Extremal subsets of $\{1, ..., n\}$ avoiding solutions to linear equations in three variables

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Abstract

We refine previous results to provide examples, and in some cases precise classifications, of extremal subsets of $\{1, ..., n\}$ containing no solutions to a wide class of non-invariant, homogeneous linear equations in three variables, i.e.: equations of the form $ax + by = cz$ with $a + b \neq c$.

1 Introduction

A well-known problem in combinatorial number theory is that of locating extremal subsets of $\{1, ..., n\}$ which contain no non-trivial solutions to a given linear equation

$$\mathcal{L}: \quad a_1 x_1 + \cdots + a_k x_k = b, \tag{1}$$

where $a_1, ..., a_k, b \in \mathbf{Z}$ and their GCD is one. Most of the best-known work concerns just three individual, homogeneous equations

$$\mathcal{L}_1: \quad x_1 + x_2 = 2x_3,$$
$$\mathcal{L}_2: \quad x_1 + x_2 = x_3 + x_4,$$
$$\mathcal{L}_3: \quad x_1 + x_2 = x_3,$$

where the corresponding subsets are referred to, respectively, as sets without arithmetic progressions, Sidon sets and sum-free sets. The idea to consider arbitrary linear equations $\mathcal{L}$ was first enunciated explicitly in a pair of articles by Ruzsa in the mid-1990s [10] [11]. The only earlier reference of note would appear to be a paper of Lucht [6] concerning homogeneous equations in three variables, though Lucht's article was only concerned with subsets of $\mathbf{N}$. Following Ruzsa, denote by $r_{\mathcal{L}}(n)$ the maximum size of a subset of $\{1, ..., n\}$ which contains no non-trivial solutions to a given equation $\mathcal{L}$. Let us pause here to recall explicitly what we mean by a 'trivial' solution to (1) (the definition is also given in [10]). Such solutions can only arise when $\mathcal{L}$ is translation-invariant, i.e.: $\sum a_i = b = 0$. Then a solution $(x_1, ..., x_k)$ to (1) is said to be trivial if there is a partition of the index set $\{1, ..., k\} = \mathcal{T}_1 \sqcup \cdots \sqcup \mathcal{T}_l$ such that $x_i = x_j$ whenever i and j are in the same part of the partition, and for each $r = 1, ..., l$ one has $\sum_{i \in \mathcal{T}_r} a_i = 0$.

When considering the function $r_{\mathcal{L}}(n)$ for arbitrary $\mathcal{L}$, one begins by observing a basic distinction between those $\mathcal{L}$ which are translation-invariant and those which are not, namely : for the former it is always the case that $r_{\mathcal{L}}(n) = o(n)$, a fact which follows easily from Szemerédi's famous theorem, whereas for the latter $r_{\mathcal{L}}(n) = \Omega(n)$ always.

This paper is concerned with non-invariant, homogeneous equations only. For simplicity the words '(linear) equation' will, for the remainder of the article, be assumed to refer to those equations with these extra properties, though some of our initial observations also apply in the inhomogeneous setting. We shall also employ the concise formulation 'A avoids $\mathcal{L}$' to indicate that a set A of positive integers contains no solutions to the equation $\mathcal{L}$. Finally we will employ the interval notation $[\alpha, \beta] := \{x \in \mathbf{Z} : \alpha \leq x \leq \beta\}$, and similarly for open intervals.

As Ruzsa observed, given an equation $\mathcal{L} : \sum a_i x_i = 0$, there are two basic ways to exhibit the fact that $r_{\mathcal{L}}(n) = \Omega(n)$:

I. Let $s := \sum a_i$ so $s \neq 0$. Let q be any positive integer not dividing s and let $A := \{x \in \mathbf{N} : x \equiv 1 \pmod{q}\}$. Then A avoids $\mathcal{L}$ and $|A \cap [1, n]| \geq \lfloor n/q \rfloor$.

II. Set

$$s_+ := \sum_{a_i > 0} a_i, \quad s_- := \sum_{a_i < 0} |a_i|$$

and assume without loss of generality that $s_+ > s_-$. For a fixed $n > 0$ let $A := \left(\frac{s_-}{s_+} n, n\right]$. Then A avoids $\mathcal{L}$ and has size $\Omega(n)$.

As in [11], set $\lambda_{0,\mathcal{L}} := \limsup_{n \to \infty} \frac{r_{\mathcal{L}}(n)}{n}$. Ruzsa asked whether the above two constructions were the prototypes for extremal $\mathcal{L}$-avoiding sets in the sense that $\lambda_0 = \max\{\rho, \frac{s_+ - s_-}{s_+}\}$, where the quantity ρ is defined as follows : for each $m > 0$ let $\rho_m \cdot m$ be the maximum size of a subset of $[1, m]$ which contains no solutions to $\mathcal{L}$ modulo m. Then $\rho := \sup_m \rho_m$.

As illustrated by Schoen [12], the answer to Ruzsa's question is no. But from what is currently known, it seems that for many equations something not much more complicated holds. One observes that the construction **II** above can be modified into something more general :

II′. For a given $\mathcal{L} : \sum a_i x_i = 0$, let notation be as above and let a denote the smallest absolute value of a negative coefficient a_i. Now for fixed $k, n > 0$ and $\xi \in [1, n]$

set

$$A_{n,k,\xi} := \bigsqcup_{j=1}^{k-1} \left(\frac{s_-}{s_+} n_j, n_j \right] \sqcup [\xi, n_k],$$

where $n_1, ..., n_k$ is any sequence of integers satisfying the recurrence

$$n_1 = n, \quad \frac{s_-}{s_+} n_k < \xi \le n_k, \quad s_+ n_{j+1} \le a \frac{s_-}{s_+} n_j + (s_- - a)\xi, \;\; j = 1, ..., k-1. \tag{2}$$

Clearly for (2) to have any solution we will need to have $k = O(\log n)$. Assuming a solution exists, the set $A_{n,k,\xi}$ avoids $\mathcal{L}$ and

$$|A_{n,k,\xi}| = (1 + o(1)) \cdot \left[\frac{s_+ - s_-}{s_+} \sum_{j=1}^{k-1} n_j + (n_k - \xi + 1) \right].$$

The important special case is when $\xi = 1 + \left\lfloor \frac{s_-}{s_+} n_k \right\rfloor$. Then Ruzsa's question can be replaced by the following :

Question *Is it always the case that*

$$\lambda_0 = \max \left\{ \rho, \sup_{n,k,\xi} \frac{|A_{n,k,\xi}|}{n} \right\},$$

where the supremum ranges over all triples n, k, ξ for which (2) has a solution where $\xi = 1 + \left\lfloor \frac{s_-}{s_+} n_k \right\rfloor$?

We will give examples in Section 3 which show that the answer to this question is still no : we are not aware of any in the existing literature. However, existing results strongly suggest that the answer is very often yes :
see [1] [2] [3] [8] [9] for example, plus further results in this paper. Also, in our counterexamples the extremal sets are a pretty obvious hybrid between the two alternatives which the question offers. We think that our question is thus a good foundation for further research in this area.

We now give a closer overwiew of the results in this paper. To identify extremal $\mathcal{L}$-avoiding sets and compute $\lambda_{0,\mathcal{L}}$ for arbitrary $\mathcal{L}$ seems a very daunting task, so an obvious strategy is to study equations in a fixed number of variables. One variable is utterly trivial and two only slightly less so. I have not been able to locate the following statement anywhere in the literature, however (though see for example [5], pp.30-34), so include it for completeness :

Proposition *Consider the equation $\mathcal{L} : ax = by$ where $a > b$ and $GCD(a, b) = 1$.*

For every $n > 0$ an extremal $\mathcal{L}$-avoiding subset of $[1, n]$ is obtained by running through the numbers from 1 to n and choosing greedily. This yields the extremal subset

$$A := \{u \cdot a^{2i} : i \geq 0 \text{ and } a \dagger u\}$$

of $\mathbf{N}$. In particular, $\lambda_{0,\mathcal{L}} = \frac{a}{a+1}$. For each $n > 0$ a complete description of the extremal $\mathcal{L}$-avoiding subsets of $[1, n]$ is given as follows :

Case I *: $b = 1$.*

For each $u \in [1, n]$ such that $u \dagger a$, let α be the largest integer such that $u \cdot a^{\alpha} \leq n$. Then an extremal set contains exactly $\lceil \alpha/2 \rceil$ of the numbers $u \cdot a^i$, for $0 \leq i \leq \alpha$, and no two numbers $u \cdot a^i$ and $u \cdot a^{i+1}$.

Case II *: $b > 1$.*

For each $u \in [1, n]$ divisible by neither a nor b and each non-negative integer α such that $u \cdot a^{\alpha} \leq n$, an extremal set contains exactly $\lceil \alpha/2 \rceil$ of the numbers $u \cdot b^i \cdot a^{\alpha-i}$, for $0 \leq i \leq \alpha$, and no two numbers $u \cdot b^i \cdot a^{\alpha-i}$ and $u \cdot b^{i+1} \cdot a^{\alpha-i-1}$.

Note that the proposition implies in particular that $\lambda_0 = \rho$ for any equation in two variables. For three variables things get more interesting and a number of papers have been entirely devoted to this situation, see [1] [2] [4] [6] [7] plus the multitude of papers on sum-free sets, of which the most directly relevant is probably [3]. The combined results of [1], [2] and [3] give, in principle, a complete classification of the extremal $\mathcal{L}$-avoiding subsets of $[1, n]$, for every $n > 0$, and $\mathcal{L} : x + y = cz$ for any $c \neq 2$. Of particular interest for us are the results of [1]. There it is shown that for every $c \geq 4$ and $n \gg_c 0$, a set $A_{n,3}$ of type $\mathbf{II'}$ is extremal, namely

$$A_{n,3} := \bigsqcup_{j=1}^{3} \left(\frac{2}{c} n_j, n_j \right],$$

where

$$n_1 = n, \quad n_{j+1} = \left\lfloor \frac{(1 + \lfloor \frac{2}{c} n_j \rfloor) + (1 + \lfloor \frac{2}{c} n_3 \rfloor)}{c} \right\rfloor, \quad j = 1, 2.$$

Moreover it is shown that, for all $n >>_c 0$, there are only a bounded number of extremal sets, all of whose symmetric differences with $A_{n,3}$ consist of a bounded number of elements (both bounds are independent of both n and c).

These results were partly extended in [4]. Here the authors considered equations $\mathcal{L} : ax + by = cz$ in two families :

Family I : $a = 1 < b$.

FAMILY II : $a = b$, $\mathrm{GCD}(b, c) = 1$.

For FAMILY I equations their main result is that, when $c > 2(b+1)^2 - \mathrm{GCD}(b+1, c)$ then sets $A_{n,2}$ of type $\mathbf{II}'$ consisting of exactly 2 intervals are extremal $\mathcal{L}$-avoiding sets in $[1, n]$ up to an error of at most $O(\log n)$ for every n. In particular, these sets give the right value of λ_0. They do not attempt any classification of the extremal sets, however. They also note that, whenever $c > (b+1)^{3/2}$, the same sets are of maximum size among all type $\mathbf{II}'$ sets, and conjecture that they are still extremal, up to the same $O(\log n)$ error.
For FAMILY II equations they simply note that when $c > (2b)^{3/2}$, then among all type $\mathbf{II}'$ sets the largest consist of three intervals. They do not discuss whether such sets are extremal or not.

Our results concern the same two families of equations. For FAMILY I we employ the methods of [1] to obtain a classification (Theorem 2.5) of the extremal $\mathcal{L}$-avoiding sets whenever

$$c > \frac{(b+1)b^2}{b-1}. \tag{3}$$

We show that, for every $n \gg_{b,c} 0$ the sets $A_{n,2}$ are actually extremal and there are only a bounded number of possibilities for the extremal subsets of $[1, n]$, all of which have a symmetric difference with $A_{n,2}$ of bounded size. Both bounds are independent of n, b and c.

We show by means of an example that the lower bound (3) on c cannot be significantly improved, which also disproves the conjecture of Dilcher and Lucht. Namely we show that when $c = b^2$ another type of $\mathcal{L}$-avoiding subset of $[1, n]$ is larger than $A_{n,2}$ by a factor of $\Omega(n)$. In some cases we can prove that these other sets are in fact extremal and conjecture that this is generally the case (conjecture 2.7).

For FAMILY II equations we describe extremal sets in $[1, n]$ for all n, and for every b, c with $b > 1$ (Theorem 3.1). Their appearance takes three different forms, for values of c in the following three ranges : (i) $c > 2b$, (ii) $2 \le c < 2b$, (iii) $c = 1$. In contrast to when $b = 1$, it is not the case for $c \gg b$ that the extremal sets consist of three intervals. Rather they are a hybrid between the two alternatives offered by our earlier Question.

2 Results for Family I equations

The methods of this section follow very closely those of [1], so we will not include full proofs of all results. Nevertheless, several technical difficulties arise and considerable care is needed to dispose of them. Thus we will give a fair amount of detail anyway, even if the resulting computations become somewhat long-winded. Let $\mathcal{L}$ be a fixed equation of the form $x + by = cz$ such that $b > 1$ and (3) holds. We prove analogues of Lemmas 2,3,4 and Theorems 1,2 in [1]. First a definition corresponding to Definition 1 of that paper :

DEFINITION 1 : Let $n \in \mathbf{N}$ and $A \subseteq [1, n]$ be $\mathcal{L}$-avoiding with smallest element $s := s_A$.

Define sequences $(r_i), (l_i), (A_i)$ by

$$A_0 := A, \quad r_1 := n,$$
$$l_i := \left\lfloor \frac{b+1}{c} r_i \right\rfloor, \quad r_{i+1} := \left\lfloor \frac{l_i + bs}{c} \right\rfloor,$$
$$A_i := (A_{i-1} \backslash (r_{i+1}, l_i]) \cup [l_i, r_i] \cap (s, n], \quad \text{for } i \geq 1.$$

Let t denote the least integer such that $r_{t+1} < s$. Observe that for all $i \geq t$,

$$A_i = A_t = [\alpha, r_t] \cup \left(\cup_{j=1}^{t-1} (l_j, r_j] \right), \tag{4}$$

where $\alpha = \alpha_A := \max\{l_t + 1, s\}$.

It is easy to see that, by construction, each set in the sequence (A_i) is $\mathcal{L}$-avoiding provided A_0 is, and that A_t is then an $\mathcal{L}$-avoiding set of type $\mathbf{II'}$ in the introduction. The crucial observation is the direct generalisation of Lemma 2(b) in [1] : because of its importance and because an apparently awkward technicality arises in dealing with one of the cases (*Case I* below), we will provide a complete proof.

Lemma 2.1 *Let $n > 0$ and $A := A_0 \subseteq [1, n]$ be $\mathcal{L}$-avoiding. Then $|A_i| \geq |A_{i-1}|$ for every $i > 0$.*

Proof : Following the same reasoning as in [1], it suffices to prove the claim for $i = 1$, and thus to prove that, for every $n > 0$ and every $\mathcal{L}$-avoiding subset of $[1, n]$, we have

$$|A| \leq |A \cap [1, r_{2,A}]| + \left\lceil \left(1 - \frac{b+1}{c} \right) n \right\rceil, \tag{5}$$

where

$$r_{2,A} := \left\lfloor \frac{\left\lfloor \frac{b+1}{c} n \right\rfloor + bs_A}{c} \right\rfloor.$$

The proof is by induction on n, the case $n = 1$ being trivial. So suppose the result holds for $1 \leq m < n$ and let A be an $\mathcal{L}$-avoiding subset of $[1, n]$. The result is again trivial if $s_A > \left(\frac{b+1}{c}\right) n$, so we may assume that $s_A \leq \left(\frac{b+1}{c}\right) n$ and thus that

$$r_{2,A} \leq \frac{\left(\frac{b+1}{c}\right) n + bs_A}{a} \leq \left(\frac{b+1}{c}\right)^2 n < \frac{n}{c},$$

because of (3).

First suppose that there exists $z \in A \cap \left(\frac{n}{c}, \frac{(b+1)n}{c}\right]$. To simplify notation, denote $A^c := [1, n] \backslash A$.

Case I : $z \leq bn/c$.

In this case we will show independently of the induction hypothesis that something stronger than (5) holds, namely that

$$|A| \leq \left\lceil \left(1 - \frac{b+1}{c}\right) n \right\rceil. \tag{6}$$

We have $cz \in (n, bn]$. Set $t := cz$. Now A contains no solutions to the equation

$$x + by = t. \tag{7}$$

Hence for every $y \in [\frac{t-n}{b}, \frac{t-1}{b}]$ at most one of the numbers y and $t - by$ lies in A. Now $t - by \equiv t \pmod b$ for every y. In this way we can locate in A^c at least as many numbers as there are numbers in the interval $[\frac{t-n}{b}, \frac{t-1}{b}]$ not congruent to $t \pmod b$. Define two parameters $u, v \in [1, b]$ as follows :

(i) $u \equiv t$ (mod b),

(ii) the total number of integers in the interval $\left[\frac{t-n}{b}, \frac{t-1}{b}\right]$ is congruent to $v \pmod b$.

Then one readily checks that the number of integers in $\left[\frac{t-n}{b}, \frac{t-1}{b}\right]$ not congruent to $t \pmod b$ is at least

$$\frac{b-1}{b}\left[\frac{n-(u-1)}{b} - \frac{b-v}{b-1}\right] = n\left(\frac{b-1}{b^2}\right) - \frac{(b-1)(u-1)+b(b-v)}{b^2},$$

and is at least one more than this when $v < b$ unless one of the first v numbers in the interval $\left[\frac{t-n}{b}, \frac{t-1}{b}\right]$ is congruent to $t \pmod b$. Set

$$f(n, b, c, u, v) := \frac{(b-1)(u-1)+b(b-v)}{b^2} - n\left(\frac{b-1}{b^2} - \frac{b+1}{c}\right).$$

Note that (3) implies that $f(n, b, c, u, v) < 2$ but, for (6) to be already satisfied we would need $f(n, b, c, u, v) < 1$. This is where things get messy. Note that certainly $f(n, b, c, u, v) < 1$ unless perhaps $u \geq 3$, $v \leq u - 2$ and one of the first v numbers in the interval $\left[\frac{t-n}{b}, \frac{t-1}{b}\right]$ is congruent to $u \pmod b$. The first assumption implies in particular that $b \geq 3$. All three together imply that the numbers 1 and 2 both lie to the left of the interval $\left[\frac{t-n}{b}, \frac{t-1}{b}\right]$, and neither is congruent to $t \pmod b$. Thus neither can have been already located in A^c via the pairing arising from (7). Since it suffices at this point to locate just one extra element in A^c we may for the remainder of this argument assume that $b \geq 3$ and that $1, 2 \in A$. The latter implies that there are no solutions in A to either of the equations

$$x + by = c, \tag{8}$$

$$x + by = 2c. \tag{9}$$

To continue the argument we go back to (7). To locate elements in A^c we paired off numbers in $\left[\frac{t-n}{b}, \frac{t-1}{b}\right]$ not congruent to $t \pmod b$ with numbers in $[1, n]$ congruent to

$t \pmod b$. It would thus suffice if we could also pair off at least one further number in the former interval, whch we call $\mathcal{I}$. It is easy to see that this can definitely be done if $\mathcal{I}$ contains a total of at least b^2 numbers. Hence we may further assume now that

$$|\mathcal{I}| \leq b^2, \tag{10}$$

and hence that $n \leq b^3$, though we will make no explicit use of this latter fact.

From (10) we want to conclude that either $n < c$ or, for an appropriate choice of the original z, that $c \equiv t \equiv 0 \pmod b$. So suppose $n \geq c$. First set

$$x_1 := 1, \quad x_2 := b+1, \quad y_1 := c-b, \quad y_2 := c - b(b+1).$$

Since A contains no solutions to (8), at most one of x_i and y_i is in A for each $i = 1, 2$. Thus $y_1 \in A^c$ since $n \geq c$ and we already know that $x_1 \in A$. From (3) it follows that $x_2 < y_2$ and also from (10) that $y_1 - y_2 > |\mathcal{I}|$, so that at least one of y_1 and y_2 must lie outside $\mathcal{I}$. Furthermore, since $u \geq 3$, neither of the x_i is congruent to $t \pmod b$. If $b+1 \notin \mathcal{I}$ it is thus already clear that, unless $c \equiv t \pmod b$, we can find amongst x_2, y_1, y_2 at least one element of A^c not previously located via (7). But similarly, if $b+1 \in \mathcal{I}$ then one easily checks that (3) implies that $y_1 \notin \mathcal{I}$ and so we have the same conclusion.

Thus we are done if $n \geq c$ unless $c \equiv t \pmod b$. To get that $c \equiv 0 \pmod b$ it would then suffice to also show that $2c \equiv t \pmod b$. If $n \geq 2c - b$ then this is immediately achieved by a similar argument to the one just given, but this time using (9) instead of (8). If $n < 2c - b$ then we just have to note that we could have from the beginning chosen $z := 2$, in which case $2c = t$, by definition.

Thus (6) holds unless either $n < c$ or $c \equiv 0 \pmod b$. By (3) the latter would imply that $c \geq b^2 + kb$ where we can take $k = 3$ when $b > 3$ and $k = 4$ when $b = 3$. Then

$$\begin{aligned} f(n,b,c,u,v) &= \frac{(b-1)(u-1) + b(b-v)}{b^2} - n\left(\frac{b-1}{b^2} - \frac{b+1}{c}\right) \\ &\leq \frac{(b-1)(2b-1)}{b^2} - n\left(\frac{b-1}{b^2} - \frac{b+1}{b^2+kb}\right). \end{aligned}$$

We'll be done unless $f(n,b,c,u,v) \geq 1$. One checks that this already forces $n < c$ when $b = 3$, and that for $b > 3$ it yields (taking $k = 3$) that

$$n \leq b^2 + 3b + 1 + \frac{6}{b-3} \leq c + 1 + \frac{6}{b-3}.$$

This in turn yields that

$$\left\lfloor \left(\frac{b+1}{c}\right) n \right\rfloor \leq b+1 \tag{11}$$

except when $b = 4, c = 28$ and $n \in \{34, 35\}$. One easily checks that (6) holds in these exceptional cases, so we're left with (11). First suppose $n \geq c$ so that $\left\lfloor \left(\frac{b+1}{c}\right) n \right\rfloor = b+1$, and consider (8). For $1 \leq i \leq b+1$ set $y_i := i$ and $x_i := c - ib$. Then at least one of

x_i and y_i is in A^c for each i. But (3) implies that $x_{b+1} < y_{b+1}$, hence $|A^c| \geq b+1$ which proves (6).

We are thus indeed left with the case when $n < c$. Now we could have chosen $z := 1$ initially and thus paired off numbers in $\mathcal{I}_1 := \left[\frac{c-n}{b}, \frac{c-1}{b}\right]$ not congruent to c (mod b) with numbers in $[1, n]$ congruent to c (mod b). As usual, it suffices to locate at least one further element in A^c. First suppose

$$\frac{2c-1}{b} \leq n. \tag{12}$$

Then similarly, by (9), we can pair off numbers in $\mathcal{I}_2 := \left[\frac{2c-n}{b}, \frac{2c-1}{b}\right]$ not congruent to $2c$ (mod b) with numbers in $[1, n]$ congruent to $2c$ (mod b). The crucial point is that, since $n < c$, the intervals $\mathcal{I}_1$ and $\mathcal{I}_2$ are disjoint. Each interval certainly contains at least three elements by (12). It is then easy to see that the $\mathcal{I}_2$-pairing will certainly locate at least one more element in A^c unless, at the very least, $2c \equiv c \equiv 0$ (mod b). But in that case the map $\phi : y \mapsto y + \frac{c}{b}$ is a bijection from $\mathcal{I}_1$ to $\mathcal{I}_2$ so that if the $\mathcal{I}_1$-pairing pairs y with x, say, then the $\mathcal{I}_2$-pairing pairs $\phi(y)$ with x. If we now choose y as the smallest multiple of b in $\mathcal{I}_1$, then we see that one of the two pairings must locate the desired extra element in A^c, unless perhaps $\frac{c}{b} \equiv 0$ (mod b) also. But then $c \equiv 0$ (mod b^2) and thus $c \geq 2b^2$ if $b > 3$ and $c \geq 3b^2$ if $b = 3$. But then, calculating as before, we'll have $f(n, b, c, u, v) < 1$ unless perhaps

$$\begin{cases} n \leq \frac{2(b^2-3b+1)}{b-3}, & \text{when } b > 3, \\ n \leq \frac{3(b^2-3b+1)}{2(b-2)}, & \text{when } b = 3. \end{cases}$$

But these inequalities contradict (12). Now we are only left with the possibility that $n < \frac{2c-1}{b}$, hence that $\left\lfloor \left(\frac{b+1}{c}\right) n \right\rfloor \in \{1, 2\}$. But in each case one may check that one can locate one or two elements of A^c as approppriate, by considering solutions of (8) with x and y close to $\frac{c}{b+1}$. This finally completes the analysis of *Case I*.

Case II : $z > bn/c$.

Then $cz \in (bn, (b+1)n]$. Put $cz := t$ again. Let $t = (b+1)n - s$ where $0 \leq s < n$. If $x + by = t$ for some integers $x, y \in [1, n]$, then $y \geq n - s/b$ and $x \geq n - s$. Since A avoids $\mathcal{L}$ we thus find, for every integer $y \in A \cap [n - \frac{s}{b}, n]$ not congruent to t (mod b), an integer $x \in A^c \cap [n - s, n]$, congruent to t (mod b). Noting in addition that at least one of n and $n - s$ is not in A, one readily verifies that hence

$$|A \cap [n - s, n]| \leq \left(1 - \frac{b-1}{b^2}\right)(s+1). \tag{13}$$

We now apply the induction hypothesis. Let $B := A \cap [1, n - s - 1]$. If B is empty then (13) and (3) immediately imply (5). Otherwise clearly $s_B = s_A$ and $r_{2,B} \leq r_{2,A}$, so the induction hyothesis gives that

$$|A| \leq |A \cap [1, r_{2,A}]| + \left\lceil \left(1 - \frac{b+1}{c}\right)(n - s - 1) \right\rceil + \left(1 - \frac{b-1}{b^2}\right)(s+1),$$

from which (5) follows by another application of (3).

We have thus completed the induction step under the assumption that $A \cap \left(\frac{n}{c}, \frac{(b+1)n}{c}\right] \neq \phi$, so we can now assume the intersection is empty. Suppose $z \in A \cap (r_{2,A}, n/c]$. Then $\left\lfloor \frac{b+1}{c} n \right\rfloor + bs_A < cz \leq n$ and $cz - bs_A \in A^c$. In other words, we can pair off elements in $A \cap (r_{2,A}, \frac{n}{c}]$ with elements in $\left(\frac{b+1}{c} n, n\right] \cap A^c$. This immediately implies (5) and completes the proof of Lemma 2.1.

Lemma 2.2 *Let A be an $\mathcal{L}$-avoiding subset of $[1,n]$ of maximum size. Let $s = s_A$ and $t := \max\{i \in \mathbf{N} : r_i \geq s\}$. If $n \gg_{b,c} 0$ then $t = 2$.*

PROOF : Just follow the reasoning in the proof of Lemma 3 in [1]. By Lemma 2.1, it suffices to know that there exists an absolute positive constant $\kappa^0_{b,c}$ such that, if $t \neq 2$ then

$$\frac{|A|}{n} \leq D(b,c) - \kappa^0_{b,c},$$

where

$$D(b,c) := \frac{(c-b-1)(c^2-b^2+1)}{c[c^2-b(b+1)]} \tag{14}$$

is such that, in the notation of eq.(4), $|A_2| = D(b,c) \cdot n + O(1)$ when $s = l_2 + 1$. The core of a proof that such a constant exists is contained in the proof of Lemma 1 in [4], though one has to be a little careful since there only sets A_t in which $s = l_t + 1$ are considered. However one can tediously check that allowing for arbitrary $s \in (l_t, r_t]$ will not change matters (I note that the authors of [4] needed this fact in Section 3 of their paper, though they do not seem to explicitly mention it anywhere).

Lemma 2.3 *Let $n \gg_{b,c} 0$. If A is an $\mathcal{L}$-avoiding subset of $[1,n]$ of maximum size then there exists an absolute positive constant $\kappa^1_{b,c}$ such that $S - \kappa^1_{b,c} \leq s_A \leq S + 2$ where $S = \left\lfloor \frac{(b+1)^2 n}{c[c^2-b(b+1)]} \right\rfloor$.*

PROOF : The proof follows that of Lemma 4 in [1]. We set

$$s' := \min\{s \in [1,n] : l_2(s) < s\}.$$

A computation similar to that in the Appendix of [1] yields that

$$l_2(s) < s \Leftrightarrow s > \frac{(b+1)^2}{c[c^2-b(b+1)]} n - (\epsilon_1 c + \epsilon_2) \left(\frac{b+1}{c^2 - b(b+1)} \right),$$

where $\epsilon_1, \epsilon_2 \in [0,1)$. By (3) it follows that

$$s' \in [S, S+1]. \tag{15}$$

Now we have

$$|A_2(s)| = \begin{cases} \left\lceil \left(1 - \frac{b+1}{c}\right) n \right\rceil + r_2(s) - s + 1, & \text{if } s \geq s', \\ \left\lceil \left(1 - \frac{b+1}{c}\right) n \right\rceil + r_2(s) - l_2(s), & \text{if } s < s'. \end{cases}$$

First suppose $s \geq s'$. We will certainly have $|A_2(s'+2)| < |A_2(s)|$ because of (3) and since $r_2(s)$ can only increase at most once in $\left\lfloor \frac{c}{b} \right\rfloor$ times. This proves that $s_A \leq S + 2$ for a maximum $\mathcal{L}$-avoiding A. Secondly, if $s < s'$ then $|A_2(s)|$ will decrease once $r_2(s) - l_2(s)$ decreases, and clearly this will happen after $\Omega(\frac{c}{b})$ steps. This proves that $S - \kappa^1_{b,c} \leq s_A$ for a maximum A and some $\kappa^1_{b,c} = \Omega(\frac{c}{b})$.

Theorem 2.4 *$r_{\mathcal{L}}(n) = D(b,c) \cdot n + O(1)$, where $D(b,c)$ is given by (14). In particular, $\lambda_{0,\mathcal{L}} = D(b,c)$.*

Proof : See Lemma 1 in [4] and Theorem 1 in [1]. Note that the second statement is the same as in Theorem 1 of [4], just with a better lower bound for c, namely (3).

We can now present the main classification result, analogous to Theorem 2 in [1]. In fact the result here is in some sense even cleaner, as the maximum $\mathcal{L}$-avoiding sets consist essentially of two rather than three intervals and there is thus even less possibility for variation.

Theorem 2.5 *Let $b, c \in \mathbf{N}$ with $b > 1$ and c satisfying (3). Let $\mathcal{L}$ be the equation $x + by = cz$. Let $n > 0$. Define S and s' as in Lemma 2.3. Let A be an $\mathcal{L}$-avoiding subset of $[1,n]$ of maximum size and with smallest element $s = s_A$. If $n \gg_{b,c} 0$ then the following holds : $s \in [S, S+2]$ and $A = \mathcal{I}_2 \cup \mathcal{I}_1$ where*

$$\mathcal{I}_2 \in \begin{cases} \{[s, r_2], [s, r_2+1]\}, & \text{if } s \geq s', \\ \{[s, r_2), [s, r_2] \backslash \{r_2 - \xi_1\}\}, & \text{if } s < s', \end{cases} \tag{16}$$

for some $\xi_1 \in [1, b]$, and

$$\mathcal{I}_1 \in \begin{cases} (l_1, n]\,, & \text{if } r_2 + 1 \notin A \text{ and } l_1 \notin A, \\ [l_1, n] \backslash \{n - \xi_2\}\,, & \text{if } r_2 + 1 \notin A \text{ and } l_1 \in A, \\ (l_1, n]\ \backslash \{l_1 + \xi_3\}, & \text{if } r_2 + 1 \in A \text{ and } l_1 \notin A, \\ [l_1, n] \backslash \{l_1 + \xi_4, n - \xi_5\}\,, & \text{if } r_2 + 1 \in A \text{ and } l_1 \in A, \end{cases} \tag{17}$$

for some $\xi_2 \in [0, n]$, $\xi_3 \in [1, n]$, $\xi_4 \in [1, b-1]$, $\xi_5 \in [0, n]$. Moreover, in all cases where they arise, the parameters ξ_i, $i = 1, ..., 5$, are uniquely determined by n and s according to the following relations :

$$cl_2 = (b+1)r_2 - \xi_1, \tag{18}$$

$$cl_1 = (b+1)n - \xi_i, \quad i \in \{2, 5\}, \tag{19}$$

$$c(r_2 + 1) = bs + (l_1 + \xi_i), \quad i \in \{3, 4\}. \tag{20}$$

Remark : As is the case in [1], Theorem 2.5 does not precisely determine the maximum $\mathcal{L}$-avoiding subsets of $[1, n]$ for every $n \gg_{b,c} 0$. For any particular n, some of the possibilities listed for A may either not be $\mathcal{L}$-avoiding or not have maximum size. But the important point is that we have a bounded number of possibilities for any n, and the symmetric difference between any two of these possibilities is also bounded in size, both bounds being independent of n, b and c. Since the periodicity phenomenon described in Section 4 of [1] easily generalises to the present setting, Theorem 2.5 thus reduces the precise classification of the extremal $\mathcal{L}$-avoiding sets to a finite computation for any given $\mathcal{L}$.

Proof of Theorem 2.5 : Again we follow the approach in [1]. On the one hand, since $t = 2$ here, there are fewer steps in the analysis. On the other hand, we will need a somewhat modified argument in one of the steps. We shall thus present the full argument quite carefully, though not in every single detail. We will need something analogous to Lemma 1 of [1]. Let $z \in \mathbf{N}$. Then $x + by = cz$ will have solutions where x and y are close to $\frac{c}{b+1}$. Indeed if $1 - \epsilon := \frac{c}{b+1} - \left\lfloor \frac{c}{b+1} \right\rfloor$ then we have a solution $(x_0, y_0) = \left(\frac{c}{b+1} - b\epsilon, \frac{c}{b+1} + \epsilon\right)$. For each $i \geq 0$ define the solution

$$(x_i, y_i) := \left(\frac{c}{b+1} - b(i+\epsilon), \frac{c}{b+1} + i + \epsilon\right).$$

Now for any $z \in \mathbf{N}$ and $d \geq 0$ let I_z^d denote the interval $[x_d, y_d]$. Then the analogue of Lemma 1 we need is

Lemma 2.6 *Let $A \subseteq [1, n]$ be $\mathcal{L}$-avoiding and $z \in A$. Then for any $d \geq 0$, $|I_z^d \backslash A| \geq d + 1$.*

Let A be a maximum $\mathcal{L}$-avoiding subset of $[1, n]$. By Lemma 2.1 we know that $|A| = |A_1| = |A_2|$ and by Lemma 2.2 that $r_3 < s$, when n is sufficiently large. The proof of Theorem 2.5 is accomplished in three steps. First, by comparing A_2 with A_1 we show that A contains almost the whole interval $(l_2, r_2]$. Here the argument entirely parallells that in [1]. In the second step we deduce that $(r_2, l_1] \cap A$ is almost empty. Here we use Lemma 2.6, but in a somewhat different way than in [1], as we will instead use an approach similar to that in the proof of Lemma 2.1. The final step is to compare A with A_1 to show that A contains almost all of $(l_1, n]$.

Step 1 : If $s > l_2$ then clearly $[s, r_2] \subseteq A$. Lemma 2.2 and (15) give $s \in [S, S + 2]$. Suppose $s \leq l_2$. We want to show that $s = l_2$. Suppose on the contrary that $B := [s, l_2)$ is non-empty. Put

$$C := I_s^1 \cup \bigcup_{b \in B \backslash \{s\}} I_b^0.$$

It is clear that $C \subseteq [l_2, r_2]$ for all $n \gg_{b,c} 0$. The crucial point is that (3) guarantees that all the intervals making up C are pairwise disjoint. Thus Lemma 2.6 implies that

$|C\backslash A| > |B|$, which contradicts the maximality of A. Thus $s = l_2$, which implies on the one hand (computation required, using (3)) that $s \geq S$, and on the other that

$$|A \cap [s, r_2]| = |(s, r_2]|. \tag{21}$$

If $r_2 \notin A$ we infer that $A \cap [s, r_2] = [s, r_2)$. If $r_2 \in A$ then $cs - br_2 = cl_2 - br_2 \notin A$, so $-c + 1 \leq cl_2 - (b+1)r_2 \leq -1$ and hence

$$\frac{-c+1}{b+1} \leq \frac{c}{b+1} l_2 - r_2 < 0.$$

But if $\frac{c}{b+1} l_2 - r_2 \leq -1$ then $I^1_{l_2} \subseteq (l_2, r_2]$ (for $n \gg_{b,c} 0$), which would contradict (21). Thus $\frac{c}{b+1} l_2 - r_2 \in (-1, 0)$, which confirms that $A \cap [s, r_2] = [s, r_2] \backslash \{r_2 - \xi_1\}$ for the unique $\xi_1 \in [1, b]$ satisfying (18). We also deduce that $r_2 + 1 \notin A$ since $cl_2 - b(r_2+1) < r_2 - b$ and thus lies in A.

This completes Step 1 and shows that A must contain a set $\mathcal{I}_2$ which is one of the possibilities given by (16).

Step 2 : We will show that if $n \gg_{b,c} 0$ then $A \cap [r_2 + 2, l_1) = \phi$. We have

$$|A| = |A_1| = |(A \backslash (r_2, l_1]) \cup (l_1, n]|.$$

So the idea is to show that if $A \cap [r_2 + 2, l_1)$ were non-empty, then it would have to have smaller cardinality than $(l_1, n] \backslash A$. Since $b > 1$ the direct analogue of the argument in [1] will not work. Instead we gain inspiration from the proof of Lemma 2.1. First suppose there exists $z \in A \cap \left(\frac{n}{c}, \frac{bn}{c}\right]$. As in the proof of Lemma 2.1, this implies that

$$|A| < \left(1 - \frac{b-1}{b^2}\right) n + 2,$$

and thus, by Theorem 2.4, A can't possibly be maximum $\mathcal{L}$-avoiding for $n \gg_{b,c} 0$. Next suppose there exists $z \in A \cap \left(\frac{bn}{c}, \frac{(b+1)n}{c}\right]$. Again, as in the proof of Lemma 2.1, this will gives us a $\sigma \in [0, n]$ such that

$$|A \cap [n - \sigma, n]| \leq \left(1 - \frac{b-1}{b^2}\right)(\sigma + 1).$$

Clearly then, by Theorem 2.4, there exists a constant $\kappa^2_{b,c} > 0$ such that A cannot possibly be maximum $\mathcal{L}$-avoiding if $\sigma > \kappa^2_{b,c}$. Thus there exists a corresponding $\kappa^3_{b,c} > 0$ such that we can now deduce that $A \cap \left(\frac{n}{c}, l_1 - \kappa^3_{b,c}\right]$ is empty.

Let $\mathcal{U}_1 := A \cap [r_2 + 2, n/c]$ and $\mathcal{U}_2 := A \cap (l_1 - \kappa^3_{b,c}, l_1)$. It remains to show that $\mathcal{U}_1$ and $\mathcal{U}_2$ are empty, so let us suppose otherwise.

For $z \in \mathcal{U}_1$, let $\mathcal{C}_z := \{cz - bs, cz - b(s+1)\}$. Also let $C_{r_2+1} := \{c(r_2+1) - bs\}$. Then $\mathcal{C}_z \cap A = \phi$ for any $z \in A \cap [r_2 + 1, n/c]$. Clearly, if $n \gg_{b,c} 0$ then $C_z \subset (l_1, n - \Omega(n)]$. Also (3) guarantees that the $\mathcal{C}_z$ are pairwise disjoint.

For $z \in \mathcal{U}_2$ let $\mathcal{D}_z := I_z^1 \cap A^c$ if z is the smallest element of $\mathcal{U}_2$ and $\mathcal{D}_z := I_z^0 \cap A^c$ otherwise. Again (3) guarantees that the $\mathcal{D}_z$ are pairwise disjoint. Clearly there exists a constant $\kappa_{b,c}^4 > 0$ such that all the $\mathcal{D}_z$ are contained in $[n - \kappa_{b,c}^4, n]$. Thus the $\mathcal{D}_z$ are also disjoint from the $\mathcal{C}_z$.

In summary, we can thus conclude that, for n sufficiently large,

$$|(l_1, n] \cap A^c| \geq \delta_1 + 2|\mathcal{U}_1| + (|\mathcal{U}_2 + 1 - \delta_2),$$

where $\delta_1 = 1$ if $r_2 + 1 \in A$ and zero otherwise, and $\delta_2 = 1$ if $\mathcal{U}_2 = \phi$ and zero otherwise. It follows immediately that $|A| < |A_1|$ unless $\mathcal{U}_1$ and $\mathcal{U}_2$ are both empty. This completes Step 2.

Step 3 : It just remains to show that the possibilities for $A \cap [l_1, n]$ are as given by (17). By Steps 1 and 2 we only have four cases to consider, according to $A \cap \{r_2 + 1, l_1\}$. To verify the various possibilities for $\mathcal{I}_1$ one considers the numbers $c(r_2 + 1) - bs$ or $cl_1 - (b + 1)n$ as appropriate, the analysis being similar to that in the latter part of Step 1 above. We omit the details and consider the proof of Theorem 2.5 as complete.

We close this section by showing that, in general, the bound (3) cannot be significantly decreased without the extremal sets avoiding $x + by = cz$ looking quite different than those described in the above theorem. In particular we have a counterexample to the conjecture in [4] that a bound of $c > (b + 1)^{3/2}$ should suffice. For a counterexample we set $c = b^2$. Let $\mathcal{L}^b$ denote the equation $x + by = b^2 z$ where $b > 1$. The constructions considered earlier in this section yield that

$$\lambda_{0,\mathcal{L}_b} \geq D(b, b) = \frac{(b^2 - b - 1)(b^4 - b^2 + 1)}{b^2[b^4 - b(b + 1)]}.$$

But for every $b > 1$ the true value of $\lambda_{0,\mathcal{L}_b}$ is larger. For let

$$A^b := \{u \cdot b^{3i} : u > 0,\ i \geq 0 \text{ and } b \dagger u\}.$$

(Here we use a superscript so as not to confuse these sets with those described in eq.(4) earlier). Then clearly A^b is an $\mathcal{L}^b$-avoiding subset of $\mathbf{N}$ and

$$d(A^b) = \frac{b^2}{b^2 + b + 1} > \frac{(b^2 - b - 1)(b^4 - b^2 + 1)}{b^2[b^4 - b(b + 1)]} \quad \forall\, b \geq 2.$$

We conjecture the following :

Conjecture 2.7 *For every $n > 0$ and every $b \geq 2$ the set $A^b \cap [1, n]$ is an $\mathcal{L}^b$-avoiding subset of $[1, n]$ of maximum size. In particular $\lambda_{0,\mathcal{L}^b} = \rho_{\mathcal{L}^b} = \frac{b^2}{b^2+b+1}$.*

We suspect in fact that for $n \gg_b 0$ any extremal $\mathcal{L}^b$-avoiding subset of $[1, n]$ must be very similar to $A^b \cap [1, n]$. Frustratingly we have not been able to verify any of these assertions in general, not even the value of $\lambda_{0,\mathcal{L}^b}$. We do have proofs of Conjecture 2.7 for $b = 2, 3$ which we now present. They employ the same idea, but things get pretty messy for $b = 3$ and we don't see how to make the same idea work for larger b.

Theorem 2.8 *Conjecture 2.7 holds for $b = 2$ and $b = 3$.*

Proof for $b = 2$: Fix $n > 0$. Put $A = A^2 \cap [1, n]$. Let B be an $\mathcal{L}^2$-avoiding subset of $[1, n]$. We must show that $\mid B \mid \leq \mid A \mid$. We will do this by exhibiting a one-to-one function

$$f : B \backslash A \to A \backslash B.$$

For $k = 1, 2$ let

$$B_k := B \cap \{u \cdot 2^{3i+k} : 2 \dagger u, i \geq 0\}.$$

Then $B \backslash A = B_1 \sqcup B_2$. For $x \in B \backslash A$, we shall define $f(x)$ according as to whether $x \in B_1$ or $x \in B_2$.

First suppose $x \in B_1$. Then $x = 2y$ for some $y \in A$. But $y \notin B$ since B avoids $\mathcal{L}^2$ and

$$4 \cdot y = 2 \cdot y + (2y).$$

So in this case we define $f(x) = y$.

Next suppose $x \in B_2$. Then $x = 4y$ for some $y \in A$. Then $3y \in A$, but $3y \notin B$ since B avoids $\mathcal{L}^2$ and

$$4(3y) = 2(4y) + (4y).$$

So in this case we define $f(x) = 3y$.

It is clear that the restrictions of f to both B_1 and B_2 are one-to-one. So it remains to show that $f(B_1) \cap f(B_2) = \phi$.

So let $y, z \in A$ and suppose that $f(2y) = f(4z)$. Thus $y = 3z$ and so $2y = 6z$. So both $4z \in B$ and $6z \in B$. But this is a contradiction, since B avoids $\mathcal{L}^2$ and

$$4(4z) = 2(6z) + (4z).$$

Proof for $b = 3$: Fix $n > 0$. Put $A = A^3 \cap [1, n]$ and let B be an $\mathcal{L}^3$-avoiding subset of $[1, n]$. As before we will describe an explicit one-to-one function $f : B \backslash A \to A \backslash B$. We define the sets B_1 and B_2 in an analogous manner to above and for $x \in B \backslash A$ will define $f(x)$ according as to whether $x \in B_1$ or B_2. This time, both the definition of f and the proof that it is one-to-one will be somewhat more complicated than before, so we divide this process into three clear steps.

Step 1 : We define f on B_1 and show that $f\,|_{B_1}$ is one-to-one.

Let $x \in B_1$. Then $x = 3y$ for some $y \in A$. Then $2y \in A$. Now it can't be the case that both y and $2y$ lie in B, since B avoids $\mathcal{L}^3$ and

$$9 \cdot y = 3(2y) + (3y).$$

Hence we define

$$f(x) = \begin{cases} y, & \text{if } y \notin B, \\ 2y, & \text{otherwise.} \end{cases}$$

We need to show that f is on-to-one on B_1. Suppose otherwise. Then there is an $x = 3y \in B_1$ such that $2x = 6y \in B_1$ and $f(x) = f(2x) = 2y$. But this implies that $y \in B$, which is a contradiction, since B avoids $\mathcal{L}^3$ and

$$9 \cdot y = 3 \cdot y + (6y).$$

Step 2 : We define f on B_2 in such a way that

$$f(B_1) \cap f(B_2) = \phi.$$

Let $x \in B_2$. Then $x = 9y$ for some $y \in A$. Now $4y \in A$, but $4y \notin B$ since B avoids $\mathcal{L}^3$ and

$$9(4y) = 3(9y) + (9y).$$

If $4y \in f(B_1)$ then either

(a) $12y \in B$, in which case $f(12y) = 4y$, or

(b) $12y \notin B$, but $6y \in B$ and $f(6y) = 4y$. In this case, the definition of f on B_1 implies that $2y \in B$. I also claim that in this case, $y \notin B \cup f(B_1)$. Suppose $y \in B$. Then, since $6y \in B$, the equation $9 \cdot y = 3 \cdot y + (6y)$ contradicts the fact that B avoids $\mathcal{L}^3$. So suppose $y \in f(B_1)$. Then either $y = f(3y)$ or $y = f(3y/2)$. But if $3y \in B$ then, since both $6y$ and $9y$ are also in B, the equation $9(3y) = 3(6y) + (9y)$ contradicts the fact that B avoids $\mathcal{L}^3$. And if $3y/2 \in B$ then the equation $9(\frac{3y}{2}) = 3(\frac{3y}{2}) + (9y)$ likewise gives a contradiction.

Hence, we begin by defining

$$f(x) = \begin{cases} 4y, & \text{if } 12y \notin B \text{ and } \{2y, 6y\} \not\subseteq B, \\ y, & \text{if } 12y \notin B \text{ and } \{2y, 6y\} \subseteq B. \end{cases}$$

It remains to define $f(x)$ when $12y \in B$. Notice that then $2y \notin B$, since $9(2y) = 3(2y) + (12y)$. If $2y \in f(B_1)$ then either $2y = f(3y)$ or $2y = f(6y)$. So we define

$$f(x) = 2y, \quad \text{if } 12y \in B,\ 3y \notin B \text{ and } 6y \notin B.$$

Next suppose $3y \notin B$ but $6y \in B$. Then $y \notin B$ since $9 \cdot y = 3 \cdot y + (6y)$. And $y \notin f(B_1)$ either, since if it were then either $y = f(3y)$ or $y = f(3y/2)$. But $3y \notin B$, by assumption, and $3y/2 \notin B$ since $9(\frac{3y}{2}) = 3(\frac{3y}{2}) + (9y)$.

Thus we may define

$$f(x) = y, \quad \text{if } 12y \in B,\ 3y \notin B \text{ and } 6y \in B.$$

Finally, it remains to define f when both $12y$ and $3y$ are in B. I claim that in this case, $8y \notin B \cup f(B_1)$. Suppose $8y \in B$. Then $9(3y) = 3(8y) + (3y)$, contradicting B's avoidance of $\mathcal{L}^3$. Suppose $8y \in f(B_1)$. Then either $8y = f(12y)$ or $8y = f(24y)$. But $f(12y) = 4y$, since $4y \notin B$, by the definition of f on B_1. Otherwise $24y \in B$, in which case $9(9y) = 3(24y) + (9y)$, provoking another contradiction.

Thus we may define

$$f(x) = 8y, \quad \text{if } 12y \in B \text{ and } 3y \in B.$$

This completes the definition of f on B_2, and it is automatic that $f(B_1) \cap f(B_2) = \phi$.

Step 3 : We show that $f\,|_{B_2}$ is one-to-one.

So suppose that there are $y, z \in A$ with $y \neq z$ but $f(9y) = f(9z)$. Without loss of generality, $f(9y)/9y < f(9z)/9z$. We then have nine cases to consider.

Case I : $f(9y) = y$, $f(9z) = 2z$ and $12y \notin B$.

Then $y = 2z$ and $\{2y, 6y\} \subseteq B$. But $2y = 4z$ and then the equation $9(4z) = 3(9z) + (9z)$ contradicts B's avoidance of $\mathcal{L}^3$.

Case II : $f(9y) = y$, $f(9z) = 2z$ and $12y \in B$.

Then $12y = 24z$ and the equation $9(9z) = 3(24z) + (9z)$ contradicts B's avoidance of $\mathcal{L}^3$.

Case III : $f(9y) = y$, $f(9z) = 4z$ and $12y \notin B$.

Then $6y = 24z \in B$ and we get the same contradiction as in Case II.

Case IV : $f(9y) = y$, $f(9z) = 4z$ and $12y \in B$.

Then we still have that $6y = 24z \in B$, so we get the same contradiction as in Case II.

Case V : $f(9y) = y$, $f(9z) = 8z$ and $12y \notin B$.

Then $12z = 3y/2 \in B$ and the equation $9(\frac{3y}{2}) = 3(\frac{3y}{2}) + (9y)$ yields a contradiction.

Case VI : $f(9y) = y$, $f(9z) = 8z$ and $12y \in B$.

Then we still have that $12z = 3y/2 \in B$, so we get the same contradiction as in Case V.

Case VII : $f(9y) = 2y$ and $f(9z) = 4z$.

Then $y = 2z$ and $12y = 24z \in B$, so we get the same contradiction as in Case II.

Case VIII : $f(9y) = 2y$ and $f(9z) = 8z$.

Then $12z = 3y \in B$, contradicting the definition of f and the fact that $f(9y) = 2y$.

Case IX : $f(9y) = 4y$ and $f(9z) = 8z$.

Then $y = 2z$ and $3z = 3y/2 \in B$, so we get the same contradiction as in Case V.

We have now completed Steps 1,2 and 3, and with that the proof of
Theorem 2.8.

3 Results for Family II equations

In this section, $\mathcal{L}$ denotes an equation $b(x + y) = cz$, where b and c are positive integers such that $b > 1$ and $\mathrm{GCD}(b, c) = 1$. These equations were briefly touched on in [4], and the case $b = 1$ was studied in detail in [1]. We present a theorem which describes extremal $\mathcal{L}$-avoiding subsets of $[1, n]$ for all values of b, c and n. The most interesting part of the theorem is part (i) which shows that the situation when $c > 2b$ is quite different from when $b = 1$, since the extremal sets we describe are a 'hybrid' between the two possibilities predicted by the Question in the introduction.

Theorem 3.1 *(i) If $c > 2b$ then, for every $n > 0$, the set*

$$A_n = \left(\frac{2bn}{c}, n\right] \cup \left\{x \in \left[1, \frac{2bn}{c}\right] : x \not\equiv 0 \ (mod\ b)\right\}$$

is an $\mathcal{L}$-avoiding subset of $[1, n]$ of maximum size. In particular, $\lambda 0, \mathcal{L} = 1 - \frac{2}{c}$.
(ii) If $2 \le c < 2b$ then, for every $n > 0$, the set

$$A'_n = \{x \in [1, n] : x \not\equiv 0 \ (mod\ b)\}$$

is an $\mathcal{L}$-avoiding subset of $[1, n]$ of maximum size. In particular, $\lambda_{0,\mathcal{L}} = 1 - \frac{1}{b}$.
(iii) If $c = 1$ then, for every $n > 0$, the set

$$A''_n = \left(\frac{n}{2b}, n\right]$$

is an $\mathcal{L}$-avoiding subset of $[1, n]$ *of maximum size. In particular,* $\lambda_{0,\mathcal{L}} = 1 - \frac{1}{2b}$.

Note that

$$|A_n| = n - \left\lfloor \frac{2n}{c} \right\rfloor, \tag{22}$$

$$|A'_n| = n - \left\lfloor \frac{n}{b} \right\rfloor, \tag{23}$$

$$|A''_n| = n - \left\lfloor \frac{n}{2b} \right\rfloor. \tag{24}$$

Proof of part (i) : We fix $c > 2b$ and proceed by induction on n. The theorem obviously holds if $n = 1$. Fix $n > 1$ and let B be any $\mathcal{L}$-avoiding subset of $[1, n]$. We must show that $|B| \le |A_n|$.

First suppose there exists a number $z \in B \cap (\frac{bn}{c}, \frac{2bn}{c}]$ which is a multiple of b. Let $z = bz_1$. Then $t := cz_1 \in (n, 2n]$ and, since B avoids $\mathcal{L}$, there are no solutions in B to the equation

$$x + y = t.$$

Now the map $f : x \mapsto t - x$ is a 1-1 mapping from the interval $I := [t - n, n]$ to itself, and for each $x \in I$, at most one of x and $f(x)$ lies in B. Define $s \in [1, n]$ so that $t - n = n - s + 1$. Then we conclude that

$$|B \cap [n - s + 1, n]| \le \lfloor s/2 \rfloor.$$

If $s = n$ then $|B| \le \lfloor \frac{n}{2} \rfloor \le |A_n|$, by (22). Otherwise, the induction hypothesis yields that

$$\begin{aligned} |B| &\le |A_{n-s}| + \left\lfloor \frac{s}{2} \right\rfloor \\ &= (n - s) - \left\lfloor \frac{2(n - s)}{a} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor \\ &< |A_n| + 1, \end{aligned}$$

and since $|B|$ is an integer, we conclude that $|B| \le |A_n|$, as desired.

Thus we may assume that B contains no multiples of b in the interval $(\frac{bn}{c}, \frac{2bn}{c}]$. If B contains no multiples of b at all in the range $[1, \frac{2bn}{c}]$, then trivially $|B| \le |A_n|$. So let's assume B does contain such a number, and let the largest such be z_0. Thus $z_0 \in [1, \frac{bn}{c}]$. Let $z_0 = bz_1$. Then $cx_1 \in [c, n]$ and there are no solutions in B to

$$x + y = cz_1. \tag{25}$$

Note that since $c > 2b$, we have $cz_1 > 2z_0$, so if (x, y) is a solution to (25), then $x \le z_0 \Rightarrow y > z_0$.

Suppose $cz_1 \equiv j \pmod{b}$. Then for every number $x \in [1, z_0]$ such that $x \not\equiv j \pmod{b}$, at most one of x and $cz_1 - x$ lies in B. But note that, for every such x, $cz_1 - x$ is not divisible by b, and is strictly greater than z_0. We conclude that at least $z_0 - z_1 \geq z_1$ numbers are missing from B which are either multiples of b in $[1, z_0]$ or not divisible by b. This implies that $|B| \leq |A_n|$ and completes the proof of part (i) of Theorem 3.1.

Proof of Part (ii) : We divide the proof into two cases.

Case I : $2 \leq c < b$.

Fix $n > 0$ and an $\mathcal{L}$-avoiding subset B of $[1, n]$. We must show that $|B| \leq |A'_n|$. If B contains no multiples of b we are done, so suppose the contrary. Let $z = bz_1$ be the largest element of B which is a multiple of b. Then it suffices to produce at least z_1 numbers in the interval $[1, z]$ which are not in B. Since B avoids $\mathcal{L}$, it contains no solutions to the equation

$$x + y = cz_1.$$

Thus B contains no more than $\left\lceil \frac{cz_1}{2} \right\rceil$ of the numbers in the interval $[1, cz_1]$. But since $2 \leq c < b$, it follows that $z_1 \leq \left\lfloor \frac{cz_1}{2} \right\rfloor$ and $cz_1 < z$. Thus we are done.

Case II : $b < c < 2b$.

We proceed by induction on n. The theorem obviously holds for $n = 1$. Now fix $n > 1$ and an $\mathcal{L}$-avoiding subset B of $[1, n]$. If B contains no multiples of b then we are done, so we may assume that B contains some such elements.

First suppose there exists $z \in B \cap (\frac{b}{c}n, n]$ which is a multiple of b. Let $z = bz_1$. Then $cz_1 \in (n, \frac{c}{b}n] \subseteq (n, 2n]$. To simplify notation, set $cz_1 := t_1$ and $t_1 - n := n - s_1 + 1$. Since B avoids $\mathcal{L}$, it contains no solutions to the equation

$$x + y = t_1.$$

The map $f : x \mapsto t_1 - x$ is a 1-1 mapping from the interval $I_1 = [n - s_1 + 1, n]$ to itself and for each $x \in I_1$ at most one of the numbers x and $f(x)$ lies in B. Thus

$$|\ B \cap I_1\ | \leq \lfloor s/2 \rfloor\ .$$

Then clearly (since $b \geq 2$) the induction argument implies that $|B| \leq |A'_n|$.

Thus we may assume that $B \cap (\frac{b}{c}n, n]$ contains no multiples of b. But then another application of the induction hypothesis yields that $|B| \leq |A'_n|$ in this case too.

Thus part (ii) of the theorem is proved.

Proof of Part (iii) : The argument is similar to that in Case I of part (ii). Let B be an $\mathcal{L}$-avoiding subset of $[1, n]$. If B contains no multiples of b, then clearly $|B| \leq |A''_n|$. So suppose $z = bz_1$ is the largest multiple of b in B. Then

$$|B \cap (z, n]| \leq (n - z) - \left\lfloor \frac{n - z}{b} \right\rfloor . \tag{26}$$

Since B avoids $\mathcal{L}$, it contains no solutions to the equation

$$x + y = z_1.$$

Thus

$$|B \cap [1, z]| \leq z - \left\lfloor \frac{z_1}{2} \right\rfloor = z - \left\lfloor \frac{z}{2b} \right\rfloor . \tag{27}$$

Clearly, (26) and (27) imply that $|B| \leq |A''_n|$ (with strict inequality unless $z_1 = \lfloor n/b \rfloor$).

This completes the proof of Theorem 3.1.

Concluding remark It is worthwhile to investigate if the proof of Theorem 3.1 can be used to obtain a stronger result, namely a classification of the extremal sets. We choose not to go into this matter in this paper, which we think already contains enough in the way of detailed, technical computations. In any case, the important thing is the 'hybrid' nature of the extremal sets in part (i) of the theorem. Note that, for each $b \geq 2$, the sets A_n have strictly greater asymptotic density than any sets of type $\mathbf{II}'$. This follows from (22), Lemma 1(b) of [4] and a straightforward computation.

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