has vector dimension 20 and that it is the unique full polarized embedding of $\mathbb{G}_3$ was already mentioned in Brouwer et al. [4, Table p. 350].

(3) Although $\mathbb{G}_n$, $n \geq 2$, admits a unique full polarized embedding, its absolutely universal embedding is not the only full embedding of $\mathbb{G}_n$ if $n \geq 3$. This follows from an easy counting argument. If $e : \mathbb{G}_n \to \Sigma$ denotes the absolutely universal embedding of $\mathbb{G}_n$, $n \geq 3$, then the number of points of Σ which are on a line of the form $e(x)e(y)$, where x and y are two distinct points of $\mathbb{G}_n$ is less than the number of points of $\Sigma \cong \mathrm{PG}(\binom{2n}{n}-1, 2)$. If x^* is a point of Σ not on any of these lines, then also e/x^* is a full embedding of $\mathbb{G}_n$.

2 A full polarized embedding of $\mathbb{G}_n$

Let $n \geq 2$, let V be a $2n$-dimensional vector space over GF(4) and let $B = \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n}\}$ be a basis of V. Let $H(2n-1, 4)$ denote the hermitian variety of PG(V) whose equation with respect to the basis B is given by $X_1^3 + X_2^3 + \ldots + X_{2n}^3 = 0$.

Let $\bigwedge^n V$ denote the n-th exterior power of V. For every maximal subspace $p = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle_V$ of $H(2n-1, 4)$, let $\wedge^n(p)$ denote the point $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \rangle_{\bigwedge^n V}$ of $\mathrm{PG}(\bigwedge^n V)$. Notice that the point $\wedge^n(p)$ of $\mathrm{PG}(\bigwedge^n V)$ is independent from the generating set $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ of p. By Cooperstein [9] and De Bruyn [14], the points $\wedge^n(p)$ are contained in a necessarily unique Baer subgeometry Σ of $\mathrm{PG}(\bigwedge^n V)$ and $\wedge^n$ defines a full polarized embedding e of $DH(2n-1, 4)$ into Σ. This embedding is called the *Grassmann-embedding* of $DH(2n-1, 4)$.

Now, let $\mathbb{G}_n$ be isometrically embedded into the dual polar space $DH(2n-1, 4)$ as described in Section 1.1. Then the Grassmann-embedding e of $DH(2n-1, 4)$ induces a full embedding e' of $\mathbb{G}_n$ into a subspace Σ' of Σ.

Proposition 2.1 *The embedding $e' : \mathbb{G}_n \to \Sigma'$ is polarized.*

Proof. Let x be an arbitrary point of $\mathbb{G}_n$. Since the singular hyperplane of $DH(2n-1, 4)$ with deepest point x arises from the Grassmann-embedding of $DH(2n-1, 4)$, there exists a hyperplane α of Σ such that the following holds:

(i) if y is a point of $DH(2n-1, 4)$ at non-maximal distance from x, then $e(y) \in \alpha$;

(ii) if y is a point of $DH(2n-1, 4)$ opposite to x, then $e(y) \notin \alpha$.

In particular, (i) and (ii) hold for points y of $\mathbb{G}_n$. Now, since the embedding of $\mathbb{G}_n$ into $DH(2n-1, 4)$ is isometric, α intersects Σ' in a hyperplane of Σ' and the singular hyperplane of $\mathbb{G}_n$ with deepest point x arises from the hyperplane $\alpha \cap \Sigma'$ of Σ'. ■

Lemma 2.2 *Let μ_0 and μ_1 be two distinct elements of a field $\mathbb{K}$ and let $m \geq 1$. Let A_m be the $(2^m \times 2^m)$-matrix over the field $\mathbb{K}$ whose rows and columns are indexed lexicographically by the elements of $\{0,1\}^m$. For all $\bar{\epsilon}, \bar{\delta} \in \{0,1\}^m$, the $(\bar{\epsilon}, \bar{\delta})$-entry of the matrix A_m is equal* *to* $\prod_{i=1}^m \mu_{\epsilon(i)}^{\delta(i)}$*, where* $\epsilon(i)$ *and* $\delta(i)$ *denote the* i*-th component of* $\bar{\epsilon}$ *and* $\bar{\delta}$*, respectively. Then* A_m *is nonsingular.*

Proof. We will prove this by induction on m. Suppose first that $m = 1$. Then

$$\det(A_1) = \det \begin{bmatrix} 1 & \mu_0 \\ 1 & \mu_1 \end{bmatrix} = \mu_1 - \mu_0 \neq 0.$$

If $m \geq 2$, then

$$A_m = \begin{bmatrix} A_{m-1} & \mu_0 \cdot A_{m-1} \\ A_{m-1} & \mu_1 \cdot A_{m-1} \end{bmatrix} = \begin{bmatrix} I_{2^{m-1}} & O_{2^{m-1}} \\ I_{2^{m-1}} & I_{2^{m-1}} \end{bmatrix} \cdot \begin{bmatrix} A_{m-1} & \mu_0 \cdot A_{m-1} \\ O_{2^{m-1}} & (\mu_1 - \mu_0) \cdot A_{m-1} \end{bmatrix},$$

where $I_{2^{m-1}}$ is the $(2^{m-1} \times 2^{m-1})$-identity matrix and $O_{2^{m-1}}$ is the $(2^{m-1} \times 2^{m-1})$-matrix with all entries equal to 0. Hence $\det(A_m) = (\mu_1 - \mu_0)^{2^{m-1}} \cdot [\det(A_{m-1})]^2$ is different from 0 by the induction hypothesis. ∎

The following proposition completes the proof of Theorem 1.1.

Proposition 2.3 *We have that* $\Sigma' = \Sigma$.

Proof. It suffices to prove that for all $i_1, \ldots, i_n \in \{1, \ldots, 2n\}$ with $i_1 < i_2 < \cdots < i_n$, there exist points $p_1, \ldots, p_k$ of $\mathbb{G}_n$ such that

$$\bar{e}_{i_1} \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_n} \in \langle \wedge^n(p_1), \wedge^n(p_2), \ldots, \wedge^n(p_k) \rangle_{\bigwedge^n V}.$$

Let $j_1, j_2, \ldots, j_n \in \{1, \ldots, 2n\}$ such that $\{i_1, i_2, \ldots, i_n\} \cup \{j_1, j_2, \ldots, j_n\} = \{1, \ldots, 2n\}$. Let μ_0 and μ_1 be two distinct elements of $\mathrm{GF}(4) \setminus \{0\}$ and let W denote the set of all 2^n vectors of the form $\bar{e}_{k_1} \wedge \bar{e}_{k_2} \wedge \cdots \wedge \bar{e}_{k_n}$, where $k_l \in \{i_l, j_l\}$ for every $l \in \{1, \ldots, n\}$. For every $\bar{\epsilon} \in \{0,1\}^n$, put

$$\bar{v}(\bar{\epsilon}) := (\bar{e}_{i_1} + \mu_{\epsilon(1)}\bar{e}_{j_1}) \wedge (\bar{e}_{i_2} + \mu_{\epsilon(2)}\bar{e}_{j_2}) \wedge \cdots \wedge (\bar{e}_{i_n} + \mu_{\epsilon(n)}\bar{e}_{j_n}) \in \bigwedge^n V$$

and

$$p(\bar{\epsilon}) := \langle \bar{e}_{i_1} + \mu_{\epsilon(1)}\bar{e}_{j_1}, \bar{e}_{i_2} + \mu_{\epsilon(2)}\bar{e}_{j_2}, \ldots, \bar{e}_{i_n} + \mu_{\epsilon(n)}\bar{e}_{j_n} \rangle_V.$$

Here, $\epsilon(i)$, $i \in \{1, \ldots, n\}$, denotes the i-th component of $\bar{\epsilon}$. Notice that $\langle \bar{v}(\bar{\epsilon}) \rangle_{\bigwedge^n V} = \wedge^n(p(\bar{\epsilon}))$. By Lemma 2.2, the matrix relating the 2^n vectors $\bar{v}(\bar{\epsilon})$, $\bar{\epsilon} \in \{0,1\}^n$, with the 2^n vectors of W is nonsingular. It follows that $W \subseteq \langle \wedge^n(p(\bar{\epsilon})) \,|\, \bar{\epsilon} \in \{0,1\}^n \rangle_{\bigwedge^n V}$. In particular, we have that $\bar{e}_{i_1} \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_n} \in \langle \wedge^n(p(\bar{\epsilon})) \,|\, \bar{\epsilon} \in \{0,1\}^n \rangle_{\bigwedge^n V}$. This proves the proposition. ∎

3 The generating rank of $\mathbb{G}_n$

Let $n \geq 2$, let V be a $2n$-dimensional vector space over GF(4) and let $B = \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n}\}$ be a basis of V. Let $H(2n-1, 4)$, $n \geq 2$, denote the hermitian variety of $\mathrm{PG}(V)$ whose equation with respect to the basis B is given by $X_1^3 + X_2^3 + \ldots + X_{2n}^3 = 0$. Let $DH(2n-1,4)$ denote the corresponding dual polar space and let $\mathbb{G}_n$ be the sub near $2n$-gon of $DH(2n-1,4)$ as defined in Section 1.1. So, the points of $\mathbb{G}_n$ are the maximal subspaces of $H(2n-1,4)$ which are generated by n points of weight 2 whose sum has weight $2n$.

The near polygon $\mathbb{G}_n$ has convex subspaces of different types. For the purposes of determining a generating set of $\mathbb{G}_n$, we are only interested in those convex subspaces of $\mathbb{G}_n$ which are big in $\mathbb{G}_n$. The maximal subspaces of $H(2n-1,4)$ which are points of $\mathbb{G}_n$ and which contain a given point $\langle \bar{e}_i + \mu \bar{e}_j \rangle_V$ of weight 2 define a big convex subspace of $\mathbb{G}_n$ which we will denote by $M[\bar{e}_i + \mu \bar{e}_j]$. If $n \geq 3$, then by De Bruyn [12, Lemma 12], every big convex subspace of $\mathbb{G}_n$ is obtained in this way and is isomorphic to $\mathbb{G}_{n-1}$.

3.1 Two lemmas

Lemma 3.1 *Let $i_1, j_1, i_2, j_2 \in \{1, \ldots, 2n\}$ and $\mu_1, \mu_2 \in \mathrm{GF}(4) \setminus \{0\}$ such that $i_1 \neq j_1$, $i_2 \neq j_2$ and $\langle \bar{e}_{i_1} + \mu_1 \bar{e}_{j_1} \rangle_V \neq \langle \bar{e}_{i_2} + \mu_2 \bar{e}_{j_2} \rangle_V$. Then $M_1 := M[\bar{e}_{i_1} + \mu_1 \bar{e}_{j_1}]$ and $M_2 := M[\bar{e}_{i_2} + \mu_2 \bar{e}_{j_2}]$ are disjoint if and only if $|\{i_1, j_1\} \cap \{i_2, j_2\}| \geq 1$.*

Suppose now that M_1 and M_2 are disjoint and put $M_3 := \mathcal{R}_{M_1}(M_2)$. If $(i_1, j_1) = (i_2, j_2)$, then $M_3 = M[\bar{e}_{i_1} + \mu_3 \bar{e}_{j_1}]$, where μ_3 is the unique element of $\mathrm{GF}(4)$ different from 0, μ_1 and μ_2. If $i_1 = i_2$ and $j_1 \neq j_2$, then $M_3 = M[\bar{e}_{j_1} + \mu_1^{-1} \mu_2 \bar{e}_{j_2}]$.

Proof. Obviously, if $|\{i_1, j_1\} \cap \{i_2, j_2\}| = 1$ or $\{i_1, j_1\} = \{i_2, j_2\}$, then $M_1 \cap M_2 = \emptyset$, since $\langle \bar{e}_{i_1} + \mu_1 \bar{e}_{j_1} \rangle_V$ and $\langle \bar{e}_{i_2} + \mu_2 \bar{e}_{j_2} \rangle_V$ are not collinear on the hermitian variety $H(2n-1,4)$. If $\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset$, then let $i_3, j_3, \ldots, i_n, j_n$ such that $\{i_1, j_1, i_2, j_2, \ldots, i_n, j_n\} = \{1, 2, \ldots, 2n\}$. Then $\langle \bar{e}_{i_1} + \mu_1 \bar{e}_{j_1}, \bar{e}_{i_2} + \mu_2 \bar{e}_{j_2}, \bar{e}_{i_3} + \bar{e}_{j_3}, \ldots, \bar{e}_{i_n} + \bar{e}_{j_n} \rangle_V$ is a point of $M_1 \cap M_2$. This proves the first part of the lemma.

Now, suppose $(i_1, j_1) = (i_2, j_2)$. Let μ_3 be the unique element of GF(4) different from 0, μ_1 and μ_2. Let $p_1 = \langle \bar{e}_{i_1} + \mu_1 \bar{e}_{j_1}, \bar{v}_2, \ldots, \bar{v}_n \rangle_V$ be an arbitrary point of $M[\bar{e}_{i_1} + \mu_1 \bar{e}_{j_1}]$, where $\bar{v}_2, \ldots, \bar{v}_n$ are vectors of weight 2. (Recall that if a maximal isotropic subspace p belongs to $\mathbb{G}_n$, then given any point x of weight 2 in p, there exists a set of $n-1$ points of weight 2 which together with x generate p.) Then $p_2 = \langle \bar{e}_{i_1} + \mu_2 \bar{e}_{j_1}, \bar{v}_2, \ldots, \bar{v}_n \rangle_V$ is the unique point of M_2 collinear with p_1 and $p_3 = \langle \bar{e}_{i_1} + \mu_3 \bar{e}_{j_1}, \bar{v}_2, \ldots, \bar{v}_n \rangle_V$ is the third point of the line $p_1 p_2$. It now readily follows that $M_3 = M[\bar{e}_{i_1} + \mu_3 \bar{e}_{j_1}]$.

Now, suppose $i_1 = i_2$ and $j_1 \neq j_2$. Let $p_1 = \langle \bar{e}_{i_1} + \mu_1 \bar{e}_{j_1}, \bar{e}_{j_2} + \mu' \bar{e}_{j_3}, \bar{v}_3, \ldots, \bar{v}_n \rangle_V$ be an arbitrary point of $M[\bar{e}_{i_1} + \mu_1 \bar{e}_{j_1}]$, where $\bar{v}_3, \ldots, \bar{v}_n$ are vectors of weight 2. Then $p_2 = \langle \bar{e}_{i_1} + \mu_2 \bar{e}_{j_2}, \mu_1 \bar{e}_{j_1} + \mu' \mu_2 \bar{e}_{j_3}, \bar{v}_3, \ldots, \bar{v}_n \rangle_V$ is the unique point of M_2 collinear with p_1. The third point p_3 on the line $p_1 p_2$ is equal to $p_3 = \langle \bar{e}_{i_1} + \mu' \mu_2 \bar{e}_{j_3}, \mu_2 \bar{e}_{j_2} + \mu_1 \bar{e}_{j_1}, \bar{v}_3, \ldots, \bar{v}_n \rangle_V$. It now readily follows that $M_3 = M[\bar{e}_{j_1} + \mu_1^{-1} \mu_2 \bar{e}_{j_2}]$. ∎

Now, let ω denote an arbitrary element of $\mathrm{GF}(4) \setminus \mathrm{GF}(2)$.

Lemma 3.2 *The smallest subspace of $\mathbb{G}_n$ containing the maxes $M[\bar{e}_1 + \bar{e}_2]$, $M[\bar{e}_1 + \omega \bar{e}_2]$ and $M[\bar{e}_i + \bar{e}_{i+1}]$, $i \in \{2, \ldots, n\}$, coincides with the whole point set of $\mathbb{G}_n$.*

Proof. We will make use of the following fact:

($*$) If S is a subspace of $\mathbb{G}_n$ and if M_1 and M_2 are two disjoint big maxes contained in S, then also $\mathcal{R}_{M_1}(M_2)$ is contained in S.

We will use ($*$) with S the smallest subspace of $\mathbb{G}_n$ containing $M[\bar{e}_1 + \bar{e}_2]$, $M[\bar{e}_1 + \omega\bar{e}_2]$ and $M[\bar{e}_i + \bar{e}_{i+1}]$, $i \in \{2, \ldots, n\}$.

Step 1: *If $\mu \in \mathrm{GF}(4) \setminus \{0\}$, then S contains $M[\mu\bar{e}_1 + \bar{e}_2]$.*
PROOF. Apply ($*$) to the maxes $M[\bar{e}_1 + \bar{e}_2]$ and $M[\bar{e}_1 + \omega\bar{e}_2]$.

Step 2: *For every $\mu \in \mathrm{GF}(4) \setminus \{0\}$ and every $i \in \{2, \ldots, n+1\}$, $M[\mu\bar{e}_1 + \bar{e}_i]$ is contained in S.*
PROOF. By induction on i. The case $i = 2$ is precisely Step 1. Suppose now that $M[\mu\bar{e}_1 + \bar{e}_{i-1}] \subseteq S$ for a certain $i \in \{3, \ldots, n+1\}$. Then applying ($*$) to $M[\mu\bar{e}_1 + \bar{e}_{i-1}]$ and $M[\bar{e}_{i-1} + \bar{e}_i]$, we find that $M[\mu\bar{e}_1 + \bar{e}_i] \subseteq S$.

Step 3: *For all $i, j \in \{1, \ldots, n+1\}$ with $i \neq j$ and all $\mu \in \mathrm{GF}(4) \setminus \{0\}$, $M[\bar{e}_i + \mu\bar{e}_j] \subseteq S$.*
PROOF. By Step 2 we may suppose that $i \neq 1 \neq j$. Then the claim follows by applying ($*$) to $M[\bar{e}_1 + \bar{e}_i]$ and $M[\bar{e}_1 + \mu\bar{e}_j]$.

Step 4: *Every point of $\mathbb{G}_n$ is contained in S.*
PROOF. Let $p = \langle \bar{e}_{i_1} + \mu_1\bar{e}_{j_1}, \bar{e}_{i_2} + \mu_2\bar{e}_{j_2}, \ldots, \bar{e}_{i_n} + \mu_n\bar{e}_{j_n} \rangle_V$ be an arbitrary point of $\mathbb{G}_n$ ($\{i_1, j_1, i_2, j_2, \ldots, i_n, j_n\} = \{1, 2, \ldots, 2n\}$ and $\mu_1, \mu_2, \ldots, \mu_n \in \mathrm{GF}(4) \setminus \{0\}$). Obviously, there exists at least one $k \in \{1, \ldots, n\}$ such that $\{i_k, j_k\} \subseteq \{1, \ldots, n+1\}$. Then $p \in M[\bar{e}_{i_k} + \mu_k\bar{e}_{j_k}] \subseteq S$ by Step 3. ∎

3.2 A recursively defined series of numbers

For every $n \in \mathbb{N} \setminus \{0, 1\}$ and every $j \in \{0, \ldots, n\}$, we will now define a number $f(n, j)$.

For $n = 2$, we define $f(2, 0) = f(2, 1) = f(2, 2) = 2$. Suppose that for some $n \geq 2$, we have defined $f(n, j)$ for all $j \in \{0, \ldots, n\}$. Then we define

$$\begin{aligned} \lambda(n) &:= \sum_{j=0}^{n} f(n, j), \\ f(n+1, 0) &:= \lambda(n), \\ f(n+1, 1) &:= \lambda(n), \\ f(n+1, 2) &:= \lambda(n), \\ f(n+1, k) &:= \sum_{j=k-1}^{n} f(n, j) \text{ for every } k \in \{3, \ldots, n+1\}. \end{aligned}$$

The above array of numbers was defined by Cooperstein in [9]. He showed the following:

Lemma 3.3 ([9, Lemma 4.2]) *Let $n \geq 2$. Then*

$$f(n, 0) = \binom{2n-2}{n-1},$$

$$
\begin{aligned}
f(n,1) &= \binom{2n-2}{n-1}, \\
f(n,j) &= 2\cdot\binom{2n-1-j}{n-j} \text{ for every } j\in\{2,\ldots,n\}.
\end{aligned}
$$

3.3 Construction of a generating set for the near polygon $\mathbb{G}_n$

In this section, we are going to construct a generating set of size $\binom{2n}{n}$ for the near $2n$-gon $\mathbb{G}_n$, $n\geq 2$. The technique we will use to achieve this goal is the one used by Cooperstein in [9] and [10]. As before, let ω denote an arbitrary element of $\mathrm{GF}(4)\setminus\mathrm{GF}(2)$. Put

$$
\begin{aligned}
M_1 &= M[\bar{e}_1+\bar{e}_2], \\
M_2 &= M[\bar{e}_1+\omega\bar{e}_2], \\
M_i &= M[\bar{e}_{i-1}+\bar{e}_i], \qquad i\in\{3,\ldots,n+1\}.
\end{aligned}
$$

Lemma 3.4 *Put $B_0=\emptyset$ and $B_j=\langle M_1,\ldots,M_j\rangle_{\mathbb{G}_n}$ for every $j\in\{1,\ldots,n+1\}$. Then for every $j\in\{0,\ldots,n\}$, there exists a set X of points in M_{j+1} satisfying:*

(i) $|X|=f(n,j)$;

(ii) $\langle(B_j\cap M_{j+1})\cup X\rangle_{\mathbb{G}_n}=M_{j+1}$.

Proof. We will prove the lemma by induction on n.

Suppose $n=2$ and $j\in\{0,1,2\}$. Then M_{j+1} is a line of the generalized quadrangle $\mathbb{G}_2\cong Q(5,2)$. So there exists a set X of size $f(2,j)=2$ such that $\langle X\rangle_{\mathbb{G}_2}=M_{j+1}$. Hence, also $\langle(B_j\cap M_{j+1})\cup X\rangle_{\mathbb{G}_2}=M_{j+1}$.

Suppose that $n\geq 3$ and that the lemma holds for smaller values of n. By the induction hypothesis and Lemma 3.2, every M_i, $i\in\{1,\ldots,n+1\}$, which is isomorphic to $\mathbb{G}_{n-1}$ can be generated by $\lambda(n-1)=\sum_{i=0}^{n-1}f(n-1,i)$ points. As a consequence, the claim holds if $j\in\{0,1,2\}$. So, suppose $j\geq 3$. The maximal subspaces of $H(2n-1,4)$ which contain $\langle\bar{e}_j+\bar{e}_{j+1}\rangle_V$ and $n-1$ other points of weight 2 are precisely the points of M_{j+1}.

Let $H(2n-3,4)$ denote the hermitian variety $X_1^3+X_2^3+\cdots+X_{j-1}^3+X_{j+2}^3+\cdots+X_{2n}^3=0$ in the subspace $X_j=X_{j+1}=0$ of $\mathrm{PG}(2n-1,4)$. The subspaces of $H(2n-3,4)$ which contain $n-1$ points of weight 2 define a near polygon $\mathbb{G}_{n-1}$. If α is such a subspace of $H(2n-3,4)$, then $\langle\bar{e}_j+\bar{e}_{j+1},\alpha\rangle_V$ is a point of M_{j+1}. In this way, we obtain an isomorphism θ between $\mathbb{G}_{n-1}$ and M_{j+1}. Now, in $\mathbb{G}_{n-1}$ we can define the n maxes $M'[\bar{e}_1+\bar{e}_2]$, $M'[\bar{e}_1+\omega\bar{e}_2]$, $M'[\bar{e}_2+\bar{e}_3]$, …, $M'[\bar{e}_{j-2}+\bar{e}_{j-1}]$, $M'[\bar{e}_{j-1}+\bar{e}_{j+2}]$, $M'[\bar{e}_{j+2}+\bar{e}_{j+3}]$, …, $M'[\bar{e}_{n+1}+\bar{e}_{n+2}]$ which we will denote by $M_1',M_2',\ldots,M_n'$. For every $k\in\{1,\ldots,j-1\}$, M_k contains $\theta(M_k')$. Hence, B_j contains $\theta(\langle M_1',M_2',\ldots,M_{j-1}'\rangle_{\mathbb{G}_{n-1}})$. Hence by Lemma 3.2 and the induction hypothesis, there exists a set X of size $f(n-1,j-1)+\cdots+f(n-1,n-1)=f(n,j)$ such that $\langle(B_j\cap M_{j+1})\cup X\rangle_{\mathbb{G}_n}=M_{j+1}$. This proves the lemma. ■

The following corollary of Lemma 3.4 is precisely Theorem 1.2.

Corollary 3.5 *The near polygon $\mathbb{G}_n$ can be generated by $\sum_{j=0}^{n}f(n,j)=\lambda(n)=f(n+1,0)=\binom{2n}{n}$ points.*

Proof. By Lemmas 3.2 and 3.4. ∎

4 Full polarized embeddings of $\mathbb{G}_n$

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a dense near polygon and let $e : \mathcal{S} \to \Sigma$ be a full polarized embedding of $\mathcal{S}$. For every point x of $\mathcal{S}$, let H_x denote the singular hyperplane of $\mathcal{S}$ with deepest point x. By Brouwer and Wilbrink [6, p. 156] (see also Shult [25, Lemma 6.1]), H_x is a maximal subspace of $\mathcal{S}$. This implies that the co-dimension of $e^*(x) := \langle e(H_x) \rangle_\Sigma$ in Σ is at most 1. Since e is polarized, $e^*(x)$ necessarily is a hyperplane of Σ. Now, put $R_e := \bigcap_{x \in \mathcal{P}} e^*(x)$. Then by De Bruyn [15], R_e satisfies the properties (C1) and (C2) of Section 1.1 and $\bar{e} := e/R_e$ is a full polarized embedding of $\mathcal{S}$. Also, if e' is a full polarized embedding of $\mathcal{S}$ such that $e \geq e'$, then $e' \geq \bar{e}$.

If $\mathcal{S}$ admits the absolutely universal embedding $\tilde{e}$ with respect to a certain division ring $\mathbb{K}$, then $e \geq \bar{\tilde{e}}$ for any full polarized embedding e of $\mathcal{S}$ with underlying division ring isomorphic to $\mathbb{K}$. We then call $\bar{\tilde{e}}$ the *minimal full polarized* $\mathbb{K}$*-embedding* of $\mathcal{S}$ or shortly the *minimal full polarized embedding* of $\mathcal{S}$ if no confusion is possible. If all lines of $\mathcal{S}$ have precisely 3 points, then $\mathbb{K} \cong \mathrm{GF}(2)$, and the minimal full polarized embedding of $\mathcal{S}$ is also called the *near polygon embedding*, see Brouwer et al. [4, p. 350] or Brouwer and Shpectorov [5].

We will now calculate the minimal full polarized embedding of $\mathbb{G}_n$, $n \geq 2$. As before, let $\mathbb{G}_n$ be isometrically embedded into the dual polar space $DH(2n-1, 4)$. Let $e : DH(2n-1,4) \to \Sigma$ denote the Grassmann-embedding of $DH(2n-1,4)$ and let e' denote the embedding of $\mathbb{G}_n$ induced by e. Then e' is isomorphic to the absolutely universal embedding of $\mathbb{G}_n$ by Corollary 1.3.

Let $x_1, x_2, \ldots, x_{\binom{2n}{n}}$ be points of $\mathbb{G}_n$ such that $\langle e(x_i) \,|\, 1 \leq i \leq \binom{2n}{n} \rangle_\Sigma = \Sigma$. Recall that for every point x of $DH(2n-1,4)$, $e^*(x)$ is a hyperplane of Σ. Similarly, for every point x of $\mathbb{G}_n$, $e'^*(x)$ is a hyperplane of Σ. Notice that if x is a point of $\mathbb{G}_n$, then $e'^*(x) = e^*(x)$. By Cardinali, De Bruyn and Pasini [8, Section 4.2], the map $e^D : DH(2n-1,4) \to \Sigma^*, x \mapsto e^*(x)$ defines a full polarized embedding of $DH(2n-1,4)$ into the dual Σ^* of Σ and $e^D \cong e$. It follows that $\bigcap_{i \in \{1,\ldots,\binom{2n}{n}\}} e'^*(x_i) = \bigcap_{i \in \{1,\ldots,\binom{2n}{n}\}} e^*(x_i) = \emptyset$. This implies that the minimal full polarized embedding of $\mathbb{G}_n$ is isomorphic to the absolutely universal embedding of $\mathbb{G}_n$. So, there exists (up to isomorphisms) only one full polarized embedding of $\mathbb{G}_n$.

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