The tripartite separability of density matrices of graphs

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Abstract

The density matrix of a graph is the combinatorial laplacian matrix of a graph normalized to have unit trace. In this paper we generalize the entanglement properties of mixed density matrices from combinatorial laplacian matrices of graphs discussed in Braunstein *et al.* [Annals of Combinatorics, **10** (2006) 291] to tripartite states. Then we prove that the degree condition defined in Braunstein *et al.* [Phys. Rev. A, **73** (2006) 012320] is sufficient and necessary for the tripartite separability of the density matrix of a nearest point graph.

1 Introduction

Quantum entanglement is one of the most striking features of the quantum formalism[1]. Moreover, quantum entangled states may be used as basic resources in quantum information processing and communication, such as quantum cryptography[2], quantum parallelism[3], quantum dense coding[4, 5] and quantum teleportation[6, 7]. So testing whether a given state of a composite quantum system is separable or entangled is in general very important.

Recently, normalized laplacian matrices of graphs considered as density matrices have been studied in quantum mechanics. One can recall the definition of density matrices of graphs from [8]. Ali Saif M. Hassan and Pramod Joag[9] studied the related issues like classification of pure and mixed states, von Neumann entropy, separability of multipartite quantum states and quantum operations in terms of the graphs associated with quantum states. Chai Wah Wu[10] showed that the Peres-Horodecki positive partial transpose condition is necessary and sufficient for separability in $C^2 \otimes C^q$. Braunstein *et al.*[11] proved that the degree condition is necessary for separability of density matrices of any graph and is sufficient for separability of density matrices of nearest point graphs and perfect matching graphs. Hildebrand *et al.*[12] testified that the degree condition is equivalent to the PPT-criterion. They also considered the concurrence of density matrices of graphs and pointed out that there are examples on four vertices whose concurrence is a rational number.

The paper is divided into three sections. In section 2, we recall the definition of the density matrices of a graph and define the tensor product of three graphs, reconsider the tripartite entanglement properties of the density matrices of graphs introduced in [8]. In section 3, we define partially transposed graph at first and then shows that the degree condition introduced in [11] is also sufficient and necessary condition for the tripartite state of the density matrices of nearest point graphs.

2 The tripartite entanglement properties of the density matrices of graphs

Recall that from [8] a *graph* $G = (V(G),\ E(G))$ is defined as: $V(G) = \{v_1,\ v_2,\ \cdots,\ v_n\}$ is a non-empty and finite set called *vertices*; $E(G) = \{\{v_i,\ v_j\} :\ v_i,\ v_j \in V\}$ is a non-empty set of unordered pairs of vertices called *edges*. An edge of the form $\{v_i,\ v_i\}$ is called as a *loop*. We assume that $E(G)$ does not contain any loops. A graph G is said to be on n vertices if $|V(G)| = n$. The *adjacency matrix* of a graph G on n vertices is an $n \times n$ matrix, denoted by $M(G)$, with lines labeled by the vertices of G and ij-th entry defined as:

$$[M(G)]_{i,j} = \begin{cases} 1, & \text{if } (v_i,\ v_j) \in E(G); \\ 0, & \text{if } (v_i,\ v_j) \notin E(G). \end{cases}$$

If $\{v_i,\ v_j\} \in E(G)$ two distinct vertices v_i and v_j are said to be *adjacent*. The *degree* of a vertex $v_i \in V(G)$ is the number of edges adjacent to v_i, we denote it as $d_G(v_i)$. $d_G = \sum_{i=1}^{n} d_G(v_i)$ is called as the *degree sum*. Notice that $d_G = 2|E(G)|$. The *degree matrix* of G is an $n \times n$ matrix, denoted as $\Delta(G)$, with ij-th entry defined as:

$$[\Delta(G)]_{i,\ j} = \begin{cases} d_G(v_i), & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

The *combinatorial laplacian matrix* of a graph G is the symmetric positive semidefinite matrix

$$L(G) = \Delta(G) - M(G).$$

The density matrix of G of a graph G is the matrix

$$\rho(G) = \frac{1}{d_G} L(G).$$

Recall that a graph is called *complete*[14] if every pair of vertices are adjacent, and the *complete graph* on n vertices is denoted by K_n. Obviously, $\rho(K_n) = \frac{1}{n(n-1)}(nI_n - J_n)$, where I_n and J_n is the $n \times n$ identity matrix and the $n \times n$ all-ones matrix, respectively. A *star graph* on n vertices $\alpha_1, \alpha_2, \cdots, \alpha_n$, denoted by $K_{1,n-1}$, is the graph whose set of edges is $\{\{\alpha_1, \alpha_i\} : i = 2, 3, \cdots, n\}$, we have

$$\rho(K_{1,n-1}) = \frac{1}{2(n-1)} \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & & & \\ -1 & & 1 & & \\ \vdots & & & \ddots & \\ -1 & & & & 1 \end{pmatrix}.$$

Let G be a graph which has only a edge. Then the density matrix of G is pure. The density matrix of a graph is a uniform mixture of pure density matrices, that is, for a graph G on n vertices $v_1, v_2, \cdots, v_n$, having s edges $\{v_{i_1}, v_{j_1}\}, \{v_{i_2}, v_{j_2}\}, \cdots, \{v_{i_s}, v_{j_s}\}$, where $1 \leq i_1, j_1, i_2, j_2, \cdots, i_k, j_k \leq n$,

$$\rho(G) = \frac{1}{s} \sum_{k=1}^{s} \rho(H_{i_k j_k}),$$

here $H_{i_k j_k}$ is the factor of G such that

$$[M(H_{i_k j_k})]_{u,\, w} = \begin{cases} 1, & \text{if } u = i_k \text{ and } w = j_k \text{ or } w = i_k \text{ and } u = j_k; \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that $\rho(H_{i_k j_k})$ is pure.

Before we discuss the tripartite entanglement properties of the density matrices of graphs we will at first recall briefly the definition of the tripartite separability:

Definition 1 The state ρ acting on $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{C}}$ is called *tripartite separability* if it can be written in the form

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i \otimes \rho_C^i,$$

where $\rho_A^i = |\alpha_A^i\rangle\langle\alpha_A^i|$, $\rho_B^i = |\beta_B^i\rangle\langle\beta_B^i|$, $\rho_C^i = |\gamma_C^i\rangle\langle\gamma_C^i|$, $\sum_i p_i = 1$, $p_i \geq 0$ and $|\alpha_A^i\rangle$, $|\beta_B^i\rangle$, $|\gamma_C^i\rangle$ are normalized pure states of subsystems A, B and C, respectively. Otherwise, the state is called *entangled*.

Now we define the tensor product of three graphs. The *tensor product of graphs* G_A, G_B, G_C, denoted by $G_A \otimes G_B \otimes G_C$, is the graph whose adjacency matrix is $M(G_A \otimes G_B \otimes G_C) = M(G_A) \otimes M(G_B) \otimes M(G_C)$. Whenever we consider a graph $G_A \otimes G_B \otimes G_C$, where G_A is on m vertices, G_B is on p vertices and G_C is on q vertices, the tripartite separability of $\rho(G_A \otimes G_B \otimes G_C)$ is described with respect to the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, where $\mathcal{H}_A$ is the space spanned by the orthonormal basis

$\{|u_1\rangle,\ |u_2\rangle,\ \cdots,\ |u_m\rangle\}$ associated to $V(G_A)$, $\mathcal{H}_B$ is the space spanned by the orthonormal basis $\{|v_1\rangle,\ |v_2\rangle,\ \cdots,\ |v_p\rangle\}$ associated to $V(G_B)$ and $\mathcal{H}_C$ is the space spanned by the orthonormal basis $\{|w_1\rangle,\ |w_2\rangle,\ \cdots,\ |w_q\rangle\}$ associated to $V(G_C)$. The vertices of $G_A \otimes G_B \otimes G_C$ are taken as $\{u_iv_jw_k,\ 1 \leq i \leq m,\ 1 \leq j \leq p,\ 1 \leq k \leq q\}$. We associate $|u_i\rangle|v_j\rangle|w_k\rangle$ to $u_iv_jw_k$, where $1 \leq i \leq m,\ 1 \leq j \leq p,\ 1 \leq k \leq q$. In conjunction with this, whenever we talk about tripartite separability of any graph G on n vertices, $|\alpha_1\rangle,\ |\alpha_2\rangle,\ \cdots,\ |\alpha_n\rangle$, we consider it in the space $C^m \otimes C^p \otimes C^q$, where $n = mpq$. The vectors $|\alpha_1\rangle,\ |\alpha_2\rangle,\ \cdots,\ |\alpha_n\rangle$ are taken as follows: $|\alpha_1\rangle = |u_1\rangle|v_1\rangle|w_1\rangle,\ |\alpha_2\rangle = |u_1\rangle|v_1\rangle|w_2\rangle,\ \cdots,\ |\alpha_n\rangle = |u_m\rangle|v_p\rangle|w_q\rangle$.

To investigate the tripartite entanglement properties of the density matrices of graphs it is necessary to recall the well known positive partial transposition criterion (i.e. Peres criterion). It makes use of the notion of *partial transpose* of a density matrix. Here we will only recall the Peres criterion for the tripartite states. Consider a $n \times n$ matrix ρ_{ABC} acting on $C^m_A \otimes C^p_B \otimes C^q_C$, where $n = mpq$. The partial transpose of ρ_{ABC} with respect to the systems $A,\ B,\ C$ are the matrices $\rho^{T_A}_{ABC},\ \rho^{T_B}_{ABC},\ \rho^{T_C}_{ABC}$, respectively, and with $(i,\ j,\ k;\ i',\ j',\ k')$-th entry defined as follows:

$$[\rho^{T_A}_{ABC}]_{i,\ j,\ k;\ i',\ j',\ k'} = \langle u_{i'}v_jw_k|\rho_{ABC}|u_iv_{j'}w_{k'}\rangle,$$
$$[\rho^{T_B}_{ABC}]_{i,\ j,\ k;\ i',\ j',\ k'} = \langle u_iv_{j'}w_k|\rho_{ABC}|u_{i'}v_jw_{k'}\rangle,$$
$$[\rho^{T_C}_{ABC}]_{i,\ j,\ k;\ i',\ j',\ k'} = \langle u_iv_jw_{k'}|\rho_{ABC}|u_{i'}v_{j'}w_k\rangle,$$

where $1 \leq i,\ i' \leq m;\ 1 \leq j,\ j' \leq p$ and $1 \leq k,\ k' \leq q$.

For separability of ρ_{ABC} we have the following criterion:

Peres criterion[13] If ρ is a separable density matrix acting on $C^m \otimes C^p \otimes C^q$, then $\rho^{T_A},\ \rho^{T_B},\ \rho^{T_C}$ are positive semidefinite.

Lemma 1 The density matrix of the tensor product of three graphs is tripartite separable.

Proof. Let G_1 be a graph on n vertices, $u_1,\ u_2,\ \cdots,\ u_n$, and m edges, $\{u_{c_1}, u_{d_1}\}, \cdots, \{u_{c_m}, u_{d_m}\}$, $1 \leq c_1,\ d_1,\ \cdots,\ c_m,\ d_m \leq n$. Let G_2 be a graph on k vertices, $v_1,\ v_2,\ \cdots,\ v_k$, and e edges, $\{v_{i_1},\ v_{j_1}\},\ \cdots,\ \{v_{i_e},\ v_{j_e}\}$, $1 \leq i_1,\ j_1,\ \cdots,\ i_e,\ j_e \leq k$. Let G_3 be a graph on l vertices, $w_1,\ w_2,\ \cdots,\ w_l$, and f edges, $\{w_{r_1},\ w_{s_1}\},\ \cdots,\ \{w_{r_f},\ w_{s_f}\}$, $1 \leq r_1,\ s_1,\ \cdots,\ r_f,\ s_f \leq l$. Then

$$\rho(G_1) = \frac{1}{m}\sum_{p=1}^{m}\rho(H_{c_pd_p}),\ \rho(G_2) = \frac{1}{e}\sum_{q=1}^{e}\rho(L_{i_qj_q}),\ \rho(G_3) = \frac{1}{f}\sum_{t=1}^{f}\rho(Q_{r_ts_t}).$$

Therefore

$$\rho(G_1 \otimes G_2 \otimes G_3)$$
$$= \frac{1}{d_{G_1\otimes G_2\otimes G_3}}[\Delta(G_1 \otimes G_2 \otimes G_3) - M(G_1 \otimes G_2 \otimes G_3)]$$

$$\begin{aligned}
&= \frac{1}{d_{G_1\otimes G_2\otimes G_3}}\sum_{p=1}^{m}\sum_{q=1}^{e}\sum_{t=1}^{f}[\Delta(H_{c_pd_p}\otimes L_{i_qj_q}\otimes Q_{r_ts_t})-M(H_{c_pd_p}\otimes L_{i_qj_q}\otimes Q_{r_ts_t})]\\
&= \frac{1}{d_{G_1\otimes G_2\otimes G_3}}\sum_{p=1}^{m}\sum_{q=1}^{e}\sum_{t=1}^{f}8\rho(H_{c_pd_p}\otimes L_{i_qj_q}\otimes Q_{r_ts_t})\\
&= \frac{1}{mef}\sum_{p=1}^{m}\sum_{q=1}^{e}\sum_{t=1}^{f}\rho(H_{c_pd_p}\otimes L_{i_qj_q}\otimes Q_{r_ts_t})\\
&= \frac{1}{mef}\sum_{p=1}^{m}\sum_{q=1}^{e}\sum_{t=1}^{f}\frac{1}{8}[\Delta(H_{c_pd_p})\otimes\Delta(L_{i_qj_q})\otimes\Delta(Q_{r_ts_t})-M(H_{c_pd_p})\otimes M(L_{i_qj_q})\otimes M(Q_{r_ts_t})]\\
&= \frac{1}{mef}\sum_{p=1}^{m}\sum_{q=1}^{e}\sum_{t=1}^{f}\frac{1}{4}[\rho(H_{c_pd_p})\otimes\rho(L_{i_qj_q})\otimes\rho(Q_{r_ts_t})\\
&\quad+\rho_+(H_{c_pd_p})\otimes\rho(L_{i_qj_q})\otimes\rho_+(Q_{r_ts_t})+\rho(H_{c_pd_p})\otimes\rho_+(L_{i_qj_q})\otimes\rho_+(Q_{r_ts_t})\\
&\quad+\rho_+(H_{c_pd_p})\otimes\rho_+(L_{i_qj_q})\otimes\rho(Q_{r_ts_t})],
\end{aligned}$$

where

$$\rho_+(H_{c_pd_p})\stackrel{\text{def}}{=}\Delta(H_{c_pd_p})-\rho(H_{c_pd_p})=\frac{1}{2}\big(\Delta(H_{c_pd_p})+M(H_{c_pd_p})\big),$$

$$\rho_+(L_{i_qj_q})\stackrel{\text{def}}{=}\Delta(L_{i_qj_q})-\rho(L_{i_qj_q})=\frac{1}{2}\big(\Delta(L_{i_qj_q})+M(L_{i_qj_q})\big),$$

$$\rho_+(Q_{r_ts_t})\stackrel{\text{def}}{=}\Delta(Q_{r_ts_t})-\rho(Q_{r_ts_t})=\frac{1}{2}\big(\Delta(Q_{r_ts_t})+M(Q_{r_ts_t})\big),$$

the fourth equality follows from $d_{G_1\otimes G_2\otimes G_3}=8mef$ and the fifth equality follows from the definition of tensor products of graphs.

Notice that $\rho_+(H_{c_pd_p}),\ \rho_+(L_{i_qj_q}),\ \rho_+(Q_{r_ts_t})$ are all density matrices. Let

$$\rho_+(G_1)=\frac{1}{m}\sum_{p=1}^{m}\rho_+(H_{c_pd_p}),\quad \rho_+(G_2)=\frac{1}{e}\sum_{q=1}^{e}\rho_+(L_{i_qj_q}),\quad \rho_+(G_3)=\frac{1}{f}\sum_{t=1}^{f}\rho_+(Q_{r_ts_t}).$$

Then

$$\begin{aligned}
\rho(G_1\otimes G_2\otimes G_3)= &\ \tfrac{1}{4}[\rho(G_1)\otimes\rho(G_2)\otimes\rho(G_3)+\rho_+(G_1)\otimes\rho(G_2)\otimes\rho_+(G_3)\\
&+\rho(G_1)\otimes\rho_+(G_2)\otimes\rho_+(G_3)+\rho_+(G_1)\otimes\rho_+(G_2)\otimes\rho(G_3)].
\end{aligned}$$

So we have that $\rho(G)$ is tripartite separable. □

Remark We associate to the vertices $\alpha_1,\ \alpha_2,\ \cdots,\alpha_n$ of a graph G an orthonormal basis $\{|\alpha_1\rangle,\ |\alpha_2\rangle,\ \cdots,\ |\alpha_n\rangle\}$. In terms of this basis, the *uw*-th elements of the matrices $\rho(H_{c_pd_p})$ and $\rho_+(H_{c_pd_p})$ are given by $\langle\alpha_u|\rho(H_{c_pd_p})|\alpha_w\rangle$ and $\langle\alpha_u|\rho_+(H_{c_pd_p})|\alpha_w\rangle$, respectively. In this basis we have

$$\rho(H_{c_pd_p})=P[\frac{1}{\sqrt{2}}(|\alpha_{c_p}\rangle-|\alpha_{d_p}\rangle)],\ \rho_+(H_{c_pd_p})=P[\frac{1}{\sqrt{2}}(|\alpha_{c_p}\rangle+|\alpha_{d_p}\rangle)].$$

Lemma 2 The matrix $\sigma = \frac{1}{4}P[\frac{1}{\sqrt{2}}(|ijk\rangle - |rst\rangle)] + \frac{1}{4}P[\frac{1}{\sqrt{2}}(|ijt\rangle - |rsk\rangle)] + \frac{1}{4}P[\frac{1}{\sqrt{2}}(|isk\rangle - |rjt\rangle)] + \frac{1}{4}P[\frac{1}{\sqrt{2}}(|rjk\rangle - |ist\rangle)]$ is a density matrix and tripartite separable.

Proof. Since the project operator is semipositive, σ is semipositive. By computing one can get $tr(\sigma) = 1$, so σ is a density matrix. Let

$$|u^{\pm}\rangle = \frac{1}{\sqrt{2}}(|i\rangle \pm |r\rangle), \quad |v^{\pm}\rangle = \frac{1}{\sqrt{2}}(|j\rangle \pm |s\rangle), \quad |w^{\pm}\rangle = \frac{1}{\sqrt{2}}(|k\rangle \pm |t\rangle).$$

We obtain

$$\sigma = \frac{1}{4}P[|u^+\rangle|v^-\rangle|w^+\rangle] + \frac{1}{4}P[|u^+\rangle|v^+\rangle|w^-\rangle] + \frac{1}{4}P[|u^-\rangle|v^-\rangle|w^-\rangle] + \frac{1}{4}P[|u^-\rangle|v^+\rangle|w^+\rangle],$$

thus σ is tripartite separable. □

Lemma 3 For any $n = mpq$, the density matrix $\rho(K_n)$ is tripartite separable in $C^m \otimes C^p \otimes C^q$.

Proof. Since $M(K_n) = J_n - I_n$, where J_n is the $n \times n$ all-ones matrix and I_n is the $n \times n$ identity matrix, whenever there is an edge $\{u_iv_jw_k,\ u_rv_sw_t\}$, there must be entangled edges $\{u_rv_jw_k,\ u_iv_sw_t\}$, $\{u_iv_sw_k,\ u_rv_jw_t\}$ and $\{u_iv_jw_t,\ u_rv_sw_k\}$. The result follows from Lemma 2. □

Lemma 4 The complete graph on $n > 1$ vertices is not a tensor product of three graphs.

Proof. It is obvious that K_n is not a tensor product of three graphs if n is a prime or a product of two primes. Thus we can assume that n is a product of three or more primes. Let $n = mpq,\ m,\ p,\ q > 1$. Suppose that there exist three graphs $G_1,\ G_2$ and G_3 on $m,\ p$ and q vertices, respectively, such that $K_{mpq} = G_1 \otimes G_2 \otimes G_3$. Let $|E(G_1)| = r,\ |E(G_2)| = s,\ |E(G_3)| = t$. Then, by the degree sum formula, $2r \leq m(m-1),\ 2s \leq p(p-1),\ 2t \leq q(q-1)$. Hence

$$2r \cdot 2s \cdot 2t \leq mpq(m-1)(p-1)(q-1) = mpq(mpq - mp - mq - pq + m + p + q - 1).$$

Now, observe that

$$|V(G_1 \otimes G_2 \otimes G_3)| = mpq, \quad |E(G_1 \otimes G_2 \otimes G_3)| = 4rst.$$

Therefore,

$$G_1 \otimes G_2 \otimes G_3 = K_{mpq} \iff mpq(mpq-1) = 2 \cdot 4rst,$$

so

$$mpq(mpq-1) = 8rst \leq mpq(mpq - mp - mq - pq + m + p + q - 1).$$

It follows that $mp + mq + pq - m - p - q \leq 0$, that is $m(p-1) + q(m-1) + p(q-1) \leq 0$. As $m, p, q \geq 1$ we get $m(p-1) + q(m-1) + p(q-1) = 0$. It yields that $m = p = q = 1$. □

Theorem 1 Given a graph $G_1 \otimes G_2 \otimes G_3$, the density matrix $\rho(G_1 \otimes G_2 \otimes G_3)$ is tripartite separable. However if a density matrix $\rho(L)$ is tripartite separable it does not necessarily mean that $L = L_1 \otimes L_2 \otimes L_3$, for some graphs $L_1,\ L_2$ and L_3.

Proof. The result follows from Lemmas 1, 3 and 4. □

Theorem 2 The density matrix $\rho(K_{1,\ n-1})$ is tripartite entangled for $n = mpq \geq 8$.

Proof. Consider a graph $G = K_{1,\ n-1}$ on $n = mpq$ vertices, $|\alpha_1\rangle,\ |\alpha_2\rangle,\ \cdots,\ |\alpha_n\rangle$. Then

$$\rho(G) = \frac{1}{n-1}\sum_{k=2}^{n}\rho(H_{1k}) = \frac{1}{n-1}\sum_{k=2}^{n}P[\frac{1}{\sqrt{2}}(|\alpha_1\rangle - |\alpha_n\rangle)].$$

We are going to examine tripartite separability of $\rho(G)$ in $C_A^m \otimes C_B^p \otimes C_C^q$, where $C_A^m,\ C_B^p$ and C_C^q are associated to three quantum systems $\mathcal{H}_A,\ \mathcal{H}_B$ and $\mathcal{H}_C$, respectively. Let $\{|u_1\rangle,\ |u_2\rangle,\ \cdots,\ |u_m\rangle\}$, $\{|v_1\rangle,\ |v_2\rangle,\ \cdots,\ |v_p\rangle\}$ and $\{|w_1\rangle,\ |w_2\rangle,\ \cdots,\ |w_q\rangle\}$ be orthonormal basis of $C_A^m,\ C_B^p$ and C_C^q, respectively. So,

$$\rho(G) = \frac{1}{n-1}\sum_{k=2}^{n}P[\frac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_{r_k}v_{s_k}w_{t_k}\rangle)],$$

where $k = (r_k-1)pq + (s_k-1) + t_k,\ 1 \leq r_k \leq m,\ 1 \leq s_k \leq p,\ 1 \leq t_k \leq q$. Hence

$$\begin{aligned}\rho(G) = \ & \frac{1}{n-1}\Big\{\sum_{i=2}^{m}P[\frac{1}{\sqrt{2}}(|u_1\rangle - |u_i\rangle)|v_1\rangle|w_1\rangle] + \sum_{j=2}^{p}P[|u_1\rangle\frac{1}{\sqrt{2}}(|v_1\rangle - |v_j\rangle)|w_1\rangle] \\ & + \sum_{k=2}^{q}P[|u_1\rangle|v_1\rangle\frac{1}{\sqrt{2}}(|w_1\rangle - |w_k\rangle)] + \sum_{i=2}^{m}\sum_{j=2}^{p}P[\frac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_iv_jw_1\rangle)] \\ & + \sum_{j=2}^{p}\sum_{k=2}^{q}P[\frac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_1v_jw_k\rangle)] + \sum_{i=2}^{m}\sum_{k=2}^{q}P[\frac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_iv_1w_k\rangle)] \\ & + \sum_{i=2}^{m}\sum_{j=2}^{p}\sum_{k=2}^{q}P[\frac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_iv_jw_k\rangle)]\Big\}.\end{aligned}$$

Consider now the following projectors:

$$P = |u_1\rangle\langle u_1| + |u_2\rangle\langle u_2|, \quad Q = |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2| \ \text{ and } \ R = |w_1\rangle\langle w_1| + |w_2\rangle\langle w_2|.$$

Then

$$\begin{aligned}&(P\otimes Q\otimes R)\rho(G)(P\otimes Q\otimes R) \\ &= \tfrac{1}{n-1}\Big\{\tfrac{n-8}{2}P[|u_1v_1w_1\rangle] + P[\tfrac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_1v_1w_2\rangle)] \\ &\quad + P[\tfrac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_1v_2w_1\rangle)] + P[\tfrac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_2v_1w_1\rangle)] \\ &\quad + P[\tfrac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_1v_2w_2\rangle)] + P[\tfrac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_2v_1w_2\rangle)] \\ &\quad + P[\tfrac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_2v_2w_1\rangle)] + P[\tfrac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_2v_2w_2\rangle)]\Big\}.\end{aligned}$$

In the basis

$$\{|u_1v_1w_1\rangle, |u_1v_1w_2\rangle, |u_1v_2w_1\rangle, |u_1v_2w_2\rangle, |u_2v_1w_1\rangle,\ |u_2v_1w_2\rangle,\ |u_2v_2w_1\rangle, |u_2v_2w_2\rangle\},$$

we have

$$[(P\otimes Q\otimes R)\rho(G)(P\otimes Q\otimes R)]^{T_A}=\frac{1}{n-1}\begin{pmatrix} \frac{n-1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0\\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0\\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0\\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0\\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

The eigenpolynomial of the above matrix is

$$\Big(\lambda-\frac{1}{2(n-1)}\Big)^5\Big(\lambda^3-\frac{n+1}{2(n-1)}\lambda^2+\frac{n-4}{2(n-1)^2}\lambda+\frac{n+4}{4(n-1)^3}\Big),$$

so the eigenvalues of the matrix are $\frac{1}{2(n-1)}$ (with multiplicity 5) and the roots of the polynomial $\lambda^3-\frac{n+1}{2(n-1)}\lambda^2+\frac{n-4}{2(n-1)^2}\lambda+\frac{n+4}{4(n-1)^3}$. Let the roots of this polynomial of degree three be $\lambda_1,\ \lambda_2$ and λ_3. Then $\lambda_1\lambda_2\lambda_3=-\frac{n+4}{4(n-1)^3}<0$, so one of the three roots must be negative, i.e., there must be a negative eigenvalue of the above matrix. Hence, by Peres criterion, the matrix $(P\otimes Q\otimes R)\rho(G)(P\otimes Q\otimes R)$ is tripartite entangled and then $\rho(G)$ is tripartite entangled. □

3 A sufficient and necessary condition of tripartite separability

Definition 2 *Partially transposed graph* $G^{\Gamma_A}=(V,\ E')$, (i.e. the partial transpose of a graph $G=(V,E)$ with respect to $\mathcal{H}_A$) is the graph such that

$$\{u_iv_jw_k,\ u_rv_sw_t\}\in E' \text{ if and only if } \{u_rv_jw_k,\ u_iv_sw_t\}\in E.$$

Partially transposed graphs G^{Γ_B} and G^{Γ_C} (with respect to $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively) can be defined in a similar way.

For tripartite states we denote $\Delta(G)=\Delta(G^{\Gamma_A})=\Delta(G^{\Gamma_B})=\Delta(G^{\Gamma_C})$ as the *degree condition*. Hildebrand *et al.*[12] proved that the degree criterion is equivalent to PPT criterion. It is easy to show that this equivalent condition is still true for the tripartite states. Thus from Peres criterion we can get:

Theorem 3 Let $\rho(G)$ be the density matrix of a graph on $n=mpq$ vertices. If $\rho(G)$ is separable in $C_A^m\otimes C_B^p\otimes C_C^q$, then $\Delta(G)=\Delta(G^{\Gamma_A})=\Delta(G^{\Gamma_B})=\Delta(G^{\Gamma_C})$.

Let G be a graph on $n = mpq$ vertices: $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n$ and f edges: $\{\alpha_{i_1},\ \alpha_{j_1}\}$, $\{\alpha_{i_2},\ \alpha_{j_2}\}, \cdots, \{\alpha_{i_f}, \alpha_{j_f}\}$. Let vertices $\alpha_s = u_i v_j w_k$, where $s = (i-1)pq + (j-1)q + k, 1 \leq i \leq m,\ 1 \leq j \leq p,\ 1 \leq k \leq q$. The vectors $|u_i\rangle' s,\ |v_j\rangle' s,\ |w_k\rangle' s$ form orthonormal bases of C^m, C^p and C^q, respectively. The edge $\{u_i v_j w_k,\ u_r v_s w_t\}$ is said to be *entangled* if $i \neq r,\ j \neq s,\ k \neq t$.

Consider a cuboid with mpq points whose length is m, width is p and height is q, such that the distance between two neighboring points on the same line is 1. A *nearest point graph* is a graph whose vertices are identified with the points of the cuboid and the edges have length 1, $\sqrt{2}$ and $\sqrt{3}$.

The degree condition is still a sufficient condition of the tripartite separability for the density matrix of a nearest point graph.

Theorem 4 Let G be a nearest point graph on $n = mpq$ vertices. If $\Delta(G) = \Delta(G^{\Gamma_A}) = \Delta(G^{\Gamma_B}) = \Delta(G^{\Gamma_C})$, then the density matrix $\rho(G)$ is tripartite separable in $C^m_A \otimes C^p_B \otimes C^q_C$.

Proof. Let G be a nearest point graph on $n = mpq$ vertices and f edges. We associate to G the orthonormal basis $\{|\alpha_l\rangle : l = 1,\ 2,\ \cdots,\ n\} = \{|u_i\rangle \otimes |v_j\rangle \otimes |w_k\rangle :\ i = 1,\ 2,\ \cdots, m;\ j = 1,\ 2,\ \cdots,\ p;\ k = 1,\ 2,\ \cdots,\ q\}$, where $\{|u_i\rangle :\ i = 1,\ 2,\ \cdots,\ m\}$ is an orthonormal basis of C^m_A, $\{|v_j\rangle :\ j = 1,\ 2,\ \cdots,\ p\}$ is an orthonormal basis of C^p_B and $\{|w_k\rangle :\ i = 1,\ 2,\ \cdots,\ q\}$ is an orthonormal basis of C^q_C. Let $i,\ r \in \{1,\ 2,\ \cdots,\ m\},\ j,\ s \in \{1,\ 2,\ \cdots,\ p\},\ k,\ t \in \{1,\ 2,\ \cdots,\ q\},\ \lambda_{ijk,\ rst} \in \{0,\ 1\}$ be defined by

$$\lambda_{ijk,\ rst} = \begin{cases} 1, & \text{if } (u_i v_j w_k,\ u_r v_s w_t) \in E(G); \\ 0, & \text{if } (u_i v_j w_k,\ u_r v_s w_t) \notin E(G), \end{cases}$$

where $i,\ j,\ k,\ r,\ s,\ t$ satisfy either of the following seven conditions:

- $i = r,\ j = s,\ k = t + 1;$
- $i = r,\ j = s + 1,\ k = t;$
- $i = r + 1,\ j = s,\ k = t;$
- $i = r,\ j = s + 1,\ k = t + 1;$
- $i = r + 1,\ j = s + 1, k = t;$
- $i = r + 1,\ j = s,\ k = t + 1;$
- $i = r + 1,\ j = s + 1,\ k = t + 1.$

Let $\rho(G),\ \rho(G^{\Gamma_A}),\ \rho(G^{\Gamma_B})$ and $\rho(G^{\Gamma_C})$ be the density matrices corresponding to the graph $G,\ G^{\Gamma_A},\ G^{\Gamma_B}$ and G^{Γ_C}, respectively. Thus

$$\begin{aligned} \rho(G) &= \tfrac{1}{2f}(\Delta(G) - M(G)), & \rho(G^{\Gamma_A}) &= \tfrac{1}{2f}(\Delta(G^{\Gamma_A}) - M(G^{\Gamma_A})), \\ \rho(G^{\Gamma_B}) &= \tfrac{1}{2f}(\Delta(G^{\Gamma_B}) - M(G^{\Gamma_B})), & \rho(G^{\Gamma_C}) &= \tfrac{1}{2f}(\Delta(G^{\Gamma_C}) - M(G^{\Gamma_C})). \end{aligned}$$

Let G_1 be the subgraph of G whose edges are all the entangled edges of G. An edge $\{u_iv_jw_k,\ u_rv_sw_t\}$ is entangled if $i \neq r,\ j \neq s,\ k \neq t$. Let G_1^A be the subgraph of G^{Γ_A} corresponding to all the entangled edges of G^{Γ_A}, G_1^B be the subgraph of G^{Γ_B} corresponding to all the entangled edges of G^{Γ_B}, and G_1^C be the subgraph of G^{Γ_C} corresponding to all the entangled edges of G^{Γ_C}. Obviously, $G_1^A = (G_1)^{\Gamma_A},\ G_1^B = (G_1)^{\Gamma_B},\ G_1^C = (G_1)^{\Gamma_C}$. We have

$$\rho(G_1) = \frac{1}{f}\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{q}\lambda_{ijk,\ rst}P[\frac{1}{\sqrt{2}}(|u_iv_jw_k\rangle - |u_rv_sw_t\rangle)],$$

where $i,\ j,\ k;\ r,\ s,\ t$ must satisfy either of the above seven conditions. We can get $\rho(G_1^A),\ \rho(G_1^B)$ and $\rho(G_1^C)$ by commuting the index of $u,\ v,\ w$ in the above equation, respectively. Also we have

$$\Delta(G_1) = \frac{1}{2f}\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{q}\lambda_{ijk,\ rst}P[|u_iv_jw_k\rangle],$$

where $i,\ j,\ k;\ r,\ s,\ t$ must satisfy either of the above seven conditions. We can get $\Delta(G_1^A),\ \Delta(G_1^B)$ and $\Delta(G_1^C)$ by commuting the index of λ with respect to the Hilbert space $\mathcal{H}_A,\ \mathcal{H}_B,\ \mathcal{H}_C$, respectively. Let $G_2,\ G_2^A,\ G_2^B$ and G_2^C be the subgraph of $G,\ G^A,\ G^B$ and G^C containing all the unentangled edges, respectively. It is obvious that $\Delta(G_2) = \Delta(G_2^{\Gamma_A}) = \Delta(G_2^{\Gamma_B}) = \Delta(G_2^{\Gamma_C})$. So $\Delta(G) = \Delta(G^{\Gamma_A}) = \Delta(G^{\Gamma_B}) = \Delta(G^{\Gamma_C})$ if and only if $\Delta(G_1) = \Delta(G_1^{\Gamma_A}) = \Delta(G_1^{\Gamma_B}) = \Delta(G_1^{\Gamma_C})$. The degree condition implies that

$$\lambda_{ijk,\ rst} = \lambda_{rjk,\ ist} = \lambda_{isk,\ rjt} = \lambda_{ijt,\ rsk},$$

for any $i,\ r \in \{1,\ 2,\ \cdots,\ m\},\ j,\ s \in \{1,\ 2,\ \cdots,\ p\},\ \ k,\ t \in \{1,\ 2,\ \cdots,\ q\}$. The above equation shows that whenever there is an entangled edge $\{u_iv_jw_k,\ u_rv_sw_t\}$ in G (here we must have $i \neq r,\ j \neq s,\ k \neq t$), there must be the entangled edges $\{u_rv_jw_k,\ u_iv_sw_t\}$, $\{u_iv_sw_k,\ u_rv_jw_t\}$ and $\{u_iv_jw_t,\ u_rv_sw_k\}$ in G. Let

$$\begin{aligned}\rho(i,\ j,\ k;\ r,\ s,\ t) = \ & \tfrac{1}{4}(P[\tfrac{1}{\sqrt{2}}(|u_iv_jw_k\rangle - |u_rv_sw_t\rangle)] + P[\tfrac{1}{\sqrt{2}}(|u_rv_jw_k\rangle - |u_iv_sw_t\rangle)]\\ & + P[\tfrac{1}{\sqrt{2}}(|u_iv_sw_k\rangle - |u_rv_jw_t\rangle)] + P[\tfrac{1}{\sqrt{2}}(|u_iv_jw_t\rangle - |u_rv_sw_k\rangle)]).\end{aligned}$$

By Lemma 2, we know $\rho(i,\ j,\ k;\ r,\ s,\ t)$ is tripartite separable in $C_A^m \otimes C_B^p \otimes C_C^q$. By Theorem 3 in [11] we can easily get $\rho(G_2)$ is tripartite separable in $C_A^m \otimes C_B^p \otimes C_C^q$. □

From Theorems 3 and 4 we can obtain the following corollary which is a sufficient and necessary criterion (we called *degree-criterion*) of the density matrix of a nearest point graph:

Corollary 1 Let G be a nearest point graph on $n = mpq$ vertices, then the density matrix $\rho(G)$ is tripartite separable in $C_A^m \otimes C_B^p \otimes C_C^q$ if and only if $\Delta(G) = \Delta(G^{\Gamma_A}) = \Delta(G^{\Gamma_B}) = \Delta(G^{\Gamma_C})$.

Example Let G be a graph on $12 = 3 \times 2 \times 2$ vertices, having a unique edge $\{u_1v_1w_1,\ u_2v_2w_2\}$. Then we have

$$\rho(G) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The partially transposed graph G^{Γ_A} is a graph on 12 vertices and has an edge $\{u_2v_1w_1,$ $u_1v_2w_2\}$. Then

$$\rho(G^{\Gamma_A}) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Obviously, the degree matrices of G and G^{Γ_A} are different. The eigenvalues of $\rho(G)^{T_A}$ are 0 (with multiplicity 8), $\frac{1}{2}$ (with multiplicity 3) and $-\frac{1}{2}$, so $\rho(G)^{T_A}$ is not positive semidefinite. According to Peres criterion, $\rho(G)$ is tripartite entangled. □

Two graphs G and H are said to be *isomorphic*, denoted as $G \cong H$, if there is an isomorphism between $V(G)$ and $V(H)$, i.e., there is a permutation matrix P such that $PM(G)P^T = M(H)$.[8]

Theorem 5 Let G and H be two graphs on $n = mpq$ vertices. If $\rho(G)$ is tripartite entangled in $C^m \otimes C^p \otimes C^q$ and $G \cong H$, then $\rho(H)$ is not necessarily tripartite entangled in $C^m \otimes C^p \otimes C^q$.

Proof. Let G be the graph introduced in the above example. Then $\rho(G)$ is tripartite entangled. Let H be a graph on 12 vertices, having an edge $\{u_1v_1w_1,\ u_1v_1w_2\}$. Obviously, G is isomorphic to H. However,

$$\rho(H) = P[\frac{1}{\sqrt{2}}(|u_1v_1w_1\rangle - |u_1v_1w_2\rangle)] = |u_1\rangle\langle u_1| \otimes |v_1\rangle\langle v_1| \otimes |w^+\rangle\langle w^+|,$$

where $|w^+\rangle = \frac{1}{\sqrt{2}}(|w_1\rangle - |w_2\rangle)$, shows that $\rho(H)$ is tripartite separable. □

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