On Subsequence Sums of a Zero-sum Free Sequence

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Abstract

Let G be a finite abelian group with exponent m, and let S be a sequence of elements in G. Let $f(S)$ denote the number of elements in G which can be expressed as the sum over a nonempty subsequence of S. In this paper, we show that, if $|S| = m$ and S contains no nonempty subsequence with zero sum, then $f(S) \geq 2m - 1$. This answers an open question formulated by Gao and Leader. They proved the same result with the restriction $(m, 6) = 1$.

1 Introduction

Let G be a finite abelian group of order n and exponent m, additively written. Let $S = (a_1, \ldots, a_k)$ be a sequence of elements in G. By $\sum(S)$ we denote the set that consists of all elements of G that can be expressed as the sum over a nonempty subsequence of S, i.e.,

$$\sum(S) = \{a_{i_1} + \ldots + a_{i_l} : \ 1 \leq i_1 < \ldots < i_l \leq k\}.$$

We write $f(S) = |\sum(S)|$. If $0 \notin \sum(S)$, we call S a zero-sum free sequence.

Let $\sum_n(S)$ denote the set that consists of all elements in G which can be expressed as the sum over a subsequence of S of length n, i.e.,

$$\sum\nolimits_n(S) = \{a_{i_1} + \ldots + a_{i_n} : \ 1 \leq i_1 < \ldots < i_n \leq k\}.$$

If U is a subsequence of S, we write SU^{-1} for the subsequence obtained by deleting the terms of U from S; if U and V are disjoint subsequences of S, we write UV for the subsequence obtained by adjoining the terms of U to V; if U is a subsequence of S, we write $U|S$.

Let $D(G)$ be the Davenport's constant of G, i.e., the smallest integer d such that every sequence S of elements in G with $|S| \geq d$ satisfies $0 \in \sum(S)$; let $s(G)$ be the smallest integer t such that every sequence of elements in G with $|S| \geq t$ satisfies $0 \in \sum_n(S)$. In 1961, Erdős, Ginzburg and Ziv proved $s(G) \leq 2n - 1$ for any finite abelian group of order n. This result is now well known as the Erdős-Ginzburg-Ziv theorem. In 1996, Gao proved $s(G) = D(G) + n - 1$ for any finite abelian group of order n. In 1999, Bollobás and Leader investigated the problem of determining the minimal cardinality of $|\sum_n(S)|$ in terms of the length of $|S|$ assuming that $0 \notin \sum_n(S)$.

For every positive integer r in the interval $\{1, \dots, D(G) - 1\}$, where $D(G)$ is the Davenport constant of G, let

$$f_G(r) = min_{S,|S|=r}|\sum(S)|,$$

where S runs over all zero-sum free sequences of r elements in G.

In 2006, Gao and Leader proved the following result:

Theorem A.[8] Let S be a sequence of elements in a finite abelian group of order n. Suppose $|S| \geq n$ and $0 \notin \sum_n(S)$. Set $r = |S| - n + 1$. Then, $|\sum_n(S)| \geq f_G(r)$. The equality can be achieved when we take $S = (\underbrace{0, \dots, 0}_{n-1}, a_1, \dots, a_r)$, where $(a_1, \dots, a_r)$ is a zero-sum free sequence in G with $f((a_1, \dots, a_r)) = f_G(r)$.

If $1 \leq r < m$, it is easy to see that $f_G(r) = r$, where m is the exponent of G. However, when $r \geq m$, the problem of determining $f_G(r)$ becomes difficult. Gao and Leader[8] proved $f_G(m) = 2m - 1$ with the restriction $(m, 6) = 1$. They also conjectured the same result without the restriction $(m, 6) = 1$. In this paper we show that $f_G(m) = 2m - 1$ still holds without that restriction.

Theorem 1. *If G is a finite non-cyclic abelian group of exponent m, then $f_G(m) = 2m-1$.*

Corollary 1 Let G be a finite abelian group of order n and exponent m, and let S be a sequence of elements in G with $|S| = n + m - 1$. Then, either $0 \in \sum_n(S)$ or $|\sum_n(S)| \geq 2m - 1$.

Proof. It follows from Theorem A and Theorem 1 immediately. □

2 Proof of Theorem 1

Lemma 2. *[2] Let G be an abelian group, and let S be a zero-sum free sequence of elements in G. Let $S_1, \dots, S_t$ be disjoint nonempty subsequences of S. Then, $f(S) \geq \sum_{i=1}^{t} f(S_i)$.*

Lemma 3. *[3] Let S be a zero-sum free sequence consisting of three distinct elements in an abelian group G. If no element in S has order 2, then $f(S) \geq 6$.*

Lemma 4. *Let S be a zero-sum free sequence in G. If there is some element g in S with order two, then $|\sum(S)| \geq 2|S| - 1$.*

Proof. Set $k = |S|$. Suppose $S = (g, a_1, \ldots, a_{k-1})$. Since S is zero-sum free and $g = -g$, we have that

$$a_1, a_1 + a_2, \ldots, a_1 + a_2 + \ldots + a_{k-1}$$
$$g, g + a_1, g + a_1 + a_2, \ldots, g + a_1 + a_2 + \ldots + a_{k-1}$$

are $2k - 1$ pairwise distinct elements in $\sum(S)$. Therefore,

$$|\sum(S)| \geq 2k - 1.$$

□

Lemma 5. *Let $S = S_1 S_2$ be a zero-sum free sequence in G. Let $H = \langle S_1 \rangle$ be the subgroup of G generated by S_1. Let ϕ be the natural homomorphism from G onto G/H. Set $h = |\phi(\{0\} \bigcup \sum(S_2))| = |(\{0\} \bigcup \sum(S_2)) + H/H|$. Then*

$$f(S) \geq hf(S_1) + h - 1.$$

Proof. Set $A = \{0\} \bigcup \sum(S_1)$. Since S is zero-sum free, we infer that $0 \notin \sum(S_1)$. Therefore,

$$|A| = 1 + f(S_1).$$

Suppose

$$\phi(\{0\} \bigcup \sum(S_2)) = \{\phi(a_0), \phi(a_1), \ldots, \phi(a_{h-1})\},$$

where $a_0 = 0$ and $a_i \in \sum(S_2)$ for $i = 1, \ldots, h - 1$. Since $A \subseteq H = \langle S_1 \rangle$, we infer that

$$A \setminus \{0\}, a_1 + A, \ldots, a_{h-1} + A$$

are pairwise disjoint subsets of $\sum(S)$. Therefore

$$\begin{aligned} f(S) &\geq |A \setminus \{0\}| + |a_1 + A| + \ldots + |a_{h-1} + A| \\ &= hf(S_1) + h - 1. \end{aligned}$$

□

For every $a \in G$, write $v_a(S)$ for the number of occurrences of a in S.

Lemma 6. *Let S be a zero-sum free sequence in G. Choose $g \in G$ so that $v_g(S) = max_{a \in S}\{v_a(S)\}$. Then $f(S) \geq 2|S| - 1$ or $v_g(S) \geq \frac{4|S| - f(S)}{6}$.*

Proof. By Lemma 4 we may assume that S contains no element with order 2.

Let $l \geq 0$ be the maximal integer t such that S contains t disjoint subsets each consisting of three distinct elements. Let $A_1, \dots, A_l$ be l disjoint 3-subsets of S such that the residual sequence $T = S(A_1 \dots A_l)^{-1}$ contains as many distinct elements as possible. Clearly, T can be written in the form

$$T = (\underbrace{a, \dots, a}_{u}, \underbrace{b, \dots, b}_{v}),$$

where $u \geq v \geq 0$ and $u + v = |T|$.

We distinguish two cases:
Case 1. $u \leq 1$. If $v = 0$, then $l = \frac{|S|-u}{3}$. Since S contains no element with order 2, by Lemma 2 and Lemma 3,

$$\begin{aligned} f(S) &\geq \sum_{i=1}^{l} f(A_i) + |T| \\ &\geq 6l + u \\ &= 2|S| - u \\ &\geq 2|S| - 1. \end{aligned}$$

Now assume that $v = 1$. Then $u = v = 1$ and $l = \frac{|S|-2}{3}$. Again by Lemmas 2 and 3,

$$\begin{aligned} f(S) &\geq \sum_{i=1}^{l} f(A_i) + f((a, b)) \\ &\geq 6l + 3 \\ &= 2|S| - 1. \end{aligned}$$

Case 2. $u \geq 2$. If $a \notin A_i$ for some $1 \leq i \leq l$, take $c \in A_i$ with $c \neq b$ and set $A_i' = (A_i \setminus \{c\}) \cup \{a\}$. Then $A_1, \dots, A_{i-1}, A_i', A_{i+1}, \dots, A_l$ are l disjoint 3-subsets of S and the residual sequence contains one more distinct elements than T does, a contradiction to the choice of $A_1, \dots, A_l$. This shows that $a \in A_i$ for every $i \in \{1, \dots, l\}$. Therefore

$$v_g(S) \geq l + u.$$

By Lemma 2 and Lemma 3, we have that

$$\begin{aligned} f(S) &\geq \sum_{i=1}^{l} f(A_i) + v f(a, b) + (u - v) f(a) \\ &\geq 6l + 3v + u - v \\ &= 6l + u + 2v. \end{aligned}$$

Hence

$$6l + u + v \leq f(S) - v. \tag{1}$$

Combining $3l + u + v = |S|$ with (1), we obtain that

$$3(2l + u + v) \geq 4|S| - f(S) + v \geq 4|S| - f(S).$$

Therefore,

$$v_g(S) \geq l + u \geq \frac{2l + u + v}{2} = \frac{3(2l + u + v)}{6} \geq \frac{4|S| - f(S)}{6}.$$

□

Lemma 7. *[12] Let $G = C_{n_1} \bigoplus C_{n_2}$ with $n_1 | n_2$. Then $D(C_{n_1} \bigoplus C_{n_2}) = n_1 + n_2 - 1$.*

Lemma 8. *[12] Every sequence S in $C_n \bigoplus C_n$ with $|S| = 3n - 2$ contains a zero-sum subsequence T with $1 \leq |T| \leq n$.*

Proof of Theorem 1. Let $S = (a_1, \ldots, a_m)$ be a zero-sum free sequence of m elements in G. We have to prove that $f(S) \geq 2m - 1$. Choose $g \in G$ so that $v_g(S) = max_{a \in S}\{v_a(S)\}$. By Lemma 6, we may assume that

$$v_g(S) \geq \frac{4|S| - f(S)}{6} \geq \frac{4m - (2m - 2)}{6} = \frac{m + 1}{3},$$

else the proof is complete.

Let H be the cyclic subgroup generated by g. Write $S = S_1 S_2$ such that all terms of S_1 are in H and no term of S_2 is in H. Hence $\langle S_1 \rangle = \langle g \rangle = H$ and $|S_1| \geq v_g(S) \geq \frac{m+1}{3}$. Let ϕ be the projection from G to G/H. Let

$$S_2 = (b_1, \ldots, b_w),$$

and set

$$\phi(S_2) = (\phi(b_1), \ldots, \phi(b_w)).$$

If there is a subsequence W of S_2 with $|W| \leq 3$ such that $|\{0\} \bigcup \sum(\phi(W)|) \geq 4$, then by Lemma 2 and Lemma 5, we have that

$$\begin{aligned} f(S) &\geq f(S_1 W) + f(S_2 W^{-1}) \\ &\geq 4f(S_1) + 3 + f(S_2 W^{-1}) \\ &\geq 4f(S_1) + 3 + |S_2| - |W| \\ &\geq 4|S_1| + 3 + |S_2| - |W| \\ &\geq 4|S_1| + |S_2| \\ &= 3|S_1| + m > 2m - 1. \end{aligned}$$

Therefore, we may assume that

$$|\{0\} \bigcup \sum \phi(W)| \leq 3 \tag{2}$$

for every subsequence W of S_2 with $|W| \leq 3$.

Let us fix $a \in S_2$. For every $b \in S_2$, since $|\sum(\phi(a), \phi(b)) \bigcup \{0\}| \leq 3$, we infer that $\phi(a) = \phi(b)$, or $\phi(a) \neq \phi(b)$ and $\phi(a) + \phi(b) = 0$. Therefore,

$$S_2 = (a + k_1 g, \ldots, a + k_u g, -a + l_1 g, \ldots, -a + l_v g),$$

where $u \geq v \geq 0$ and $u \geq 1$ and $k_i, l_j \in \{0, 1, \ldots, m-1\}$.

Let $G_0 = \langle a, g \rangle$ be the subgroup of G generated by a and g. Clearly, $|G_0| = |\langle \phi(a) \rangle||\langle g \rangle| = ord(\phi(a))ord(g)$. Observe that S is a zero-sum free sequence in $\langle S \rangle = G_0$. We distinguish two cases:

Case 1: $ord(\phi(a)) = 2$, i.e., $2a \in \langle g \rangle = H$. Since S is zero-sum free we have $v_g(S) < ord(g)$. Therefore, $ord(g) > \frac{m+1}{3}$. Hence $ord(g) = m$ or $ord(g) = \frac{m}{2}$. If $ord(g) = \frac{m}{2}$, then $|G_0| = m$ and $D(G_0) \leq m = |S|$, a contradiction to the fact that S is zero-sum free. Therefore, $ord(g) = m$ and

$$G_0 \cong C_2 \bigoplus C_m.$$

By Lemma 7, it follows that $D(G_0) = m + 1$.
For an arbitrary $g' \in G_0 \setminus \{0\}$, set $T = S(-g')$. Then $|T| = m + 1 = D(G_0)$. Therefore, T contains a nonempty zero-sum subsequence W. Since S is zero-sum free, $W = W_0(-g')$ with $W_0|S$. Therefore, $\sigma(W_0) + (-g') = 0$, or $g' = \sigma(W_0) \in \sum(S)$. This shows that $\sum(S) = G_0 \setminus \{0\}$. Therefore,

$$f(S) = |\sum(S)| = |G_0| - 1 = 2m - 1.$$

Case 2: $ord(\phi(a)) \geq 3$. Hence $m \geq 3$. If $u = 1$ and $v = 0$, then by Lemma 5 it follows that

$$f(S) \geq 2f(S_1) + 1 \geq 2|S_1| + 1 = 2m - 1.$$

If $u = 2$ and $v = 0$, then since $ord(\phi(a)) \geq 3$, it follows that

$$|\sum(\phi(a + k_1 g), \phi(a + k_2 g)) \cup \{0\}| = 3.$$

Hence, since $m \geq 3$, it follows in view of Lemma 5 that

$$f(S) \geq 3f(S_1) + 2 \geq 3|S_1| + 2 = 3(m - 2) + 2 \geq 2m - 1.$$

Now assume that either $u \geq 3$, or else $u = 2$ and $v \geq 1$. Hence, if $ord(\phi(a)) \geq 4$, then either

$$|\{0\} \cup \sum(\phi(a + k_1 g), \phi(a + k_2 g), \phi(a + k_3 g))| \geq 4,$$

or

$$|\{0\} \cup \sum(\phi(a + k_1 g), \phi(a + k_2 g), \phi(-a + l_1 g))| \geq 4,$$

contradicting inequality (2) in both cases. Therefore, we conclude that

$$ord(\phi(a)) = 3.$$

Hence,

$$|G_0| = 3(ord(g)) \quad and\ 3|m.$$

From the proof of Case 1, we know that $ord(g) = m$ or $ord(g) = \frac{m}{2}$. If $ord(g) = \frac{m}{2}$, then $|G_0| = \frac{3m}{2}$. It follows from $exp(G_0)|m$ that $G_0 = C_3 \bigoplus C_{\frac{m}{2}}$. Hence by Lemma 7, it follows that $D(G_0) = \frac{m}{2} + 2 \leq m = |S|$, a contradiction. Hence $ord(g) = m$ and

$$G_0 = C_3 \bigoplus C_m.$$

From $ord(\phi(a)) = 3$, we infer that $3a = kg$ for some $k \geq 0$. Therefore, $\frac{m}{3}kg = ma = 0$. Hence, $m|\frac{m}{3}k$. This gives that $3|k$. Set $q = \frac{k}{3}$. Thus $3a = 3qg$. Set $a' = a - qg$. Hence $3a' = 0$ and $ord(\phi(a')) = 3$. Clearly,

$$S_2 = (a' + k'_1 g, \ldots, a' + k'_u g, 2a' + l'_1 g, \ldots, 2a' + l'_v g),$$

where $k'_i = k_i + q$ and $l'_j = l_j - q$.
Now we have that

$$G_0 = \langle a' \rangle \oplus \langle g \rangle.$$

Let $H_0 = \langle a' \rangle \bigoplus \langle \frac{m}{3} g \rangle$. Note $H_0 \cong C_3 \bigoplus C_3$. Let ρ be the homomorphism from G_0 onto H_0 defined by :

$$\rho(ra' + sg) = ra' + \frac{m}{3}sg.$$

Clearly, $ker(\rho) = \langle 3g \rangle \cong C_{\frac{m}{3}}$.
Since $v_g(S) \geq \frac{m+1}{3}$ and $m \geq 3$, it follows that $v_g(S) \geq 2$. Set $S_0 = S(a' + k'_1 g, a' + k'_2 g, g, g)^{-1}$. Hence,

$$S = (a' + k'_1 g, a' + k'_2 g)(g, g)S_0.$$

Suppose $m \geq 9$. Hence applying Lemma 8 to the sequence $\rho(S_0)$ in $H_0 \cong C_3 \bigoplus C_3$, one can find $\frac{m}{3} - 3$ disjoint subsequences $T_1, \ldots, T_{\frac{m}{3}-3}$ of S_0 such that

$$\sigma(\rho(T_i)) = 0 \ and\ 1 \leq |T_i| \leq 3.$$

The residual sequence $S_0(T_1 \dots T_{\frac{m}{3}-3})^{-1}$ has length

$$\begin{aligned} |S_0(T_1 \dots T_{\frac{m}{3}-3})^{-1}| &= |S_0| - |T_1 \dots T_{\frac{m}{3}-3}| \\ &\geq m - 4 - 3(\frac{m}{3} - 3) \\ &= 5 \\ &= D(C_3 \oplus C_3) = D(H_0). \end{aligned}$$

Therefore, $S_0(T_1 \dots T_{\frac{m}{3}-3})^{-1}$ contains a nonempty subsequence $T_{\frac{m}{3}-2}$ (say) such that $\sigma(\rho(T_{\frac{m}{3}-2})) = 0$. Now we have

$$\sigma(T_i) \in ker(\rho) = \langle 3g \rangle \cong C_{\frac{m}{3}}$$

for every $i \in \{1, 2, \dots, \frac{m}{3} - 2\}$.
Since S is zero-sum free, we know that $(a + k_1'g, a + k_2'g, g, g, \sigma(T_1), \dots, \sigma(T_{\frac{m}{3}-2}))$ is also zero-sum free. By Lemma 5 and Lemma 2, we have that

$$\begin{aligned} f((g, g)(\sigma(T_1), \dots, \sigma(T_{\frac{m}{3}-2}))) &\geq 3f(\sigma(T_1), \dots, \sigma(T_{\frac{m}{3}-2})) + 2 \\ &\geq 3(\frac{m}{3} - 2) + 2 \\ &= m - 4. \end{aligned}$$

Again, by Lemma 5 and Lemma 2, we have that

$$\begin{aligned} & f((a + k_1'g, a + k_2'g, g, g, \sigma(T_1), \dots, \sigma(T_{\frac{m}{3}-2}))) \\ & \geq 3f((g, g)(\sigma(T_1), \dots, \sigma(T_{\frac{m}{3}-2}))) + 2 \\ & \geq 3(m - 4) + 2 \\ & = 3m - 10. \end{aligned}$$

Since $m \geq 9$, it follows that $f(S) \geq 3m - 10 \geq 2m - 1$.
So, we may assume that $m \leq 8$. Consequently, since $3|m$, it follows that $m = 3$ or $m = 6$.
Note that $v_g(S) \geq \frac{m+1}{3}$ and $u \geq 2$. Therefore, $\frac{m+1}{3} + 2 \leq |S| = m$. Hence $m > 3$. Thus, $m = 6$.
Since $v_g(S) \geq \frac{m+1}{3}$, we have that $|S_1| \geq 3$. Thus by Lemma 5,

$$\begin{aligned} f(S) &\geq f(S_1(a' + k_1'g, a' + k_2'g)) \\ &\geq 3f(S_1) + 2 \\ &\geq 3|S_1| + 2 \\ &\geq 3 \cdot 3 + 2 = 2 \cdot 6 - 1. \end{aligned}$$

This proves that $f(S) \geq 2m - 1$.

The following example shows that $f_G(m) = 2m - 1$. Let a, b be elements in G with $ord(a) = m$ and $b \notin \langle a \rangle$. Let $S = (\underbrace{a, \ldots, a}_{m-1}, b)$. Clearly, S is zero-sum free and $f(S) = 2m - 1$. This completes the proof. $\square$

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