Unextendible Sequences in Finite Abelian Groups

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Abstract

Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ be a finite abelian group with $r = 1$ or $1 < n_1 | \ldots | n_r$, and let $S = (a_1, \ldots, a_t)$ be a sequence of elements in G. We say S is an unextendible sequence if S is a zero-sum free sequence and for any element $g \in G$, the sequence Sg is not zero-sum free any longer. Let $L(G) = \lceil \log_2 n_1 \rceil + \ldots + \lceil \log_2 n_r \rceil$ and $d^*(G) = \sum_{i=1}^{r}(n_i - 1)$, in this paper we prove, among other results, that the minimal length of an unextendible sequence in G is not bigger than $L(G)$, and for any integer k, where $L(G) \le k \le d^*(G)$, there exists at least one unextendible sequence of length k.

1 Introduction

Let G be an additively written finite abelian group, $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ its direct decomposition into cyclic groups, where $r = 1$ or $1 < n_1 | \ldots | n_r$. Set $e_i = (0, \ldots, 0, \underbrace{1}_{i-th}, 0, \ldots, 0)$ for all $i \in [1, r]$, then $(e_1, \ldots, e_r)$ is a basis of G. We set

$$L(G) = \lceil \log_2 n_1 \rceil + \ldots + \lceil \log_2 n_r \rceil,$$

and

$$d^*(G) = \sum_{i=1}^{r}(n_i - 1).$$

Let $\mathcal{F}(G)$ denote the free abelian monoid over G with monoid operation written multiplicatively and given by concatenation, i.e., $\mathcal{F}(G)$ consists of all multi-sets over G, and an element $S \in \mathcal{F}(G)$, which we refer to as *a sequence*, is written in the form

$$S = \prod_{i=1}^{k} g_i = \prod_{g \in G} g^{v_g(S)},$$

with $g_i \in G$, where $v_g(S) \in \mathbb{N}_0$ is the *multiplicity of g in S* and k is *the length of S*, denoted by $|S| = k$. A sequence T is *a subsequence of S* if $v_g(T) \leq v_g(S)$ for every $g \in G$, denoted by $T|S$, and ST^{-1} denote the sequence obtained by deleting the terms of T from S. By $\sigma(S)$ we denote *the sum of S*, that is $\sigma(S) = \sum_{i=1}^{k} g_i = \sum_{g\in G} v_g(S)g \in G$. For every $l \in \{1, \ldots, k\}$, let $\sum_l(S) = \{g_{i_1} + \ldots + g_{i_l} | 1 \leq i_1 < \ldots < i_l \leq k\}$, and $\sum(S) = \cup_{i=1}^{k} \sum_i(S)$.

Let S be a sequence in G, we call S a *zero-sum sequence* if $\sigma(S) = 0$; a *zero-sum free sequence* if for any subsequence W of S, $\sigma(W) \neq 0$.

In inverse zero-sum problems, for example, when we study the structure of the zero-sum free sequences in C_n, set S be a zero-sum sequence of length $D(G) - k$, where $D(G)$ is the well-known Davenport constant of G, that is the smallest integer $l \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq l$ has a zero-sum subsequence. If $k = 1$, then $S = g^{n-1}$; if $k = 2$, then $S = g^{n-2}$ or $S = g^{n-3} \cdot (2g)$, where $g \in S$ and $(g, n) = 1$ (see [1],[3] and [2] etc.). The sequences g^{n-1} and $g^{n-3} \cdot (2g)$ have the same character as pointed out as follows.

Definition 1.1 *Let S be a zero-sum free sequence of elements in an abelian group G, we say S is an unextendible sequence if for any element $g \in G$, the sequence Sg is not zero-sum free any longer.*

In other words, S is an unextendible sequence if and only if $\sum(S) = G\backslash\{0\}$. In fact, if $\sum(S) = G\backslash\{0\}$, then for any element $g \in G$, we get $0 \in \sum(Sg)$ and S is an unextendible sequence; conversely, if S is unextendible, then for any $g \in G$ and $g \neq 0$, $0 \in \sum(Sg)$ but $0 \notin \sum(S)$, and thus $-g \in \sum(S)$, that is $\sum(S) = G\backslash\{0\}$.

For any finite abelian group G, it is obvious that the maximal size of an unextendible sequence is $D(G) - 1$.

Definition 1.2 *For a finite abelian group G, we define $u(G)$ to be the minimal size of an unextendible sequence S in G.*

2 On Unextendible Sequences

We begin by describing $u(G)$ for cyclic group C_n and for any finite abelian group.

For some real number $x \in \mathbb{R}$, let $\lceil x \rceil = \min\{m \in \mathbb{Z} | m \geq x\}$.

Lemma 2.1 *Let G be a finite abelian group of order n, then $u(G) \geq \lceil \log_2 n \rceil$.*

Proof. Let $S \in \mathcal{F}(G)$ be an unextendible sequence of length $u(G)$, then $|\sum(S)| = n-1$, and note that S contains at most $2^{u(G)} - 1$ nonempty subsequences, therefore we get $u(G) \geq \lceil \log_2 n \rceil$. □

Theorem 2.2 *For any cyclic group C_n, $u(C_n) = L(C_n) = \lceil \log_2 n \rceil$.*

Proof. By Lemma 2.1, it is sufficient to prove $u(C_n) \le \lceil \log_2 n \rceil$.

If $n = 2^m$, where m is a positive integer. Note that the sequence $S = \prod_{i=0}^{m-1} 2^i$ is an unextendible sequence of length $|S| = m = \log_2 n$, since each integer which is smaller than 2^m can be expressed as the sum of some subsequence of S.

Now we assume $2^t < n < 2^{t+1}$ for some positive integer t, consider the sequence $S = \prod_{i=0}^{t-1} 2^i \cdot (n-2^t)$. Using the same method as above, it is easy to check that $\sum(S) = C_n \backslash \{0\}$ and thus S is an unextendible sequence of length $|S| = t+1 = \lceil \log_2 n \rceil$.

Therefore $u(C_n) \le \lceil \log_2 n \rceil$, and thus $u(C_n) = \lceil \log_2 n \rceil$. □

Corollary 2.3 *For any finite abelian group $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 | \ldots | n_r$, $u(G) \le L(G)$.*

Proof. Consider the sequence

$$S = \prod_{i=1}^{r} ((n_i - 2^{\lceil \log_2 n_i \rceil - 1}) e_i \prod_{j=0}^{\lceil \log_2 n_i \rceil - 2} 2^j e_i),$$

according to the proof of Theorem 2.2, S is an unextendible sequence of length $|S| = L(G) = \lceil \log_2 n_1 \rceil + \ldots + \lceil \log_2 n_r \rceil$. Therefore $u(G) \le |S| = L(G)$. □

Now we discuss the existence of an unextendible sequence with certain length.

Theorem 2.4 *Let k be an integer satisfying $\lceil \log_2 n \rceil \le k \le n-1$, then there exists an unextendible sequence $S \in \mathcal{F}(C_n)$ such that $|S| = k$.*

Proof. We distinguish two cases:

Case 1. $\lceil \frac{n}{2} \rceil \le k \le n-1$. Consider the sequence

$$S = 1^{k-1} \cdot (n - 1 - (k-1)),$$

clearly, $\sum(S) = C_n \backslash \{0\}$, and therefore S is an unextendible sequence of length $|S| = k$.

Case 2. $\lceil \log_2 n \rceil \le k < \lceil \frac{n}{2} \rceil$. In this case, $n \ge 7$. Set $t = k - \lceil \log_2 n \rceil$, then $0 \le t < \lceil \frac{n}{2} \rceil - \lceil \log_2 n \rceil$. If $t = 0$, by Theorem 2.2 we are done. Now suppose $0 < t < \lceil \frac{n}{2} \rceil - \lceil \log_2 n \rceil$, set $2^i \le t < 2^{i+1}$ where $i \in \mathbb{N}_0$, we consider the following subcases:

Subcase 1. $2^i \le t < 2^{i+1}$, and $i + 1 \le \lceil \log_2 n \rceil - 2$, then

$$S = \prod_{j=0}^{\lceil \log_2 n \rceil - 2} 2^j \cdot 2^{-(i+1)} \cdot (n - 2^{\lceil \log_2 n \rceil - 1}) \cdot 1^t \cdot (2^{i+1} - t)$$

is an unextendible sequence of length $|S| = t + \lceil \log_2 n \rceil = k$.

Subcase 2. $2^{\lceil \log_2 n \rceil - 2} \le t < \lceil \frac{n}{2} \rceil - \lceil \log_2 n \rceil$. If $t < n - 2^{\lceil \log_2 n \rceil - 1}$, we consider

$$S = \prod_{j=0}^{\lceil \log_2 n \rceil - 2} 2^j \cdot 1^t \cdot (n - 2^{\lceil \log_2 n \rceil - 1} - t),$$

otherwise, $t \geq n - 2^{\lceil \log_2 n \rceil - 1}$. Noting that

$$2^{\lceil \log_2 n \rceil - 2} + n - 2^{\lceil \log_2 n \rceil - 1} = n - 2^{\lceil \log_2 n \rceil - 2} > \lceil \frac{n}{2} \rceil - \lceil \log_2 n \rceil > t,$$

we take

$$S = \prod_{j=0}^{\lceil \log_2 n \rceil - 3} 2^j \cdot 1^{t+1} \cdot (2^{\lceil \log_2 n \rceil - 2} + n - 2^{\lceil \log_2 n \rceil - 1} - t - 1).$$

Then S is an unextendible sequence of length $|S| = t + \lceil \log_2 n \rceil = k$. This completes the proof. □

For any finite abelian group G, we have the following theorem.

Theorem 2.5 *Let G be a finite abelian group. Then for every $k \in [L(G), d^*(G)]$, there exists an unextendible sequence $S \in \mathcal{F}(G)$ of length $|S| = k$.*

Theorem 2.5 comes from the following observation.

Lemma 2.6 *Let $G = G_1 \oplus G_2$, and $S_i \in \mathcal{F}(G_i)$ for $i \in \{1, 2\}$. Then $S_1 S_2 \in \mathcal{F}(G)$ is unextendible if and only if both S_1 and S_2 are unextendible.*

Proof. It is obvious. □

Proof of Theorem 2.5. Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ be its direct decomposition into cyclic groups. Write k in the form $k = k_1 + \ldots + k_r$, where $\lceil \log_2 n_i \rceil \leq k_i \leq n_i - 1$. By Theorem 2.4, there exist unextendible sequences $S_i \in \mathcal{F}(C_{n_i})$ of length k_i, and put $S = S_1 \cdot \ldots \cdot S_r$, then by Lemma 2.6, $S \in \mathcal{F}(G)$ is an unextendible sequence of lengt $|S| = k$. □

It is evident that any zero-sum free sequence S in G can be extended into an unextendible sequence. We can extend S by choosing a series elements $g_1, \ldots, g_r$ in a natural process such that $|\sum(Sg_1)| = \max\{|\sum(Sg)| | g \in G\}$, and $|\sum(Sg_1 \ldots g_i g_{i+1})| = \max\{|\sum(Sg_1 \ldots g_i g)| | g \in G\}$ for any $i = 2, \ldots, r$.

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