A note on neighbour-distinguishing regular graphs total-weighting

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Abstract

We investigate the following modification of a problem posed by Karoński, Łuczak and Thomason [J. Combin. Theory, Ser. B 91 (2004) 151-157]. Let us assign positive integers to the edges and vertices of a simple graph G. As a result we obtain a vertex-colouring of G by sums of weights assigned to the vertex and its adjacent edges. Can we obtain a proper colouring using only weights 1 and 2 for an arbitrary G?

We know that the answer is yes if G is a 3-colourable, complete or 4-regular graph. Moreover, it is enough to use weights from 1 to 11, as well as from 1 to $\lfloor \frac{\chi(G)}{2} \rfloor + 1$, for an arbitrary graph G. Here we show that weights from 1 to 7 are enough for all regular graphs.

Keywords: neighbour-distinguishing total-weighting, regular graph

1 Introduction

A *k-total-weighting* of a simple graph G is an assignment of an integer weight, $w(e), w(v) \in \{1, \ldots, k\}$ to each edge e and each vertex v of G. A k-total-weighting is *neighbour-distinguishing* (or *vertex colouring*, see [1, 2]) if for every edge uv, $w(u) + \sum_{e \ni u} w(e) \neq w(v) + \sum_{e \ni v} w(e)$. If such a weighting exists, we say that G *permits* a neighbour-distinguishing k-total-weighting.

A similar parameter, but in the case of an *edge* (not total) weighting, was introduced and studied in [3] by Karoński, Łuczak and Thomason. They asked if each simple connected graph that is not simply a single edge permits a *neighbour-distinguishing* 3-*edge-weighting*, and showed that this statement holds for 3-colourable graphs. Then Addario-Berry, Dalal and Reed showed that it is enough to use numbers from 1 to 16 to construct a neighbour-distinguishing edge-weighting for an arbitrary graph (not containing a single edge as a component), see [2].

In [4] we conjectured that numbers 1 and 2 in turn are enough to distinguish neighbours of each graph by a total-weighting. We verified this conjecture for some classes of graphs and established the following upper bounds.

Theorem 1 *([4]) All complete, 3-colourable and 4-regular graphs permit neighbour-distinguishing 2-total-weightings.*

Theorem 2 *([4]) Each simple graph permits a neighbour-distinguishing 11-total-weighting and a neighbour-distinguishing* $(\lfloor \frac{\chi(G)}{2} \rfloor + 1)$*-total-weighting.*

Note that a graph permits a neighbour-distinguishing 1-total weighting iff every two neighbours have distinct degrees in this graph. Here we deal then with the most difficult, in a way, case and show that the weights $1, \ldots, 7$ are enough for each regular graph, see Theorem 7.

2 Lemmas

To prove our main result we shall need the following lemmas. Then Corollary 6 will provide us with a construction of a neighbour-distinguishing total-weighting of each regular graph by weights from 1 to 8, which will be then reduced to 7 by Lemma 4.

Given a sequence of numbers $(a_1, \ldots, a_k)$, we shall call $(b_1, \ldots, b_l)$ a *block* of this sequence iff there exists $0 \leqslant j \leqslant k - l$ such that $b_i = a_{j+i}$, $i = 1, \ldots, l$.

Lemma 3 *Assume that* $s = (a_1, \ldots, a_k)$ *is a sequence of nonnegative integers such that* $a_1 + \ldots + a_k \leqslant k$*. Then there is an element* $a_j = 0$ *of that sequence such that* $a_{j-1} + a_{j+1} \leqslant 3$ *(where* $a_0, a_{k+1} := 0$*), unless* s *consists exclusively of blocks* $(1, 0, 3, 0, 1)$ *and* $(1, \ldots, 1)$*.*

Proof. Let us call the sequences consisting of blocks $(1, 0, 3, 0, 1)$ and $(1, \ldots, 1)$ (which may intersect) *forbidden*. The lemma is obvious for $k \leqslant 3$. It is also easy to verify it for $k = 4$, hence let us argue by induction on k. Take $k \geqslant 5$ and assume the proposition does not hold for some (not forbidden) sequence $s = (a_1, \ldots, a_k)$, hence if $a_i = 0$, then $a_{i-1} + a_{i+1} \geqslant 4$. If there are two consecutive elements a_r, a_{r+1} of s that are either both positive or both equal to 0, then either the sequence $(a_1, \ldots, a_r)$ or $(a_{r+1}, \ldots, a_k)$ is not forbidden and complies with the assumptions of the lemma, hence, by induction, there is an element $a_j = 0$ such that $a_{j-1} + a_{j+1} \leqslant 3$, a contradiction.

Therefore, we may assume every second element of s is positive and every second one equals 0. Let a_t be the second element that is equal to 0 in the sequence s (hence $t = 3$ or 4). By the inequality $a_{i-1} + a_{i+1} \geqslant 4$ for the first of such elements, $a_1 + \ldots + a_t \geqslant 4$. Therefore the sequence $(a_{t+1}, \ldots, a_k)$ complies with the assumptions of the lemma (and is not a forbidden one), hence we again obtain a contradiction by induction. ■

Let a k-*vertex-colouring* of $G = (V, E)$ be a proper vertex-colouring $c : V \to C$ (i.e. $c(u) \neq c(v)$ if $uv \in E$) by the colours from a colour set C with $|C| = k$. Note that we do not require c to be surjective, hence not all the colours have to be used.

Lemma 4 *Let G be a k-regular graph which is neither a complete graph nor an odd cycle. There is a k-vertex-colouring with colour classes $V_1, \ldots, V_k$ such that $d_{V_i}(v) \leqslant 3$ for each $v \in V_{i-1}$, $i = 2, \ldots, k$.*

Proof. Let $E(U, W)$ denote the set of edges between subsets U, W of the vertex set of G. Let also $e(U, W) = |E(U, W)|$. By Brooks' Theorem, there is a k-vertex-colouring of G. Let us choose such a k-vertex-colouring and such an ordering of its colour classes $V_1, \ldots, V_k$ that minimizes the sum $\sum_{l=2}^{k} e(V_{l-1}, V_l)$. We argue that it complies with our requirements.

Assume it is not so; hence there is $2 \leqslant i \leqslant k$ and $v \in V_{i-1}$ such that $d_{V_i}(v) \geqslant 4$. Denote $a_l = d_{V_l}(v)$, $l = 1, \ldots, k$ $(a_0, a_{k+1} := 0)$. Then $a_i \geqslant 4$ $(a_{i-1} = 0)$ and, since G is k-regular, $a_1 + \ldots + a_k = k$. By Lemma 3, there is $1 \leqslant j \leqslant k$ such that $a_j = 0$ and $a_{j-1} + a_{j+1} \leqslant 3$, hence $d_{V_j}(v) = 0$ and we may move v from V_{i-1} to V_j, and thus at the same time reduce the minimized sum by at least four and add to it at most three (since v has at most three neighbours in $V_{j-1} \cup V_{j+1}$), a contradiction. ■

Led $\delta(G)$ denote the minimal degree of a vertex in a graph G. We make use of the following Theorem 5 by Addario-Berry, Dalal and Reed (see [2]) to obtain a similar to their Corollary 6.

Theorem 5 *([2]) Given a graph $G = (V, E)$ and for all $v \in V$, integers a_v^-, a_v^+ such that $a_v^- \leqslant \lfloor \frac{d(v)}{2} \rfloor \leqslant a_v^+ < d(v)$, and*

$$a_v^+ \leqslant \min\Big(\frac{d(v) + a_v^-}{2} + 1, 2a_v^- + 3\Big), \tag{1}$$

there exists a spanning subgraph H of G such that $d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$ for all $v \in V$.

Corollary 6 *Given a graph $G = (V, E)$ with $\delta(G) > 4$, and for each $v \in V$, integers $a_v^- \in [\lfloor \frac{d(v)}{4} \rfloor, 2\lfloor \frac{d(v)}{4} \rfloor]$ and $a_v^+ := a_v^- + \lfloor \frac{d(v)}{4} \rfloor + 1$, there exists a spanning subgraph H of G such that $d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$ for all $v \in V$.*

Proof. We have $a_v^- \leqslant 2\lfloor \frac{d(v)}{4} \rfloor \leqslant \lfloor \frac{d(v)}{2} \rfloor$, $\lfloor \frac{d(v)}{2} \rfloor \leqslant 2\lfloor \frac{d(v)}{4} \rfloor + 1 \leqslant a_v^+$ and $a_v^+ \leqslant 3\lfloor \frac{d(v)}{4} \rfloor + 1 < d(v)$, hence, by Theorem 5, it is enough to prove (1) for all $v \in V$. Note then that $a_v^+ = \frac{a_v^-}{2} + \frac{a_v^-}{2} + \lfloor \frac{d(v)}{4} \rfloor + 1 \leqslant \frac{a_v^-}{2} + \lfloor \frac{d(v)}{4} \rfloor + \lfloor \frac{d(v)}{4} \rfloor + 1 \leqslant \frac{a_v^-}{2} + \frac{d(v)}{2} + 1$ and $a_v^+ = a_v^- + \lfloor \frac{d(v)}{4} \rfloor + 1 \leqslant a_v^- + a_v^- + 1$, thus (1) holds. ■

3 Main Result

For a given total-weighting w of G, let $c_w(v) := w(v) + \sum_{e \ni v} w(e)$ (or $c(v)$ for short if the weighting w is obvious), define the resulting colouring for each $v \in V(G)$. We shall call $c(v)$ a *colour* or a *total weight* of v. Our aim, in fact, is to find such a weighting that this vertex-colouring is proper.

Theorem 7 *Each regular graph admits a neighbour-distinguishing* 7*-total-weighting.*

Proof. Let G be a k-regular graph. By Theorem 1, we may assume that G is not a complete graph and, by Theorem 2 (and Brooks' Theorem), that $k \geqslant 14$. By Lemma 4 there is a k-vertex-colouring with colour classes $V_1, \ldots, V_k$ such that $d_{V_{4i}}(v) \leqslant 3$ for each $v \in V_{4(i-1)}$, $i = 2, \ldots, \lfloor \frac{k}{4} \rfloor$. We shall make use of this fact in the second part of the proof.

Let $s_i = k + 4\lfloor \frac{k}{4} \rfloor + 4 + i$ and $b_i = k + 8\lfloor \frac{k}{4} \rfloor + 8 + i$, and let $L_i = \{s_i, b_i\}$ be a list of admissible colours (total weights) assigned to the vertex set V_i, $i = 1, \ldots, k$. In the first part of the proof we construct an 8-total-weighting such that $c_w(v) \in L_i$ for each $v \in V_i$, $i = 1, \ldots, k$. This way, since $s_1 < \ldots < s_k < b_1 < \ldots < b_k$, this weighting will be neighbour-distinguishing. In fact we will use only weights 1 and 5 for the edges. In the second part of the proof we will reduce the weights of some vertices and increase some of the edge weights, so that $w(e) \in \{1, 2, 5, 6\}$ and $1 \leqslant w(v) \leqslant 7$ for all $e \in E$ and $v \in V$, and so that the lists of admissible colours remained the same for all colour classes but those of the form V_{4j}, $1 \leqslant j \leqslant \lfloor \frac{k}{4} \rfloor$. In these classes, we will admit colours in $L'_{4j} = \{s_{4j} - 4, s_{4j}, b_{4j} - 4, b_{4j}\}$ instead of L_{4j}, $j = 1, \ldots \lfloor \frac{k}{4} \rfloor$. Since $s_{4j} - 4 = s_{4(j-1)}$ and $b_{4j} - 4 = b_{4(j-1)}$, the total weights of the vertices in V_{4j}, $j = 1, \ldots \lfloor \frac{k}{4} \rfloor$, will have to be constructed carefully, so that the weighting remains neighbour-distinguishing. Note in particular that $s_4 - 4 < s_1$ and $s_k < b_4 - 4 < b_1$, hence colouring with L'_4 (instead of L_4) does not produce any new conflicts.

Let us then first weight all the edges of G with 1 and set a temporary weight 0 for each vertex of this graph. This way, each vertex gets a temporary colour k. Now for each $v \in V_{4j+l}$ set $a_v^- = \lfloor \frac{k}{4} \rfloor + j$ and $a_v^+ = a_v^- + \lfloor \frac{k}{4} \rfloor + 1$, $j = 0, \ldots, \lfloor \frac{k}{4} \rfloor$, $l = 1, \ldots, 4$, (hence $a_v^- \in [\lfloor \frac{k}{4} \rfloor, 2\lfloor \frac{k}{4} \rfloor])$. Then by Corollary 6 there exists a spanning subgraph H of G such that $d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$ for all $v \in V$. Let us then add 4 to the weight of each edge of this subgraph (hence $w(e) \in \{1, 5\}$ for $e \in E$). Now each vertex $v \in V_{4j+l}$ has a temporary colour in the set $\{k + 4\lfloor \frac{k}{4} \rfloor + 4j, k + 4\lfloor \frac{k}{4} \rfloor + 4j + 4, k + 8\lfloor \frac{k}{4} \rfloor + 4j + 4, k + 8\lfloor \frac{k}{4} \rfloor + 4j + 8\} = \{s_{4j+l} - 4 - l, s_{4j+l} - l, b_{4j+l} - 4 - l, s_{4j+l} - l\}$. Therefore by setting either $w(v) = l + 4$ or l, we obtain $c(v) \in L_i$ and $1 \leqslant w(v) \leqslant 8$ for all $v \in V_i$, $i = 1, \ldots, k$. This finishes the first part of the proof.

Note that we may have $w(v) = 8$ only for vertices in V_{4j}, $j = 1, \ldots, \lfloor \frac{k}{4} \rfloor$. We shall reduce these weights in the following manner. Process the vertex sets of the form V_{4j} one after another in the reversed order, starting from $V_{4\lfloor \frac{k}{4} \rfloor}$ and ending at V_4. For a given V_{4j}, process all its vertices in an arbitrary order. We introduce some changes only if $v \in V_{4j}$ is weighted with 8. Namely, if it has any neighbour in $V_{4(j-1)}$, we choose one such neighbour arbitrarily (call it u), and reduce the weights of u and v by 1 (it is each time possible since u has at most 3 neighbours in V_{4j}, and had a weight 4 or 8 after the first part of the construction), and add 1 to the weight of the edge uv (changing it to 2 or 6), hence the total weights of v and u remain unchanged. On the other hand, if v has no neighbour in $V_{4(j-1)}$ (or ($j = 1$)), we reduce the weight of v by 4, hence $c(v) \in L'_{4j}$. Since v has no neighbour in $V_{4(j-1)}$ (for $j > 1$) and $s_4 - 4 < s_1$, $s_k < b_4 - 4 < b_1$, no conflict will appear. After processing all the vertices as described, we therefore obtain a neighbour-distinguishing 7-total-weighting. ■

References

[1] L. Addario-Berry, R.E.L. Aldred, K. Dalal, B.A. Reed, *Vertex Colouring Edge Partitions*, J. Combin. Theory, Ser. B 94 (2) (2005) 237-244.

[2] L. Addario-Berry, K. Dalal, B.A. Reed, *Degree constrained subgraphs*, Proceedings of GRACO2005, volume 19 of Electron. Notes Discrete Math., Amsterdam (2005), 257-263 (electronic), Elsevier.

[3] M. Karoński, T. Łuczak, A. Thomason, *Edge weights and vertex colours*, J. Combin. Theory, Ser. B 91 (2004) 151-157.

[4] J. Przybyło, M. Woźniak *1,2 Conjecture, II*, Preprint MD 026 (2007), `http://www.ii.uj.edu.pl/preMD/index.php`.