On domination in 2-connected cubic graphs

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Abstract

In 1996, Reed proved that the domination number, $\gamma(G)$, of every n-vertex graph G with minimum degree at least 3 is at most $3n/8$ and conjectured that $\gamma(H) \leq \lceil n/3 \rceil$ for every connected 3-regular (cubic) n-vertex graph H. In [1] this conjecture was disproved by presenting a connected cubic graph G on 60 vertices with $\gamma(G) = 21$ and a sequence $\{G_k\}_{k=1}^{\infty}$ of connected cubic graphs with $\lim_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{1}{3} + \frac{1}{69}$. All the counter-examples, however, had cut-edges. On the other hand, in [2] it was proved that $\gamma(G) \leq 4n/11$ for every connected cubic n-vertex graph G with at least 10 vertices. In this note we construct a sequence of graphs $\{G_k\}_{k=1}^{\infty}$ of 2-connected cubic graphs with $\lim_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{1}{3} + \frac{1}{78}$, and a sequence $\{G'_l\}_{l=1}^{\infty}$ of connected cubic graphs where for each G'_l we have $\frac{\gamma(G'_l)}{|V(G'_l)|} > \frac{1}{3} + \frac{1}{69}$.

1 Introduction

A set D of vertices is *dominating* in a graph G if every vertex of $G \setminus D$ is adjacent to a vertex in D. An arbitrary set A of vertices in a graph G *dominates* itself and the vertices at distance one from it. The *domination number*, $\gamma(G)$, of a graph G is the minimum size of a dominating set in G.

Ore [8] proved that $\gamma(G) \leq n/2$ for every n-vertex graph without isolated vertices (i.e., with $\delta(G) \geq 1$). Blank [3] proved that $\gamma(G) \leq 2n/5$ for every n-vertex graph with $\delta(G) \geq 2$. Blank's result was also discovered by McCuaig and Shepherd [6]. Reed [9] proved that $\gamma(G) \leq 3n/8$ for every n-vertex graphs with $\delta(G) \geq 3$. All these bounds are best possible. However, Reed [9] conjectured that the domination number of each connected 3-regular (cubic) n-vertex graph is at most $\lceil n/3 \rceil$. In [1] this conjecture was disporved by exhibiting a connected cubic graph G on 60 vertices with $\gamma(G) = 21$ and a sequence $\{G_k\}_{k=1}^{\infty}$ of connected cubic graphs with $\lim_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{1}{3} + \frac{1}{69}$. All the counter-examples in [1] had cut-edges. In [2] Reed's upper bound of $\gamma(G) \leq 3n/8$ was

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improved to $\gamma(G) \le 4n/11$ for every connected cubic n-vertex graph G with at least 10 vertices by using by using Reed's techniques and examining some problematic cases more carefully and by adding a discharging argument. Kawarabayashi, Plummer, and Saito [5] proved that Reed's conjecture is at least close to the truth for cubic graphs with large girth by showing that if G is a connected cubic n-vertex graph that has a 2-factor of girth at least $g \ge 3$, then

$$\gamma(G) \le n\left(\frac{1}{3} + \frac{1}{9\lfloor g/3 \rfloor + 3}\right).$$

In [2] this result of Kawarabayashi, Plummer, and Saito was improved by proving that if G is a cubic connected n-vertex graph of girth g, then

$$\gamma(G) \le n\left(\frac{1}{3} + \frac{8}{3g^2}\right).$$

Also recently result Lowenstein and Rautenbach [7] further improved these resuls related to girth and showed that Reeds conjecture is true for girth at least 83.

In this note, we present a sequence of 2-connected counter-examples to Reed's conjecture and improve the lowerbound of $\gamma(G)$. We will contruct two sequences, with the first sequence being $\{G_k\}_{k=1}^{\infty}$ of 2-connected cubic graphs with $\lim_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \ge \frac{1}{3} + \frac{1}{78}$, and the second sequence being $\{G'_l\}_{l=1}^{\infty}$ of connected cubic graphs where for each G'_l we have $\frac{\gamma(G'_l)}{|V(G'_l)|} > \frac{1}{3} + \frac{1}{69}$. Note that (G'_1) is a connected cubic graph on 80 vertices and has the same ratio of $\frac{\gamma(G'_1)}{|V(G'_1)|} = \frac{1}{3} + \frac{1}{60}$ with the graph G on 60 vertices in [1], but has 20 more vertices. In the next section we construct the examples and in the last small section briefly discuss the results.

Note that Kelmans [10] has recently constructed a sequence $\{G_j\}_{j=1}^{\infty}$ of 2-connected cubic graphs with $\lim_{j\to\infty} \frac{\gamma(G_j)}{|V(G_j)|} \ge \frac{1}{3} + \frac{1}{60}$, and a connected cubic graph G^* with $\frac{\gamma(G^*)}{|V(G^*)|} \ge \frac{1}{3} + \frac{1}{54}$.

2 Examples

Our basic building block is the graph H_1 in Fig. 1.

The following claims in were proved [1].

Claim 1 [1] $\gamma(H_1) = \gamma(H_1 - v_6) = \gamma(H_1 - v_7) = 3$.

Claim 1 is easy to check. This claim has the following immediate consequence.

Corollary 1 [1] *For every cubic graph G containing H_1 and any dominating set D of G, either $|D \cap V(H_1)| \ge 3$ or both v_6 and v_7 are dominated from the outside of H_1.*

The bigger block, H_2 in Fig. 2, is constructed using two copies of H_1 and two additional vertices.

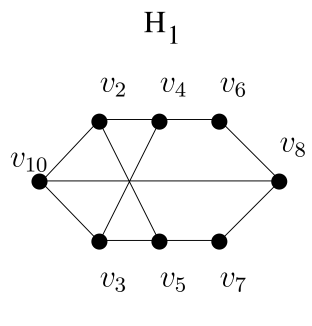

Figure 1

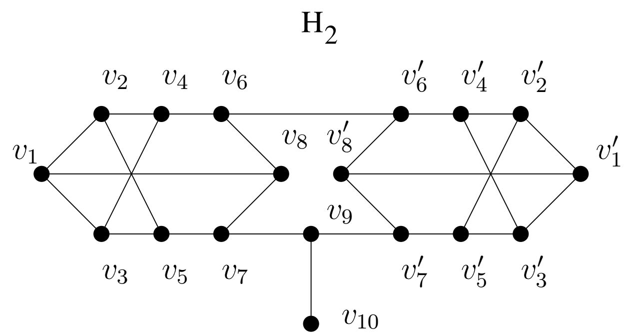

Figure 2

Claim 2 [1] $\gamma(H_2) = \gamma(H_2 - v_{10}) = \gamma(H_2 - v_9 - v_{10}) = 6$. *In particular, every dominating set in any cubic graph containing* $V(H_2)$ *has at least* 6 *vertices in* $V(H_2) - v_{10}$.

The above claim is easy to check using Claim 1.

Our yet bigger block on 36 vertices, H_3, is obtained from two copies H_2 and H'_2 of H_2 by identifying v_{10} with v'_{10} into a new vertex v^*_{10} and adding a new vertex v_0 adjacent only to v^*_{10} The following property immediately follows from Claim 2.

Claim 3 [1] *Every dominating set in any cubic graph containing* $V(H_3)$ *has at least* 12 *vertices in* $V(H_3) - v^*_{10} - v_0$.

Theorem 1 *There is a sequence* $\{G_k\}_{k=1}^{\infty}$ *of cubic* 2*connected graphs such that for every* k, $|V(G_k)| = 26k$ *and* $\gamma(G_k) \geq 9k$ *so that* $\lim\limits_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{9}{26}$.

Proof. Our big block, F_i, for constructing G_k consists of three copies of H_1 which are labeled, H, H' and H'', and two special vertices, x_i and y_i, where x_i is adacent to v_6 in H and v'_6 in H', and y_i is adacent to v_7 in H and v''_6 in H''. Furthermore, v'_7 in H' is adjacent to v''_7 in H'' (see Figure 3). This block has 26 vertices and exactly two of them, x_i and y_i, are of degree two. The main property of F_i that we will prove and use is:

(P1) *For every cubic graph* G *containing* F_i *and any dominating set* D *in* G, *the set* D *has at least* 9 *vertices in* $V(F_i)$.

If D contains neither x_i nor y_i, then by Claim 1 D must contain 3 vertices in each of $V(H)$, $V(H')$, and $V(H'')$. If D contains x_i but does not contain y_i, then by Claim 1, D must contain 3 vertices in $V(H)$, 3 vertices in $V(H'')$, and at least 2 vertices in $V(H')$. The case where D contains y_i but not x_i is symmetric. If D contains both x_i and y_i, then again by Claim 1, D has at least 2 vertices in $V(H)$, and least 5 vertices in $V(H' \cup H'')$. As a result in all the cases D contains at least 9 verices in $V(F_i)$. This proves (P1).

The graph G_k consists of disjoint graphs $F_1, \ldots F_k$, where y_i is connected by an edge to x_{i+1} for $i = 1, \ldots, k-1$, and y_k is connected by an edge to x_1. Clearly, $|V(G_k)| = 26k$ and, by (P1), $\gamma(G_k) \geq 9k$. In F_i, any copy of H_1 is connected by 2 edges to the rest of the graph. Since H_1 is 2-connected and since F_i has an edge connecting it to F_{i-1} and another edge connecting it to F_{i+1}, the graph G_k is 2-connected. □

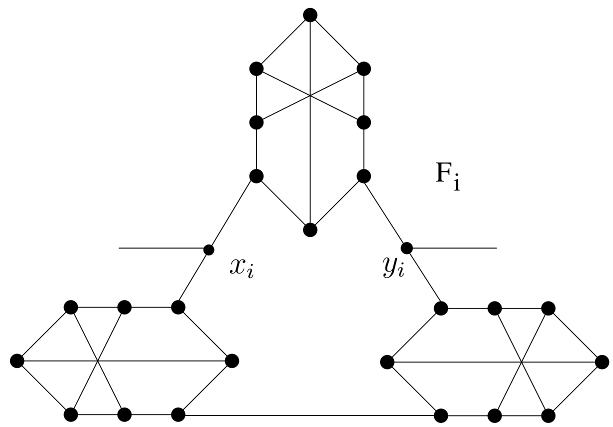

Figure 3

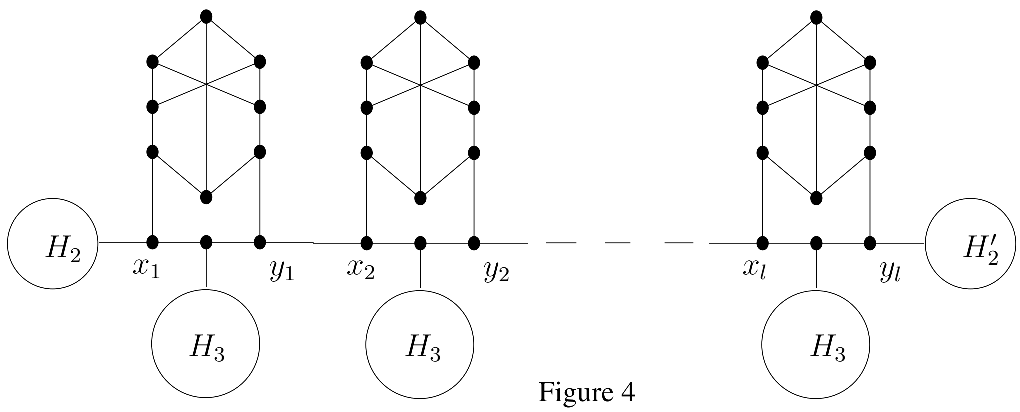

Figure 4

Theorem 2 *There is a sequence* $\{G_l'\}_{l=1}^{\infty}$ *of cubic connected graphs such that for every* l, $|V(G_l')| = 46l + 34$ *and* $\gamma(G_l') \geq 16l + 12$ *and, as a result,* $\frac{\gamma(G_l')}{|V(G_l')|} > \frac{8}{23}$. *Furthermore,* (G_1') *is a connected cubic graph on* 80 *vertices with* $\frac{\gamma(G_1')}{|V(G_1')|} = \frac{1}{3} + \frac{1}{60}$

Proof. The big block, F_j, for constructing G_l consists of a copy of H_1, a copy of H_3 and two special vertices, x_j and y_j, where x_j is adacent to v_6 in H_1 and v_0 in H_3 and y_j is adacent to v_7 in H_1 and v_0 in H_3. This block has 46 vertices and exactly two of them, x_j and y_j, are of degree two. The main property of F_j, which was proved in [1], that we will use is:

(P2) [1] *For every cubic graph* G *containing* F_j *and any dominating set* D *in* G, *the set* D *has at least* 16 *vertices in* $V(F_j)$.

Now, the graph G_l consists of disjoint graphs $F_1, \ldots F_l$, where y_l is connected by an edge to x_{l+1} for $j = 1, \ldots, l-1$, and to each of x_1 and y_l we attach one copy of H_2, let us call them H_2 and H_2'. We identify x_1 with vertex v_{10} of H_2 and identify y_l with vertex v_{10}' of H_2'. By Claim 2 any dominating set D must contain 12 vertices in $V(H_2 \cup H_2') - x_1 - y_l$, and by (P2) D must contain 16 vertices in each $V(F_j)$. This completes our proof. $\square$

3 Comments

It is not clear what the supremum of $\frac{\gamma(G)}{|V(G)|}$ over connected cubic graphs is. The situation we face now $\frac{4}{11} \geq \sup \frac{\gamma(G)}{|V(G)|} \geq \frac{1}{3} + \frac{1}{69}$. We believe that both the upper and lower bounds could be improved. The upper bound was proved in [2] by exploiting Reed's techniques in [9] and examining some of the cases in Reed's proof more carefully and adding a discharging argument. However, exploting Reed's ideas further seems difficult (but possible) as the number of cases to be analyzed grows quickly.

It would also be interesting to find out whether 3-connected counter-examples to Reed's conjecture exist.

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