Small integral trees

A. E. Brouwer

Dept. of Math., Techn. Univ. Eindhoven
P.O. Box 513, 5600MB Eindhoven, Netherlands
aeb@cwi.nl

Submitted: Dec 31, 2007; Accepted: Jan 18, 2008; Published: Jan 28, 2008
Mathematics Subject Classification: 05C05, 05C30, 05E99

Abstract

We give a table with all integral trees on at most 50 vertices, and characterize integral trees with a single eigenvalue 0.

1 Integral trees

A finite graph is called *integral* if the spectrum of its adjacency matrix has only integral eigenvalues. A *tree* is a connected undirected graph without cycles.

In this note we give a table with all integral trees on at most 50 vertices, and a further table with all known integral trees on at most 100 vertices. (For an on-line version, possibly with updates, see [1].) In particular, we find the smallest integral trees of diameter 6, and the smallest known integral tree of diameter 8.

The nicest result about integral trees is that by Watanabe [12] that says that an integral tree different from K_2 does not have a complete matching. Here we give a generalization.

All 'starlike' integral trees, that is, all integral trees with at most one vertex of degree larger than 2, were given by Watanabe and Schwenk [13].

All integral trees of diameter at most 3 were given in [13, 4]. See also [10, 3].

Several people have worked on constructing integral trees with large diameter, and examples with diameters 0–8 and 10 are known, see [13, 9, 3, 8, 6, 7]. It is unknown whether integral trees can have arbitrarily large diameter.

The *spectral radius* of a nonempty graph is the maximum absolute value of an eigenvalue. In [2] all integral trees with spectral radius at most 3 are determined.

1.1 Names of general trees

In the tables below, we need a notation to name trees. Given a tree, pick some vertex and call it the *root*. Now walk along the tree (depth-first), starting at the root, and when a

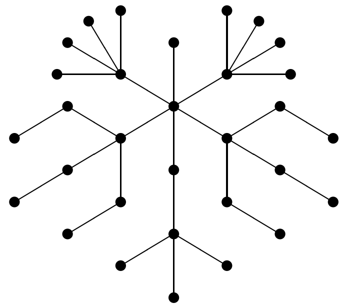

Figure 1: #17: $0123^3(1(23)^3)^2(12^4)^21$.

vertex is encountered for the first time, write down its distance to the root. The sequence of integers obtained is called a *level sequence* for the tree. A tree is uniquely determined by any level sequence. The parent of a vertex labeled m is the last vertex encountered earlier that was labeled $m-1$.

For example, the graph $K_{1,4}$ gets level sequence 01111 if the vertex of degree 4 is chosen as root, and 01222 otherwise.

We use exponents to indicate repetition: 01111 can be written 01^4 and 0121212 as $0(12)^3$.

1.2 Names of special trees

In the below we shall have use for some more or less standard tree names. Some of the constructions below require a rooted tree as input, and we indicate the root.

A) K_1 is a single vertex. K_2 is a single edge. (Any of the two vertices can serve as root.)

B) $K_{1,m}$ is the star on $m+1$ vertices. Its root is the central vertex.

C) If G is a tree, then SG is the subdivision of G (with one new vertex in the middle of each edge of G). If G is rooted, then so is SG, with the same root.

D) If G and H are rooted trees, then $G \sim H$ is the tree obtained from the disjoint union of G and H by joining the roots. The resulting tree does not have a designated root.

E) Consider for rooted trees $G_1, ..., G_s$ the rooted tree $C(G_1, ..., G_s)$ obtained by adding a new vertex to the disjoint union of the G_i, joining the new vertex to the root of each G_i, and taking this new vertex to be the root of $C(G_1, ..., G_s)$. For example, $C(mK_1)$ is $K_{1,m}$ and $C(mK_2)$ is $SK_{1,m}$. (Here mH denotes the disjoint union of m copies of H.)

F) Define rooted trees $T(n_k, ..., n_1)$ by induction on k as follows: $T()$ is K_1 and $T(n_k, ..., n_1) = C(n_k T(n_{k-1}, ..., n_1))$ for $k > 0$. For example, $T(m)$ is $K_{1,m}$, and $T(m, 1)$ is $SK_{1,m}$.

G) If G and H are rooted trees, then $G * H$ is the rooted tree obtained from the disjoint union of G and H by identifying the roots. (The resulting vertex is the new root.) For example, $T(s) * T(m,t)$ is $C(sK_1 + mK_{1,t})$. (Here $G + H$ denotes the disjoint union of G and H.)

Now tree #17 depicted above is $T(1) * T(2,4) * T(1,1,3) * T(2,3,1)$.

1.3 Families of integral trees

We give some families of integral trees for later reference. All except the last one occur in the literature. Spectra are usually given with multiplicities written as exponents.

(i) The spectrum of the complete bipartite graph $K_{1,m}$ (the tree 01^m) is $\pm\sqrt{m}$, 0^{m-1}. It follows that $K_{1,m}$ is integral when m is a square (Harary & Schwenk [5]).

(ii) The spectrum of $SK_{1,m}$ (the tree $0(12)^m$), is $\pm\sqrt{m+1}$, $\pm 1^{m-1}$, 0. It follows that $SK_{1,m}$ is integral when $m+1$ is a square.

Watanabe & Schwenk [13] showed that the graphs in (i) and (ii) are the only integral trees with a single vertex of degree more than two.

(iii) The spectrum of $K_{1,m} \sim K_{1,r}$ (the tree $01^{m+1}2^r$), the result of joining the centers of $K_{1,m}$ and $K_{1,r}$, consists of 0^{m+r-2} together with the four roots of

$$X^4 - (m+r+1)X^2 + mr = 0.$$

These are integral for example when $m = r = a(a+1)$ for some positive integer a (and then the positive roots are a and $a+1$). There are also other solutions - the smallest is $K_{1,50} \sim K_{1,98}$ on 150 vertices. The question which m and r give integral solutions was settled by Graham [4].

(iv) The spectrum of $K_{1,m} \sim SK_{1,r}$ (the tree $01^{m+1}(23)^r$), the result of joining the centers of $K_{1,m}$ and $SK_{1,r}$, consists of 0^m and $\pm 1^{r-1}$ together with the four roots of

$$X^4 - (m+r+2)X^2 + mr + m + 1.$$

These are integral for example for $m+1 = r = a(a+1)$ and for $m-1 = r+2 = a(a+1)$ (and in both cases the positive roots are a and $a+1$). There are also other solutions - the smallest is $K_{1,287} \sim SK_{1,144}$ on 577 vertices. One can find all solutions by the method of Graham [4].

Watanabe & Schwenk [13] also studied the situation with two adjacent vertices of degree more than two, and proved that in that situation one must have one of the examples given under (iii) and (iv).

(v) The spectrum of $T(r,m)$ is $\pm\sqrt{r+m}$, $\pm\sqrt{m}^{r-1}$, 0^{mr-r+1}. Thus, this graph is integral precisely when both m and $r+m$ are squares (Watanabe & Schwenk [13]).

Many other families have been studied, especially by Chinese mathematicians, but most have not more than one or two representatives in the tables below, and we refer directly to the literature. It remains to give a uniform explanation for many of the remaining graphs.

Lemma 1 *Consider the tree* $T(a) * T(b,1) * T(c,4) * T(d,1,3) * T(e,3,1)$*, also known as* $01^a(12)^b(12222)^c(12333)^d(1232323)^e$*. It is integral when it is* 01^a *with* $a = t^2$ *(that is,* $K_{1,a}$*, case (i) above), or* $0(12)^b$ *with* $b = t^2 - 1$ *(that is,* $SK_{1,b}$*, case (ii) above), or* $0(12222)^c$ *with* $c = t^2 - 4$ *(that is,* $T(c,4)$*, part of case (v) above), or when* $b = 0$*,* $c = 3a + 2d - 3$*,* $e = t^2 - 4a - 3d$*. In this last case the nonnegative eigenvalues are* 0, 1, 2, t*, with multiplicities* $10a + 9d + e - 10$, $2e + 1$, $3a + 3d + e - 4$, 1*, respectively.* □

For example, for $t = 3$ and $(a, d) = (2, 0)$, (1,1), (0,3), (1,0), (0,2), this yields examples #12, 17, 18, 22, 23.

1.4 Tables

The first table gives all integral trees on at most 50 vertices. Here n is the number of vertices and d is the diameter. Since trees are bipartite, the spectrum is symmetric around 0, and it suffices to give the nonnegative half. Multiplicities are written as exponents. The references (i)-(v) refer to the families described above and due to Harary & Schwenk [5] (for (i)) and Watanabe & Schwenk [13] (for (ii)–(v)). Graphs #1–8 were already mentioned in [5]. The reference Wang refers to Wang [11].

The second table gives all further known integral trees on at most 100 vertices.

2 Discussion

There remains the question how one can compute a table of all integral trees on at most 50 vertices. There are 10545233702911509534 nonisomorphic trees on 50 vertices, more than 10^{19}, so testing them one by one would not work.

Our approach is via interlacing. If x is a vertex of a graph G, and $G \setminus x$ the result of deleting x from G, then the eigenvalues of $G \setminus x$ interlace those of G. It follows that if $G \setminus x$ has two eigenvalues strictly between two successive integers a and $a+1$, then G has an eigenvalue strictly between a and $a + 1$ and hence is not integral.

Now the spectrum of a disconnected graph is the union of the spectra of the components, so one can conclude that $G \setminus x$ has two eigenvalues strictly between a and $a + 1$ from the fact that this is true for a component, or for the union of some of its components. This is what we used: use a recursive procedure (depth first) that constructs all trees. Each time for some vertex x one or more components of $G \setminus x$ have been completed, compute the spectrum of the union of those components, and discard that branch of the computation when two eigenvalues strictly between two successive integers are found.

#	n	d	tree	spectrum	ref
1	1	0	0	0	(i)
2	2	1	01	1	(i)
3	5	2	01111	$2, 0^3$	(i)
4	6	3	012211	$2, 1, 0^2$	(iii)
5	7	4	0121212	$2, 1^2, 0$	(ii)
6	10	2	01^9	$3, 0^8$	(i)
7	14	3	$012^6 1^6$	$3, 2, 0^{10}$	(iii)
8	17	2	01^{16}	$4, 0^{15}$	(i)
9	17	4	$0(12)^8$	$3, 1^7, 0$	(ii)
10	17	4	$012^7(12)^4$	$3, 2, 1^3, 0^7$	(iv)
11	19	4	$012^5(12)^6$	$3, 2, 1^5, 0^5$	(iv)
12	25	5	$01(23333)^3 22(12)^3$	$3, 2^3, 1^3, 0^{11}$	Wang, p. 58
13	26	2	01^{25}	$5, 0^{24}$	(i)
14	26	4	$0(12222)^5$	$3, 2^4, 0^{16}$	(v)
15	26	3	$012^{12}1^{12}$	$4, 3, 0^{22}$	(iii)
16	31	4	$0(12)^{15}$	$4, 1^{14}, 0$	(ii)
17	31	6	$0123^3(1232323)^2(12222)^2 1$	$3, 2^4, 1^5, 0^{11}$	new
18	31	6	$0(12333)^3(12222)^3$	$3, 2^5, 1, 0^{17}$	Wang, p. 68
19	35	4	$012^{13}(12)^{10}$	$4, 3, 1^9, 0^{13}$	(iv)
20	37	2	01^{36}	$6, 0^{35}$	(i)
21	37	4	$012^{11}(12)^{12}$	$4, 3, 1^{11}, 0^{11}$	(iv)
22	37	6	$0(1232323)^5 1$	$3, 2^4, 1^{11}, 0^5$	Yao [14]
23	37	6	$0(12333)^2(1232323)^3 12^4$	$3, 2^5, 1^7, 0^{11}$	new
24	42	3	$012^{20}1^{20}$	$5, 4, 0^{38}$	(iii)
25	46	4	$012^{14}(12222)^6$	$4, 3, 2^5, 0^{32}$	Yuan [15]
26	49	4	$0(12)^{24}$	$5, 1^{23}, 0$	(ii)
27	50	2	01^{49}	$7, 0^{48}$	(i)
28	50	4	$0(12222)^9 1^4$	$4, 2^8, 1, 0^{30}$	Watanabe [12]

Table 1: The integral trees on at most 50 vertices.

n	d	tree	spectrum	ref
55	5	$01(23^9)^3 2^7 (12)^8$	$4,\ 3^3,\ 2,\ 1^7,\ 0^{31}$	Wang, p. 57
56	6	$0111(12222)^8 123331232323$	$4,\ 2^9,\ 1^3,\ 0^{30}$	new
59	4	$012^{21}(12)^{18}$	$5,\ 4,\ 1^{17},\ 0^{21}$	(iv)
61	4	$012^{19}(12)^{20}$	$5,\ 4,\ 1^{19},\ 0^{19}$	(iv)
61	4	$0(12222)^{12}$	$4,\ 2^{11},\ 0^{37}$	(v)
62	3	$012^{30}1^{30}$	$6,\ 5,\ 0^{58}$	(iii)
62	4	$012^{10}(12222)^{10}$	$4,\ 3,\ 2^9,\ 0^{40}$	Yuan [15]
62	6	$0111(12222)^6(1232323)^4$	$4,\ 2^9,\ 1^9,\ 0^{24}$	Wang, p. 76
62	6	$011(12222)^7(12333)^2(1232323)^2$	$4,\ 2^{10},\ 1^5,\ 0^{30}$	new
62	6	$01(12222)^8(12333)^4$	$4,\ 2^{11},\ 1,\ 0^{36}$	Wang, p. 77
65	2	01^{64}	$8,\ 0^{63}$	(i)
68	6	$011(12222)^5(12333)(1232323)^5$	$4,\ 2^{10},\ 1^{11},\ 0^{24}$	new
68	6	$01(12222)^6(12333)^3(1232323)^3$	$4,\ 2^{11},\ 1^7,\ 0^{30}$	new
68	6	$0(12222)^7(12333)^5(1232323)$	$4,\ 2^{12},\ 1^3,\ 0^{36}$	new
71	4	$0(12)^{35}$	$6,\ 1^{34},\ 0$	(ii)
71	4	$0(12^9)^7$	$4,\ 3^6,\ 0^{57}$	(v)
71	6	$01^6(12^9)^2(1(23)^8)^2 123^8$	$4,\ 3^4,\ 2,\ 1^{14},\ 0^{31}$	new
74	6	$011(12222)^3(1232323)^8$	$4,\ 2^{10},\ 1^{17},\ 0^{18}$	Wang, p. 76
74	6	$01(12222)^4(12333)^2(1232323)^6$	$4,\ 2^{11},\ 1^{13},\ 0^{24}$	new
74	6	$0(12222)^5(12333)^4(1232323)^4$	$4,\ 2^{12},\ 1^9,\ 0^{30}$	new
78	4	$01^{25}12^{18}12^{32}$	$6,\ 5,\ 4,\ 0^{72}$	Wang, p. 44
80	6	$01(12222)^2(12333)(1232323)^9$	$4,\ 2^{11},\ 1^{19},\ 0^{18}$	new
80	6	$0(12222)^3(12333)^3(1232323)^7$	$4,\ 2^{12},\ 1^{15},\ 0^{24}$	new
81	6	$0(123^8)^2(12^9)^6$	$4,\ 3^7,\ 1,\ 0^{63}$	Wang, p. 68
82	2	01^{81}	$9,\ 0^{80}$	(i)
86	3	$012^{42}1^{42}$	$7,\ 6,\ 0^{82}$	(iii)
86	6	$01(2343434)^{12}$	$4,\ 2^{11},\ 1^{25},\ 0^{12}$	Yao [14]
86	6	$012222(12333)^2(1232323)^{10}$	$4,\ 2^{12},\ 1^{21},\ 0^{18}$	new
87	6	$01^6 2^9(1(23)^8)^3(123^8)^2$	$4,\ 3^5,\ 2,\ 1^{21},\ 0^{31}$	new
89	4	$012^{31}(12)^{28}$	$6,\ 5,\ 1^{27},\ 0^{31}$	(iv)
89	5	$01^6(12222)^{15}1232323$	$5,\ 2^{15},\ 1^3,\ 0^{51}$	Wang, p. 57
91	4	$012^{29}(12)^{30}$	$6,\ 5,\ 1^{29},\ 0^{29}$	(iv)
91	6	$01^4(12^9)^3 1(23^4)^5(123^8)^3$	$4,\ 3^6,\ 2^5,\ 0^{67}$	new
94	4	$012^{22}(12222)^{14}$	$5,\ 4,\ 2^{13},\ 0^{64}$	Yuan [15]
95	6	$01^5(12222)^{14}(12333)(1232323)^2$	$5,\ 2^{16},\ 1^5,\ 0^{51}$	new
95	6	$01^4(12222)^{15}(12333)^3$	$5,\ 2^{17},\ 1,\ 0^{57}$	Wang, p. 77
97	4	$0(12)^{48}$	$7,\ 1^{47},\ 0$	(ii)
98	8	$0(12)^3(12^9)^4(122(34)^7)^3$	$4,\ 3^6,\ 2,\ 1^{23},\ 0^{36}$	new

Table 2: Further integral trees on at most 100 vertices.

3 Integral trees with few eigenvalues 0

Theorem 2 (Watanabe [12]) *An integral tree cannot have a complete matching, that is, must have an eigenvalue* 0*, unless it is* K_2*.*

Proof Suppose T is a tree with a complete matching. Then that matching is unique, since the union of two distinct complete matchings contains a cycle. Now the constant term of the characteristic polynomial is, up to sign, the number of complete matchings. It is also the product of all eigenvalues. If this constant term is ± 1 and the tree is integral, then all eigenvalues are ± 1 and the path of length 2 is not an induced subgraph, so we have K_2. □

This argument can be extended a little.

Theorem 3 *If an integral tree has* 0 *as eigenvalue with multiplicity* 1*, then the tree is* $SK_{1,m}$ *for some* m*.*

Proof Suppose T is a tree on n vertices with eigenvalue 0 of multiplicity 1. Then it has almost matchings: coverings by m pairwise disjoint edges and a single point, where $n = 2m + 1$. The number of such almost matchings is, up to sign, the product of the nonzero eigenvalues. On the other hand, the number of such almost matchings is precisely the number of nonzero entries of the (up to a nonzero constant) unique eigenvector u for 0.

(If we delete a vertex where u is zero, then the resulting graph has eigenvalue 0 and hence no matchings. Suppose that u is nonzero at a vertex p, and the graph $T \setminus p$ has no matching. Then it has eigenvalue 0, and since $n - 1 = 2m$ is even, this eigenvalue has multiplicity at least 2, so there is an eigenvector v of $T \setminus p$ for 0 that sums to 0 on the neighbours of p. But then v extended by a 0 on p is another eigenvector for 0 of T, contradiction.)

This number of nonzero entries is at most $(n+1)/2 = m+1$, since the nonzero entries form a coclique, and no vertex with zero entry is adjacent to more than two vertices with nonzero entries.

Let t run over the positive eigenvalues of the adjacency matrix A of T. Considering the trace of A^2 (which equals twice the number of edges) we see that $\sum t^2 = 2m$. By the above we have $\prod t^2 \leq m + 1$.

Since T is integral these m eigenvalues t are all at least 1, and the extremal situation is when all except one are 1 and the last one has $t^2 = m + 1$. Since equality holds we must be in this extremal situation and know the spectrum, it is that of $SK_{1,m}$.

Now $A^2 - I$ has rank 3 (eigenvalue m with multiplicity 2 and -1 with multiplicity 1) and hence has rank 1 on one bipartite half of T. The Perron-Frobenius eigenvector is positive everywhere, so yields an eigenvector of $A^2 - I$ for both components, with eigenvalue m. The vector u vanishes on one bipartite half. That bipartite half is connected for steps of size 2, and has a rank 1 matrix, so has diameter 2, where each vertex has a unique path of length 2 to every other vertex, and has degree 2 itself. This forces the structure, and T must be $SK_{1,m}$. □

Remark The trees $SK_{1,m}$ are determined by their spectrum as trees, not as graphs. For example, $SK_{1,3}$ is cospectral with $C_6 + K_1$.

References

[1] A. E. Brouwer, `http://www.win.tue.nl/~aeb/graphs/integral_trees.html`

[2] A. E. Brouwer and W. H. Haemers, *The integral trees with spectral radius 3*, Lin. Alg. Appl., to appear.

[3] Z. F. Cao, *Integral trees of diameter R* ($3 \leq R \leq 6$) (in Chinese), Heilongjiang Daxue Ziran Kexue Xuebao **2** (1988) 1–3, 95.

[4] R. L. Graham, *On a Diophantine Equation Arising in Graph Theory*, Europ. J. Comb. **1** (1980) 107–112.

[5] F. Harary and A. J. Schwenk, *Which graphs have integral spectra?*, In: Graphs and Combinatorics, Lecture Notes in Mathematics 406, Springer-Verlag, Berlin (1974), 45–51.

[6] P. Híc and R. Nedela, *Balanced integral trees*, Math. Slovaca **48** (1998) 429–445.

[7] P. Híc and M. Pokorný, *On integral balanced rooted trees of diameter 10*, Acta Univ. M. Belii Math **10** (2003) 9–15.

[8] P. Híc and M. Pokorný, *There are integral trees of diameter 7*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **18** (2007) 59–63.

[9] R. Y. Liu, *Integral trees of diameter 5* (in Chinese), J. Systems Sci. Math. Sci. **8** (1988), no.4, 357–360.

[10] A. J. Schwenk, *Trees with integral eigenvalues*, Problem 23 in: *A collection of open problems*, M. Capobianco, S. Maurer, D. McCarthy and J. Molluzzo (eds.), pp. 565–590 in: Proc. Second International Conference on Combinatorial Mathematics (A. Gewirtz and L. V. Quintas, Eds.), Ann. New York Acad. Sci. **319** (1980) 582–583.

[11] Ligong Wang, *Integral graphs and integral trees*, Ph. D. thesis, Univ. Twente, 2005.

[12] M. Watanabe, *Note on integral trees*, Math. Rep. Toyama Univ. 2 (1979), 95–100.

[13] M. Watanabe and A. J. Schwenk, *Integral starlike trees*, J. Austral. Math. Soc. (A) **28** (1979) 120–128.

[14] X. Y. Yao, *Integral trees and integral digraphs* (in Chinese), MSc. Thesis, Northwestern Polytechnical University, China, 2001.

[15] P. Z. Yuan, *Integral trees of diameter 4* (in Chinese), J. Systems Sci. Math. Sci. 18 (1998), no.2, 177–181.