Maximal projective degrees for strict partitions

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Abstract

Let λ be a partition, and denote by f^λ the number of standard tableaux of shape λ. The asymptotic shape of λ maximizing f^λ was determined in the classical work of Logan and Shepp and, independently, of Vershik and Kerov. The analogue problem, where the number of parts of λ is bounded by a fixed number, was done by Askey and Regev – though some steps in this work were assumed without a proof. Here these steps are proved rigorously. When λ is strict, we denote by g^λ the number of standard tableau of shifted shape λ. We determine the partition λ maximizing g^λ in the strip. In addition we give a conjecture related to the maximizing of g^λ without any length restrictions.

1 Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of n. We shall write $\lambda \vdash n$. As usual, we draw the Young diagram of a partition left and top justified. Let f^λ denote the number of standard tableaux of shape λ. Note that f^λ is the number of paths in the Young graph Y from its origin (1) to λ. Also, f^λ is the dimension of the Specht module, that is the degree of the corresponding irreducible character χ^λ of the symmetric group S_n.

The partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is *strict* if $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$ for some r. If the partition λ is strict and $|\lambda| = n$, we write $\lambda \models n$. The strict partitions form precisely the

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subgraph SY in the Young graph Y. The number of paths in that subgraph from (1) to λ is denoted by g^λ. By a theorem of I. Schur, g^λ equals the degree of the corresponding projective representation of S_n.

The problem of determining the asymptotic shape of the partition λ which maximizes f^λ, as $|\lambda|$ goes to infinity, is classical, and was solved in [11, 12]. This problem is closely related to that of the expected value of the length of the longest increasing subsequences in permutations, see also [3]. Let $H(k, 0; n)$ denote the set of partitions of n with at most k parts, namely

$$H(k, 0; n) = \{(\lambda_1, \lambda_2, \ldots) \vdash n \mid \lambda_{k+1} = 0\} = \{\lambda \vdash n \mid \ell(\lambda) \leq k\}.$$

We say that these partitions lie in the k strip. The asymptotic problem of maximizing f^λ in the k-strip was essentially solved in [1]. The solution in [1] tacitly assumed that there exist $a, \delta > 0$ such that as $n \to \infty$, a maximizing λ in the k-strip does belong to the subsets $H(k, 0; n, a, \delta) \subseteq H(k, 0; n, a)$ of $H(k, 0; n)$; see Equations (4), (5) and (6) below for the definitions of these subsets. Later, one of these assumptions, namely that λ lies in $H(k, 0; n, a)$, was rigorously verified in [2] and in [6]. We call this the a-condition. In Section 5 of this paper we verify the additional "δ-condition", namely λ lies in $H(k, 0; n, a, \delta)$, thus completing the rigorous proof of the results in [1]. The a-condition and the δ-condition also play a role in the problem of maximizing g^λ in the strip: In Section 4 we verify the "a-condition", and in Section 5 we verify the "δ-condition", both for λ maximizing g^λ in the strip. In Section 6 we show that in the strip, the λ maximizing either g^λ or $2^{|\lambda|-\ell(\lambda)}(g^\lambda)^2$, have the same asymptotic shape which equals the shape maximizing f^λ given in [1].

A natural question arises which is to maximize g^λ over all strict partitions λ (not just in a k-strip). This problem is open, so far without even a conjecture of what the asymptotic shape of such maximizing λ might be. Based on some combinatorial identities, we suggest here an approach to study the asymptotic shape of such λ. Our strategy is as follows: It seems that the strict partition λ maximizing g^λ is almost the same as the strict partition maximizing $2^{|\lambda|-\ell(\lambda)} \cdot (g^\lambda)^2$, and asymptotically they might be the same, see Conjecture 8.2. In Section 8 we give a possible strategy for maximizing $2^{|\lambda|-\ell(\lambda)} \cdot (g^\lambda)^2$: We relate the latter to the problem of maximizing f^μ for a certain subset of almost symmetric partitions μ and argue why this in turn possibly is the same as maximizing f^λ for any partition λ.

2 Degrees formulas

We recall the Young-Frobenius formula and the hook formula for f^λ.

The Young-Frobenius formula. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n then

$$f^\lambda = \frac{n!}{\ell_1! \cdots \ell_k!} \cdot \prod_{1 \leq i < j \leq k} (\ell_i - \ell_j) \tag{1}$$

where $\ell_i = \lambda_i + k - i$.

The hook-formula. Again, let λ be a partition of n, then

$$f^\lambda = \frac{n!}{\prod_{x\in\lambda} h_\lambda(x)} \tag{2}$$

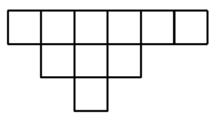
where $h_\lambda(x)$ is the hook number corresponding to the cell x in the Young diagram λ.

Both these formulas have analogues for g^λ where λ is a strict partition. Consider a strict partition $\lambda = (\lambda_1, \dots, \lambda_h)$, that is $\lambda_1 > \dots > \lambda_h > 0$. The analogue of the Young-Frobenius formula is due to I. Schur [9].

The Schur formula. Let $\lambda \vdash n$ be strict, then

$$g^\lambda = \frac{n!}{\lambda_1! \cdots \lambda_h!} \cdot \frac{\prod_{1\le i<j\le h}(\lambda_i - \lambda_j)}{\prod_{1\le i<j\le h}(\lambda_i + \lambda_j)}. \tag{3}$$

For the analogue hook formula for g^λ we need some notations. Recall that for a strict partition, one can also draw its *shifted* diagram. For example, the shifted diagram of $\lambda = (6, 3, 1)$ is



Definition 2.1 *Let $\lambda = (\lambda_1, \dots, \lambda_r) \models n$ be a strict partition with $\lambda_r > 0$. We define a partition $\mu = \mu(\lambda)$ of $2n$ (using the Frobenius notation for partitions) by*

$$\mu = \mu(\lambda) = proj(\lambda) := (\lambda_1, \lambda_2, \dots, \lambda_r \mid \lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_r - 1).$$

Conversely, given the partition $\mu = (\lambda_1, \dots, \lambda_r \mid \lambda_1 - 1, \dots, \lambda_r - 1) \vdash 2n$ in the Frobenius notation, then $\lambda_1 > \lambda_2 > \dots > \lambda_r > 0$ and we denote

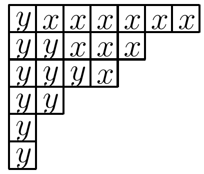
$$\sqrt{\mu} := (\lambda_1, \lambda_2, \dots) \models n,$$

see [7]. We say $\mu \vdash 2n$ is shift-symmetric if there exists $\lambda \models n$ such that $\mu = \mu(\lambda)$.

Note that if $\mu \vdash 2n$ is shift-symmetric then $\mu_i = \mu'_i + 1$ for $1 \le i \le \ell(\lambda)$. Note also that when n is large, the diagram of a shift-symmetric partition is nearly symmetric.

Figure 1 shows the diagram of a partition $\mu(\lambda)$ of $2n$. Area A_1 in this diagram is the shifted diagram of the partition λ and area A_2 is the (shifted) conjugate of A_1. For example, when $\lambda = (6, 3, 1)$, then $\mu(\lambda) = proj(6, 3, 1) = (7, 5, 4, 2, 1, 1)$ and $\sqrt{(7, 5, 4, 2, 1, 1)} = (6, 3, 1)$:

$\mu(\lambda) =$



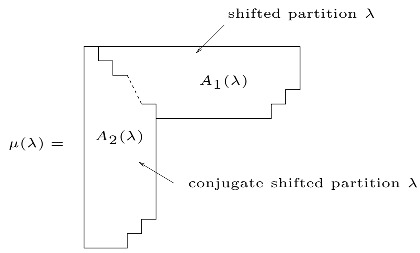


Figure 1

The projective analogue of the hook formula is due to I. G. Macdonald, and is as follows (see [4], page 267 – with the slight correction that $D(\lambda) = (\lambda_1, \lambda_2, \ldots \mid \lambda_1 - 1, \lambda_2 - 1, \ldots)$ in the Frobenius notation). Fill $\mu = \mu(\lambda)$ with its (ordinary) hook numbers $\{h_\mu(x) \mid x \in \mu\}$. Then:

Theorem 2.2 *[4] Let λ be a strict partition with $\mu = proj(\lambda)$, then*

$$g^\lambda = \frac{|\lambda|!}{\prod_{x \in A_1(\lambda)} h_\mu(x)}$$

where $A_1(\lambda)$ is defined as in Figure 1.

3 Maximal degrees in the strip

Recall that $H(k, 0; n)$ denotes the partitions λ of n with $\ell(\lambda) \leq k$. Denote by $SH(k, 0; n)$ the subset of strict partitions in $H(k, 0; n)$. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n, define for $1 \leq j \leq k$ the numbers $c_j(\lambda)$ via the equation

$$\lambda_j = \frac{n}{k} + c_j(\lambda) \cdot \sqrt{n}. \tag{4}$$

Thus $c_j(\lambda)$ parameterizes the deviation of λ_j from the average value $\frac{n}{k}$. Fix a real number a, and let

$$H(k, 0; n, a) \quad = \quad \{\lambda \in H(k, 0; n) \mid \text{all } |c_j(\lambda)| \leq a \ \}. \tag{5}$$

With a fixed, n large and with $\lambda \in \{k, 0; n, a\}$, all λ_j are approximately $\frac{n}{k}$.

In addition, also fix some $\delta > 0$, then denote

$$H(k, 0; n, a, \delta) \quad = \quad \{\lambda \in H(k, 0; n) \mid \text{all } |c_j(\lambda)| \leq a,\ c_i(\lambda) - c_{i+1}(\lambda) \geq \delta\}. \tag{6}$$

Note that if $\lambda \in H(k, 0; n, a, \delta)$ then λ is a strict partition of length either $k - 1$ or k.

The problem. For a fixed k, and for each n, we look for partitions $\lambda_{fmax} = \lambda^{(n)}_{fmax(k)}$ and $\lambda_{gmax} = \lambda^{(n)}_{gmax(k)}$ such that

$$\begin{aligned} f^{\lambda_{fmax}} &= \max\{f^\nu \mid \nu \in H(k, 0; n)\}, \\ g^{\lambda_{gmax}} &= \max\{g^\nu \mid \nu \in SH(k, 0; n)\}. \end{aligned}$$

The asymptotics of λ_{fmax} – that is the shape obtained when n goes to infinity – is given in [1], and we briefly describe it here. Let $\mathcal{H}_k(x)$ denote the k-th Hermit polynomial. It is defined via the equation

$$\frac{d^k}{dx^k}\left(e^{-x^2}\right) = (-1^k)\mathcal{H}_k(x)e^{-x^2}.$$

For example, $\mathcal{H}_0(x) = 1$, $\mathcal{H}_1(x) = 2x$, $\mathcal{H}_2(x) = 4x^2 - 2$, $\mathcal{H}_3(x) = 4x(2x^2 - 3)$, $\mathcal{H}_4(x) = 16x^4 - 48x^2 + 12$, etc. The degree of $\mathcal{H}_k(x)$ is k, and it is known that its roots are real and distinct, denoted by

$$x_1^{(k)} < x_2^{(k)} < \cdots < x_k^{(k)}.$$

Also, $x_1^{(k)} + x_2^{(k)} + \cdots + x_k^{(k)} = 0$. The following theorem is proved in [1]:

Theorem 3.1 *[1] As $n \to \infty$, the maximum $\max\{f^\lambda \mid \lambda \in H(k, 0; n)\}$ occurs when*

$$\lambda = \lambda_{fmax} \sim \left(\frac{n}{k} + x_k^{(k)}\sqrt{\frac{n}{k}}, \ldots, \frac{n}{k} + x_1^{(k)}\sqrt{\frac{n}{k}}\right).$$

Recall that for two sequences a_n, b_n, then $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$.

As was already mentioned, the proof of Theorem 3.1 in [1] tacitly assumed that there exist $a, \delta > 0$ such that for all large n, partition λ_{fmax} lies in $H(k, 0; n, a, \delta)$. This a-condition was verified in [2] and was further simplified in [6]. In Section 5 we verify the δ-condition for λ_{fmax}, thus completing the rigorous proof of Theorem 3.1. In Sections 4 and 5 we also verify the corresponding a-condition and δ-condition for λ_{gmax}. Thus, Equation (7) of the following lemma shows that λ_{fmax} and λ_{gmax} both have the same asymptotics.

Lemma 3.2 *Let $0 < a, \delta$ be fixed and let $\lambda \in H(k, 0; n, a, \delta)$. Then, as n goes to infinity,*

$$g^\lambda \sim 2^{-k(k-1)/2} \cdot f^\lambda, \qquad \textit{and also} \tag{7}$$

$$g^\lambda \sim b_\lambda \cdot \left[\prod_{1\le i<j\le k}(c_i - c_j)\right] \cdot e^{-(k/2)(\sum c_i^2)} \cdot \left(\frac{1}{n}\right)^{(k-1)(k+2)/4} \cdot k^n, \tag{8}$$

where

$$b_\lambda = \left(\frac{1}{2}\right)^{k(k-1)/2} \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \cdot k^{k^2/2}.$$

Proof. (1) In the following arguments we only use the condition $|c_i| \le a$. Show first that $\lambda_1' = k$, namely in Equation (3) we have $h = k$: Assume not, then $\lambda_k = 0$. By Equations (4) and (6), all other parts $\lambda_i \le \frac{n}{k} + a\sqrt{n}$, so $n = \lambda_1 + \cdots + \lambda_{k-1} \le (k-1)\cdot(\frac{n}{k} + a\sqrt{n}) < n$ for n large, contradiction. So $\lambda_1' = k$.

Calculate f^λ/g^λ by applying Equations (1) and (3) with $h = k$. Note that if $x \in \{\lambda_j, \lambda_j + 1, \ldots, \ell_j\}$ then $x \sim n/k$ (using $|c_i| \le a$), and hence $\ell_j!/\lambda_j! \sim (n/k)^{k-j}$. Therefore

$$\frac{\ell_1!\cdots\ell_k!}{\lambda_1!\cdots\lambda_k!} \sim \left(\frac{n}{k}\right)^{k(k-1)/2}.$$

Similarly $\ell_i + \ell_j \sim 2 \cdot \frac{n}{k}$, hence

$$\prod_{1 \le i < j \le k} (\ell_i + \ell_j) \sim 2^{k(k-1)/2} \cdot \left(\frac{n}{k}\right)^{k(k-1)/2}.$$

(2) In the following argument we use the condition $c_i - c_j \ge \delta$: Since $\delta > 0$, we have

$$\lambda_i - \lambda_j = (c_i - c_j)\sqrt{n} \sim (c_i - c_j)\sqrt{n} + j - i = \ell_i - \ell_j,$$

hence $\prod(\lambda_i - \lambda_j) \sim \prod(\ell_i - \ell_j)$.

(3) The proof now follows from parts (1) and (2). Combined with Equation (F.1.1) in [5], this implies the second approximation. ■

4 The a-condition for λ_{gmax}

The a-condition for λ_{fmax} – namely that λ_{fmax} lies in $H(k, 0; n, a)$ – was verified in [2] via a certain algorithm , and that algorithm was further simplified in [6]. As a result the following Proposition was obtained, see Theorem 2.2 in [6].

Proposition 4.1 *As n goes to infinity, the partitions $\lambda \in H(k, 0; n)$ maximizing f^λ occur in the subsets $H(k, 0; n, a)$ where $a = (k-1)\sqrt{2}$.*

In this section we verify, by a similar algorithm, the analogue a-condition for the partitions λ maximizing g^λ (as well as $2^{n-\ell(\lambda)}(g^\lambda)^2$) in the strip. That is:

Proposition 4.2 *As n goes to infinity, the partitions $\lambda \in SH(k, 0; n)$ maximizing g^λ – and $2^{n-\ell(\lambda)}(g^\lambda)^2$ – occur in the subsets $H(k, 0; n, a)$ where $a = (k-1)\sqrt{3}$. In particular, when n is large, $\lambda_j \sim n/k$ for $j = 1, \ldots, k$.*

The rest of this section is devoted to the proof of Proposition 4.2. The proof is based on the algorithm given in [6] – with the slight modification that $\sqrt{3n}$ replaces $\sqrt{2n}$. We first recall the algorithm, and then prove that when applying the algorithm, starting with an arbitrary strict partition $\lambda \in SH(k, 0; n)$, the output is a strict partition $\mu \in SH(k, 0; n)$ satisfying $g^\lambda \le g^\mu$ and $\mu_i - \mu_{i+1} \le \sqrt{3n}$ for $i = 1, \ldots, k-1$. This, together with Lemma 4.5, clearly proves Proposition 4.2.

The Algorithm. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n. Assume that for some (say, minimal) $t \le k-1$, $\lambda_t - \lambda_{t+1} \ge \sqrt{3n}$. Then the algorithm changes λ to $\lambda^{(1)}$, where

$$\lambda_i^{(1)} = \begin{cases} \lambda_i & \text{if } i \ne t, t+1, \\ \lambda_t - 1 & \text{if } i = t, \\ \lambda_{t+1} + 1 & \text{if } i = t+1. \end{cases}$$

Now take λ to be $\lambda^{(1)}$ and repeat the above step. If at some point no such $t \le k-1$ exists, the algorithm stops, and we denote the corresponding partition by μ.

Lemma 4.3 *Let $n > 3$ and $\lambda \in SH(k,0;n)$. Assume after one step of the above algorithm we obtain a partition $\lambda^{(1)}$. Then $\lambda^{(1)}$ is strict.*

Proof. Note that in one step of the algorithm, say from λ to $\lambda^{(1)}$, the differences $\lambda_i - \lambda_{i+1}$ increase except for $i = t$. More precisely,

$$\begin{aligned} \lambda_i^{(1)} - \lambda_{i+1}^{(1)} &\geq \lambda_i - \lambda_{i+1} \quad \text{if } i \neq t, \\ \lambda_t^{(1)} - \lambda_{t+1}^{(1)} &= \lambda_t - \lambda_{t+1} - 2 \geq \sqrt{3n} - 2 \geq 3 - 2 \end{aligned}$$

where the last inequality holds if $n \geq 3$. Hence if λ is strict, then also $\lambda^{(1)}$ obtained after one step of the algorithm is strict, provided $n \geq 3$. ■

Lemma 4.4 *Let $n > 3$ and $\lambda \in SH(k,0;n)$. Assume after one step of the above algorithm we obtain a partition $\lambda^{(1)}$. Then $g^\lambda \leq g^{\lambda^{(1)}}$.*

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_h)$ with $h = \lambda'_1 \leq k$. By Equation (3), $g^\lambda / g^{\lambda^{(1)}} = A \cdot B$ where

$$A = \left(\frac{\lambda_{t+1} + 1}{\lambda_t} \right) \cdot \left(\frac{\lambda_t - \lambda_{t+1}}{\lambda_t - \lambda_{t+1} - 2} \right)$$

and

$$B = \prod_{i \neq t,\, t+1} \frac{(\lambda_i - \lambda_t)(\lambda_i - \lambda_{t+1})(\lambda_i + \lambda_t - 1)(\lambda_i + \lambda_{t+1} + 1)}{(\lambda_i + \lambda_t)(\lambda_i + \lambda_{t+1})(\lambda_i - \lambda_t + 1)(\lambda_i - \lambda_{t+1} - 1)}.$$

We show first that $B < 1$ by showing that each factor x_i / y_i in B satisfies

$$\frac{x_i}{y_i} = \frac{(\lambda_i - \lambda_t)(\lambda_i - \lambda_{t+1})(\lambda_i + \lambda_t - 1)(\lambda_i + \lambda_{t+1} + 1)}{(\lambda_i + \lambda_t)(\lambda_i + \lambda_{t+1})(\lambda_i - \lambda_t + 1)(\lambda_i - \lambda_{t+1} - 1)} < 1.$$

Start by checking that $x_i, y_i > 0$. Indeed, if $i < t$ then $\lambda_i > \lambda_t \geq \lambda_{t+1} + \sqrt{3n}$ and all the factors in both x_i and in y_i are > 0. If $i > t+1$ then the four factors involving $\lambda_i - \lambda_t$ and $\lambda_i - \lambda_{t+1}$ are < 0, while the other four factors are obviously > 0, and again $x_i, y_i > 0$. Thus, to show that $B < 1$ it suffices to show that each $y_i - x_i > 0$. This follows since, by elementary manipulations, $y_i - x_i = 2\lambda_i(\lambda_t + \lambda_{t+1})(\lambda_t - \lambda_{t+1} - 1)$. But $\lambda_t - \lambda_{t+1} \geq \sqrt{3n} > 1$, so $y_i - x_i > 0$ and $B < 1$.

Show next that $A \leq 1$. Write $A = x/y$ where $x = (\lambda_{t+1} + 1)(\lambda_t - \lambda_{t+1})$ and $y = \lambda_t(\lambda_t - \lambda_{t+1} - 2)$. We need to show that $y - x \geq 0$. This follows since $y - x = (\lambda_t - \lambda_{t+1})^2 - 3\lambda_t + \lambda_{t+1} \geq (\lambda_t - \lambda_{t+1})^2 - 3\lambda_t \geq 0$ since $(\lambda_t - \lambda_{t+1})^2 \geq 3n$ while $\lambda_t \leq n$. ■

Lemma 4.5 *Let $b > 0$ and let $\mu \in H(k,0;n)$ satisfy $\mu_i - \mu_{i+1} \leq b\sqrt{n}$ for $i = 1, \ldots, k-1$. Write $\mu_j = \frac{n}{k} + c_j\sqrt{n}$, then $|c_j| \leq (k-1)b$ for all $1 \leq j \leq k$.*

Proof. Since μ is a partition of n and by the assumption we have

$$\begin{aligned} n &= k\mu_k + (k-1)(\mu_{k-1} - \mu_k) + (k-2)(\mu_{k-2} - \mu_{k-1}) + \cdots + (\mu_1 - \mu_2) \\ &\leq k\mu_k + \frac{k(k-1)}{2} b\sqrt{n}. \end{aligned}$$

Therefore

$$\frac{n}{k} - \frac{(k-1)}{2} b\sqrt{n} \leq \mu_k.$$

Also $\mu_1 = (\mu_1 - \mu_2) + (\mu_2 - \mu_3) + \cdots + (\mu_{k-1} - \mu_k) + \mu_k \leq \frac{n}{k} + (k-1)b\sqrt{n}$ since $\mu_k \leq \frac{n}{k}$. Thus $\frac{n}{k} - \frac{(k-1)}{2} b\sqrt{n} \leq \mu_k \leq \mu_j \leq \mu_1 \leq \frac{n}{k} + (k-1)b\sqrt{n}$ for all $1 \leq j \leq k$, which implies the proof. ∎

The proof of Proposition 4.2. Let $\lambda \in SH(k, 0; n)$ and apply the above algorithm to obtain a partition μ. Then $\mu_i - \mu_{i+1} \leq \sqrt{3n}$ for $i = 1, \ldots, k-1$, and hence by Lemma 4.3 and Lemma 4.4, the partition μ is strict with $g^\lambda \leq g^\mu$. By Lemma 4.5, such a partition μ lies in $H(k, 0; n, (k-1)\sqrt{3})$. The second claim is true whenever we work with partitions in a set $H(k, 0; n, a)$ with fixed $a > 0$. ∎

5 The δ-condition for λ_{fmax} and λ_{gmax} in the strip

In this section we prove the δ-condition for maximizing f^λ and g^λ in the strip. More precisely, we show:

Proposition 5.1 *For all large n, if $\lambda \in H(k, 0; n)$ and $f^\lambda = \max\{ f^\nu | \nu \in H(k, 0; n) \}$, then $\lambda \in H(k, 0; n, a, \delta)$ where $a = (k-1)\sqrt{2}$ and $\delta = \frac{1}{2k^3}$.*

Proposition 5.2 *For all large n, if $\lambda \in SH(k, 0; n)$ and $g^\lambda = \max\{ g^\nu | \nu \in SH(k, 0; n) \}$, then $\lambda \in H(k, 0; n, a, \delta)$ where $a = (k-1)\sqrt{3}$ and $\delta = \frac{1}{4k^3\sqrt{3}}$.*

Proof of Proposition 5.1. Suppose that $\lambda \in H(k, 0; n, a) \setminus H(k, 0; n, a, \delta)$. By Proposition 4.1, it suffices to show that in this case, f^λ is not maximal. Let $t = \min\{ 1 \leq i < k \mid \lambda_i - \lambda_{i+1} < \delta\sqrt{n} \}$, and let

$$r = \begin{cases} t & \text{if } t \leq k/2, \\ k - t & \text{otherwise.} \end{cases}$$

Note that $r \leq \frac{k}{2}$. Let $\mu = (\mu_1, \ldots, \mu_k) \in H(k, 0; n)$ be such that

$$\mu = \begin{cases} (\lambda_1 + 1, \ldots, \lambda_r + 1, \lambda_{k-r+1} - 1, \ldots, \lambda_k - 1) & \text{if } t = k/2, \\ (\lambda_1 + 1, \ldots, \lambda_r + 1, \lambda_{r+1}, \ldots, \lambda_{k-r}, \lambda_{k-r+1} - 1, \ldots, \lambda_k - 1) & \text{otherwise.} \end{cases}$$

Clearly μ is a partition of n into k parts. By the Young-Frobenius formula (1),

$$\frac{f^\lambda}{f^\mu} = \prod_{i=1}^{r} \frac{\lambda_i + k - i + 1}{\lambda_{k-i+1} + i - 1} \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{\lambda_i - \lambda_j + j - i + \Delta_{i,j}}$$

where

$$\Delta_{i,j} = \begin{cases} 0, & \text{if } i < j \leq r \text{ or } j > i > k-r, \\ 1, & \text{if } i \leq r < j \leq k-r \text{ or } j > k-r \geq i > r, \\ 2, & \text{if } i \leq r \text{ and } j > k-r. \end{cases}$$

For all $i < j$, then $\frac{\lambda_i - \lambda_j + j - i}{\lambda_i - \lambda_j + j - i + \Delta_{i,j}} \leq 1$, and since $\Delta_{t,t+1} \geq 1$, also $\frac{\lambda_t - \lambda_{t+1} + 1}{\lambda_t - \lambda_{t+1} + 1 + \Delta_{t,t+1}} < \frac{\delta\sqrt{n}+1}{\delta\sqrt{n}+2}$. Thus

$$\begin{aligned} \frac{f^\lambda}{f^\mu} &< \left(\prod_{i=1}^{r} \frac{\lambda_i + k - i + 1}{\lambda_{k-i+1} + i - 1} \right) \frac{\delta\sqrt{n}+1}{\delta\sqrt{n}+2} \leq \left(\frac{\lambda_1 + k}{\lambda_k} \right)^r \frac{\delta\sqrt{n}+1}{\delta\sqrt{n}+2} \\ &\leq \left(\frac{\frac{n}{k} + a\sqrt{n} + k}{\frac{n}{k} - a\sqrt{n}} \right)^r \frac{\delta\sqrt{n}+1}{\delta\sqrt{n}+2} = \frac{\alpha_0 n^{r+1/2} + \alpha_1 n^r + O(n^{r-1/2})}{\beta_0 n^{r+1/2} + \beta_1 n^r + O(n^{r-1/2})}. \end{aligned}$$

where

$$\alpha_0 = \beta_0 = (\frac{1}{k})^r \delta > 0, \qquad \begin{aligned} \alpha_1 &= \alpha_0 (r \frac{a}{1/k} + \frac{1}{\delta}), \\ \beta_1 &= \beta_0 (-r \frac{a}{1/k} + \frac{2}{\delta}). \end{aligned}$$

We have $\alpha_1 - \beta_1 = \alpha_0(2rak - \frac{1}{\delta}) \leq \alpha_0(\sqrt{2}\,k^3 - 2k^3) < 0$, so $\alpha_1 < \beta_1$. Thus $\frac{f^\lambda}{f^\mu} < 1$ for all sufficiently large n. ∎

Proof of Proposition 5.2. Suppose that $\lambda \in SH(k,0;n)$ maximizes g^λ. By Proposition 4.2, partition λ lies in $H(k,0;n,a)$. Suppose that $\lambda \notin H(k,0;n,a,\delta)$. Let $t = \min\{\, 1 \leq i < k \mid \lambda_i - \lambda_{i+1} < \delta\sqrt{n} \,\}$, and let

$$r = \begin{cases} t & \text{if } t \leq k/2, \\ k-t & \text{otherwise.} \end{cases}$$

Note that $r \leq \frac{k}{2}$. Let $\mu = (\mu_1, \ldots, \mu_k) \in SH(k,0;n)$ be such that

$$\mu = \begin{cases} (\lambda_1+1, \ldots, \lambda_r+1, \lambda_{k-r+1}-1, \ldots, \lambda_k-1) & \text{if } t = k/2, \\ (\lambda_1+1, \ldots, \lambda_r+1, \lambda_{r+1}, \ldots, \lambda_{k-r}, \lambda_{k-r+1}-1, \ldots, \lambda_k-1) & \text{otherwise.} \end{cases}$$

Clearly μ is a partition of n into k parts. By formula (3),

$$\frac{g^\lambda}{g^\mu} = \prod_{i=1}^{r} \frac{\lambda_i + 1}{\lambda_{k-i+1}} \prod_{i<j} \left(\frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_j + \Delta_{i,j}} \cdot \frac{\lambda_i + \lambda_j + \Gamma_{i,j}}{\lambda_i + \lambda_j} \right)$$

where

$$\Delta_{i,j} = \begin{cases} 0, & \text{if } i < j \leq r \text{ or } j > i > k-r, \\ 1, & \text{if } i \leq r < j \leq k-r \text{ or } j > k-r \geq i > r, \\ 2, & \text{if } i \leq r \text{ and } j > k-r \end{cases}$$

and

$$\Gamma_{i,j} = \begin{cases} -2, & \text{if } k-r < i < j, \\ -1, & \text{if } r < i \le k-r < j, \\ 0, & \text{if } i \le r \le k-r < j \text{ or } r < i < j \le k-r, \\ 1, & \text{if } i \le r < j \le k-r, \\ 2, & \text{if } i < j \le r. \end{cases}$$

For all $i < j$, then $\frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_j + \Delta_{i,j}} \le 1$, and since $\Delta_{t,t+1} \ge 1$, also $\frac{\lambda_t - \lambda_{t+1}}{\lambda_t - \lambda_{t+1} + \Delta_{t,t+1}} < \frac{\delta\sqrt{n}}{\delta\sqrt{n}+1}$. Thus

$$\begin{aligned} \frac{g^\lambda}{g^\mu} &< \left(\prod_{i=1}^{r} \frac{\lambda_i + 1}{\lambda_{k-i+1}}\right) \frac{\delta\sqrt{n}}{\delta\sqrt{n}+1} \prod_{i<j} \frac{\lambda_i + \lambda_j + \Gamma_{i,j}}{\lambda_i + \lambda_j} \\ &\le \left(\frac{\lambda_1 + 1}{\lambda_k}\right)^r \frac{\delta\sqrt{n}}{\delta\sqrt{n}+1} \left(\frac{2\lambda_k + 2}{2\lambda_k}\right)^{k(k-1)/2} \\ &\le \left(\frac{\frac{n}{k} + a\sqrt{n} + 1}{\frac{n}{k} - a\sqrt{n}}\right)^r \frac{\delta\sqrt{n}}{\delta\sqrt{n}+1} \left(\frac{\frac{n}{k} - a\sqrt{n} + 1}{\frac{n}{k} - a\sqrt{n}}\right)^{k(k-1)/2} \\ &= \frac{\alpha_0 n^{r+1/2+k(k-1)/2} + \alpha_1 n^{r+k(k-1)/2} + O(n^{r-1/2+k(k-1)/2})}{\beta_0 n^{r+1/2+k(k-1)/2} + \beta_1 n^{r+k(k-1)/2} + O(n^{r-1/2+k(k-1)/2})} \end{aligned}$$

where

$$\alpha_0 = \beta_0 = (\frac{1}{k})^{r+k(k-1)/2}\delta > 0, \qquad \begin{aligned} \alpha_1 &= \alpha_0 \left((r - k(k-1)/2)\frac{a}{1/k}\right), \\ \beta_1 &= \beta_0 \left((-r - k(k-1)/2)\frac{a}{1/k} + \frac{1}{\delta}\right). \end{aligned}$$

We have $\alpha_1 - \beta_1 = \alpha_0(2rak - \frac{1}{\delta}) \le \alpha_0(2k^3\sqrt{3} - 4k^3\sqrt{3}) < 0$, so $\alpha_1 < \beta_1$. Thus $\frac{g^\lambda}{g^\mu} < 1$ if n is sufficiently large, in contradiction to the maximality of g^λ. ■

6 Maximal g^λ in the strip

Recall that λ_{fmax} is the partition maximizing f^λ, and λ_{gmax} the partition maximizing g^λ. Denote by λ_{2gmax} the partition maximizing $2^{|\lambda|-\ell(\lambda)}(g^\lambda)^2$. Here in all three cases, maximizing means with respect to the corresponding k-strip. The main theorem of this section is:

Theorem 6.1 *As $n \to \infty$, the maximizing partitions in the k-strip λ_{2gmax}, λ_{gmax}, and λ_{fmax} are asymptotically equal. Thus*

$$\lambda_{fmax},\, \lambda_{gmax},\, \lambda_{2gmax} \sim \left(\frac{n}{k} + x_k^{(k)}\sqrt{\frac{n}{k}}, \ldots, \frac{n}{k} + x_1^{(k)}\sqrt{\frac{n}{k}}\right),$$

where $x_1^{(k)} < \cdots < x_k^{(k)}$ are the roots of the kth Hermit polynoial, see Theorem 3.1.

Proof. (i) Define the sets

$$\begin{aligned} SH(k,0;n,a) &= SH(k,0;n)\cap H(k,0;n,a),\\ SH(k,0;n,a,\delta) &= SH(k,0;n)\cap H(k,0;n,a,\delta). \end{aligned}$$

By Proposition 4.2, it follows that maximizing g^λ with $\lambda \in SH(k,0;n)$ is the same as maximizing g^λ with $\lambda \in SH(k,0;n,a_1)$ where $a_1 = (k-1)\sqrt{3}$. Let now n be large. Then by the previous section, this is the same as maximizing g^λ with $\lambda \in SH(k,0;n,a_1,\delta_1)$ for $\delta_1 = \frac{1}{4k^3\sqrt{3}}$.

(ii) The same phenomena occurs when maximizing f^λ for $\lambda \in H(k,0;n)$ when n is large: a maximizing partition λ lies in $H(k,0;n,a_2,\delta_2)$ for $a_2 = (k-1)\sqrt{2}$ and $\delta_2 = \frac{1}{2k^3}$. See Proposition 4.1 for the a-condition and Proposition 5.1 for the δ-condition.

(iii) Let $a = \max\{a_1,a_2\}$ and $\delta = \min\{\delta_1,\delta_2\}$. Then the partitions μ and ν maximizing f^λ and g^λ respectively, lie in the same set $H(k,0;n,a,\delta)$. Equation (7) implies that the partitions maximizing f^λ and the partitions maximizing g^λ have the same asymptotics when n goes to infinity. Hence λ_{gmax} and λ_{fmax} are asymptotically the same.

(iv) We show next that λ_{2gmax} and λ_{gmax} are asymptotically the same. Clearly, the problem of maximizing $2^{n-\ell(\lambda)}(g^\lambda)^2$ is the same as that of maximizing $2^{-\ell(\lambda)}(g^\lambda)^2$. By part (1) of the proof of Lemma 3.2, a maximizing λ_{2gmax} must satisfy $\ell(\lambda_{2gmax}) = k$ for large n, and therefore it also maximizes g^λ. ∎

7 Some combinatorial identities

Recall the following two well-known identities for f^λ and g^λ.

$$(a)\quad \sum_{\lambda\vdash n}(f^\lambda)^2 = n! \qquad \text{and} \qquad (b)\quad \sum_{\lambda\models n} 2^{n-\ell(\lambda)}\cdot(g^\lambda)^2 = n! \tag{9}$$

For a bijective proof of identity (a) by the RSK, see [8], and for a bijective proof of identity (b) by a modified RSK, see [7, 13].

Proposition 7.1 *Let $\lambda \models n$ and let $\mu = \mu(\lambda) = proj(\lambda)$, so $\mu \vdash 2n$. Then*

$$f^{\mu(\lambda)} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\cdot 2^{n-\ell(\lambda)}\cdot(g^\lambda)^2, \tag{10}$$

$$\sum_{\lambda\models n} f^{\mu(\lambda)} = \sum_{\lambda\models n} f^{proj(\lambda)} = 1\cdot 3\cdot 5\cdots(2n-1). \tag{11}$$

The proof of this proposition follows from the following lemma.

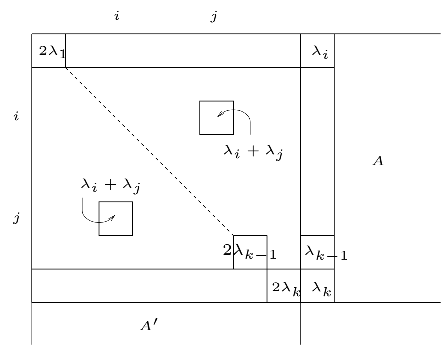


Figure 2

Lemma 7.2 *Let $\lambda \models n$ and let $\mu = \mu(\lambda) = proj(\lambda)$, with $A_1(\lambda)$ and $A_2(\lambda)$ as in Figure 1. Then*

$$\prod_{x \in \mu} h_\mu(x) = 2^{\ell(\lambda)} \cdot \left(\prod_{x \in A_1(\lambda)} h_\mu(x) \right)^2 . \tag{12}$$

Proof. Check that μ with its hook numbers looks as in Figure 2: here the part A' is the conjugate of the part A and hence has the same hook numbers. The area A_1 (in Figure 1) contains A together with the North-East half of the corner rectangle (a $k \times (k+1)$-rectangle). Similarly for A_2. Verify that the hook numbers in the corner rectangle are those indicated in Figure 2. This implies the proof of the lemma. ∎

The proof of Proposition 7.1 now follows from Lemma 7.2 and Theorem 2.2:

$$f^\mu = \frac{(2n)!}{\prod_{x \in \mu} h_\mu(x)} = \frac{(2n)!}{2^{\ell(\lambda)} \cdot \left(\prod_{x \in A_1(\lambda)} h_\mu(x) \right)^2} = \frac{(2n)!}{n!n!} \cdot 2^{-\ell(\lambda)} (g^\lambda)^2.$$

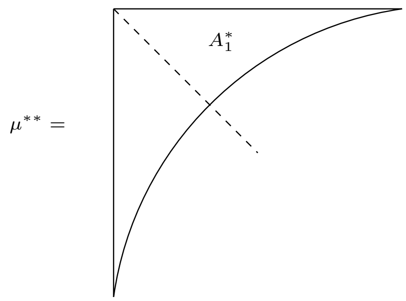
Equation (11) follows from Equation (9b), summing Equation (10) over all $\lambda \models n$. ∎

8 A strategy for maximizing $2^{|\lambda| - \ell(\lambda)} \cdot (g^\lambda)^2$

It is a natural question to ask which strict partitions λ maximize g^λ, without restricting to the k-strip. We conjecture that these partitions are very close to the strict partitions λ maximizing $2^{|\lambda| - \ell(\lambda)} \cdot (g^\lambda)^2$. In this section, we give a strategy of how one possibly may find the limit shape of those strict partitions λ maximizing $2^{|\lambda| - \ell(\lambda)} \cdot (g^\lambda)^2$. Denote

$$LP(2n) = \{\mu \vdash 2n \mid \mu \text{ is shift-symmetric}\}.$$

Proposition 7.1 shows that the strict partition λ maximizes $2^{|\lambda| - \ell(\lambda)} \cdot (g^\lambda)^2$ if and only if the shift-symmetric partition $\mu = \mu(\lambda)$ maximizes $\{f^\mu \mid \mu \in LP(2n)\}$. Thus, we need to find asymptotically which shift-symmetric $\mu \vdash 2n$ maximizes $\{f^\mu \mid \mu \in LP(2n)\}$. Note the following, not necessarily rigorous, arguments:



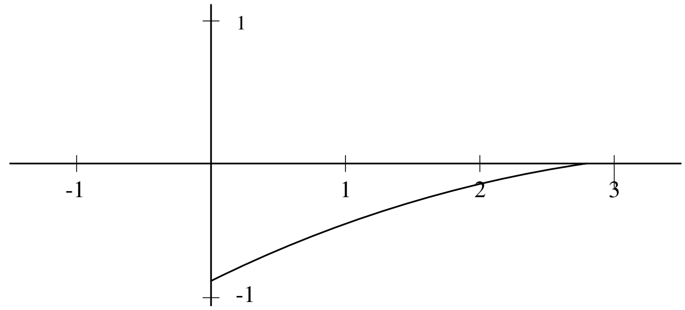


Figure 3 and Figure 4

1. For large n, a shift-symmetric diagram μ is nearly symmetric.
2. By [3], [11, 12] the asymptotic shape of the general ν maximizing f^ν (with no restrictions) is symmetric.
3. Small changes in large diagrams ν result in small changes in the hook numbers, hence in the degrees f^ν.

It is therefore reasonable to conjecture that such a shift-symmetric partition $\mu = \mu(\lambda) \vdash 2n$ maximizing f^μ is asymptotically very close to the $\nu \vdash 2n$ maximizing f^ν in the general case, that is without any restrictions on the partitions ν. Such ν is given by the classical work of Logan-Shepp [3] and Vershik and Kerov [11, 12], which we briefly describe: Given $\nu \vdash n$, we take the area of each box of the diagram ν to be one. Re-scale the boxes by multiplying each of the x-axis and the y-axis by $1/\sqrt{n}$, and denote the re-scaled diagram by $\bar{\nu}$. Thus the area of $\bar{\nu}$ equals one. For each n let $\nu_{max}^{(n)}$ denote a partition $\nu \vdash n$ with maximal f^ν: $f^{\nu_{max}^{(n)}} = max\{f^\nu \mid \nu \vdash n\}$. Although $\nu_{max}^{(n)}$ might not be unique for some values n, when n goes to infinity, $\bar{\nu}_{max}^{(n)}$ has a unique asymptotic shape ν^* given by Theorem 8.1 below. Similarly, consider $\mu(\lambda) = proj(\lambda) \vdash 2n$, and denote by $\tilde{\mu}(\lambda) = proj(\bar{\lambda})$ the rescaling of $\mu(\lambda)$ by $1/\sqrt{n}$; hence $\tilde{\mu}(\lambda)$ is of area two. If $n \to \infty$ then $\bar{\lambda}$ tends to the limit shape λ^* (of area one) if and only if $\tilde{\mu}(\lambda)$ tends to the symmetric limit shape $\mu^{**} = \mu(\lambda)^{**}$ (of area two).

Theorem 8.1 ([3], [11, 12]) *The limit shape ν^* of the re-scaled diagrams $\bar{\nu}_{max}^{(n)}$ exists, and is given by the two axes and by the parametric curve*

$$x = \left(\frac{2}{\pi}\right)(\sin\theta - \theta\cos\theta) + 2\cos\theta, \quad y = -\left(\frac{2}{\pi}\right)(\sin\theta - \theta\cos\theta), \quad 0 \le \theta \le \pi. \tag{13}$$

The curve in Equation (13) is given in Figure 3; it is symmetric with respect to $y = -x$. The last theorem, together with the discussion at the beginning of this section, leads to the following conjecture.

Conjecture 8.2 *The limit shape λ^* of $\lambda \models n$ maximizing $2^{n-\ell(\lambda)} \cdot (g^\lambda)^2$ (and possibly maximizing g^λ) is given by the two axes and by the parametric curve (Figure 4)*

$$x = 2\sqrt{2} \cdot \cos\theta, \qquad y = \left(\frac{2\sqrt{2}}{\pi}\right) \cdot (\theta\cos\theta - \sin\theta), \qquad 0 \leq \theta \leq \frac{\pi}{2}. \tag{14}$$

Conjecture 8.2 follows from the assumption that to maximize f^μ over the shift-symmetric partitions is asymptotically the same as maximizing f^ν over all partitions ν. By Proposition 7.1 the λ maximizing $2^{|\lambda|-\ell(\lambda)} \cdot (g^\lambda)^2$ satisfies $\lambda = \sqrt{\mu}$. So $\lambda^* = \sqrt{\mu^{**}}$ for the limit shapes, where the limit shape μ^{**} is of area two. We therefore dilate the curve (13) by multiplying both the x-values and the y-values by $\sqrt{2}$. This yields the limit shape μ^{**} of area two, given by the axes and by the curve

$$x = \sqrt{2}\left[\left(\frac{2}{\pi}\right)(\sin\theta - \theta\cos\theta) + 2\cos\theta\right],$$
$$y = -\sqrt{2}\left(\frac{2}{\pi}\right)(\sin\theta - \theta\cos\theta), \quad 0 \leq \theta \leq \pi.$$

To obtain λ^*, first obtain its shifted shape A_1^* by cutting μ^{**} into two halves along the line $y = -x$, see Figure 3: A_1^* is bounded by the x-axis, by the line $y = -x$ and by the part of the (dilated) LSVK curve with $0 \leq \theta \leq \frac{\pi}{2}$. To obtain λ^* from A_1^*, pull the line $y = -x$ to the left, until it equals the (negative) y-axis. Thus each point (x, y) in A_1^* is transformed to $(x - |y|, y)$ in λ^*. Under this transformation the x-axis stays invariant, the line $y = -x$ becomes the (negative) y-axis, and (half of) the dilated LSVK curve becomes the curve (14) of Conjecture 8.2.

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