Identifying codes of Cartesian product of two cliques of the same size

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Abstract

We determine the minimum cardinality of an identifying code of $K_n \square K_n$, the Cartesian product of two cliques of same size. Moreover we show that this code is unique, up to row and column permutations, when $n \geq 5$ is odd. If $n \geq 4$ is even, we exhibit two distinct optimal identifying codes.

1 Introduction

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *Cartesian product* of G_1 and G_2, denoted by $G_1 \square G_2$, is the graph on vertex set $V_1 \times V_2$ such that

- $(x, y)(x, y') \in E(G_1 \square G_2)$ if and only if $yy' \in E(G_2)$,
- $(x, y)(x', y) \in E(G_1 \square G_2)$ if and only if $xx' \in E(G_1)$,
- and $(x, y)(x', y') \notin E(G_1 \square G_2)$ if $x \neq x'$ and $y \neq y'$.

In coding theory, it is somehow natural to focus on the Cartesian product, since the most studied metrics (respectively the Hamming and the Lee metrics) in d-dimensional spaces can be defined as the iterated Cartesian product of, respectively, cliques and cycles. In this way, the (generalized) hypercube and the torus can be seen as Cartesian products of cliques and cycles, respectively.
In this manuscript, we are interested on *identifying codes* in Cartesian product of graphs.

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Given a graph $G = (V, E)$, let us denote by $N(x)$ the *neighbourhood* of $x \in V$, that is, the set of vertices adjacent to x. The *closed neighbourhood* $N[x]$ of $x \in V$ is the union of x and $N(x)$. A subset C of V is said to be an *identifying code* of G if the sets $N[x] \cap C$ are non-empty and distinct for all $x \in V$.

The notion of identifying code was introduced by Karpovsky, Chakrabarty and Levitin [6] to model a fault-detection problem in multiprocessor systems. For another application to sensor networks consult [9]. The challenging problem is, given a graph G, to find a minimum size identifying code of G. Identifying codes are closely related to other types of codes, like covering codes (which can be used to construct identifying codes in Hamming spaces, see *e.g.* [2, 3, 6]). In [7], the authors propose a construction of codes identifying sets of vertices in Cartesian products of graphs. There is a large and fast-growing bibliography on identifying codes, which can be found on Antoine Lobstein's webpage [10].

Even for special structures, determining the minimum cardinality of an identifying code is still an open problem. For instance, only recently, it was proven that the minimum density of an identifying code of the two dimensional grid graph (which can be see as the Cartesian product of two infinite paths) is equal to $\frac{7}{20}$ [4, 1]. Nevertheless, only partial results are known in the finite case [5].

Additionally, Blass, Honkala and Litsyn proposed in [3] the following natural conjecture : *the cardinality of a smallest identifying code in the hypercube will increase with the dimension.* This conjecture was only partially solved in [8].

In this note, we determine the size of a minimum identifying code of the Cartesian product of two cliques of the same size. Moreover we show that, up to row and column permutations, there exists a unique minimum identifying code in the case where the size of the cliques is odd.

Theorem 1 *Let C be a minimum identifying code of $K_n \square K_n$. Then $|C| = \lfloor \frac{3n}{2} \rfloor$. Moreover, if $n \geq 5$ is odd there is a unique (up to row and column permutations) identifying code with cardinality $\lfloor \frac{3n}{2} \rfloor$.*

2 Proof of Theorem 1

We will note $\{1, \ldots, n\}$ the vertex set of the complete graph K_n on n vertices.

First we exhibit identifying codes of cardinality $\lfloor \frac{3n}{2} \rfloor$ (see Figure 1 and Figure 2).

Let $D = \{(x, x) | x = 1, \ldots, n\}$. If n is odd let $A = \{(n - x + 1, x) | x = 1, ..., \frac{n-1}{2}\}$. Otherwise, let $A = \{(n - x + 1, x) | x = 1, ..., \frac{n}{2}\}$.

We will prove that $D \cup A$ is an identifying code of $K_n \square K_n$. First observe that D is a dominating set since it contains an element of each row. Now we check that each pair of vertices $(x, y), (a, b) \in V(K_n \square K_n)$ is separated.

If $a \neq x$ and $a \neq y$ (respectively $b \neq x$ and $b \neq y$) then $(a, a) \in N((a, b)) \setminus N((x, y))$ (resp. $(b, b) \in N((a, b)) \setminus N((x, y))$).

So, since $(a, b) \neq (x, y)$, without loss of generality, we have either $a = b = x$ or ($a = y$ and $b = x$ (with $x \neq y$)).

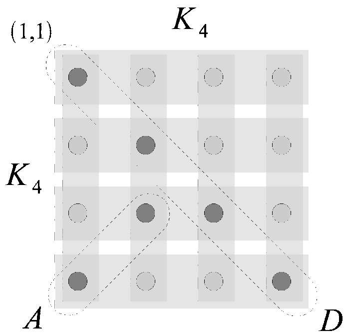

Figure 1: The code $D \cup A$ in the case n even.

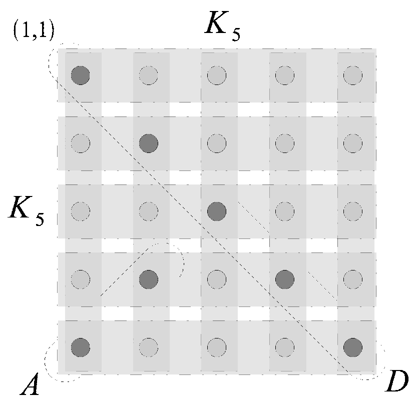

Figure 2: The code $D \cup A$ in the case n odd.

If $a = b = x$ then $(y, y) \in N((x, y)) \setminus N((a, b))$.
Assume, now that $(a, b) = (y, x)$ and $x < y$. First, when $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$, $(n + 1 - x, x) \in N((y, x)) \setminus N((x, y))$. Finally, when $x \geq \lceil \frac{n}{2} \rceil$ then $(y, n + 1 - y) \in N((y, x)) \setminus N((x, y))$. This proves that C is an identifying codes of cardinality $|D| + |A| = n + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{3n}{2} \rfloor$.

Let C be an identifying code of $K_n \Box K_n$. We will prove that $|C| \geq \lfloor \frac{3n}{2} \rfloor$. Additionally, we show that if $n \geq 5$ is odd and C is a minimum size identifying code then, up to row and column permutations, $C = D \cup A$.
We will need some additional definitions. A *row* R_x for some $x \in \{1, \ldots, n\}$ (respectively *column* C_x) of $K_n \Box K_n$ is the vertex set $\{(x, i)$ for $i = 1, \ldots, n\}$ (resp. $\{(i, x)$ for $i = 1, \ldots, n\}$). Let $I(x, y) = N[(x, y)] \cap C$. Remark that if C is an identifying code then $I(x, y) \neq I(u, v)$ for all pairs $(x, y) \neq (u, v)$. A vertex (x, y) which is in C is an $[a, b]$-vertex if $|R_x \cap C| = a$ and $|C_y \cap C| = b$.
We need a simple and useful Lemma :

Lemma 1 *There are no two* $[*, 1]$*-vertices in the same row. By symmetry, there are no two* $[1, *]$*-vertices in the same column.*

Proof: Suppose that the vertices $(1, x)$ and $(1, y)$ $(x \neq y)$ are $[*, 1]$-vertices, that is to say $(1, x)$ is a $[a, 1]$-vertex, and $(1, y)$ is a $[a', 1]$-vertex. Then $I(1, x) = I(1, y)$, a contradiction. ■

As a direct consequence of Lemma 1, one can observe that if there is a row containing exactly one vertex (x, y) of C then the row of any other vertex of $C_y \cap C$ contains at least two vertices of C.

If $n = 1, 2$ then trivially the result holds. For $n = 3$, it is easy to see that $|C| \geq 4$. It is worth to note that there is another identifying code of cardinality 4 than $D \cup A$ (see Figure 3).

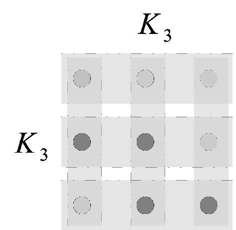

Figure 3: Another optimal identifying code for $n = 3$.

So assume now that $n \geq 4$.

Case 1 : there exists a row R_x such that $R_x \cap C = \emptyset$.
Without loss of generality, we may suppose that $x = 1$. If there exists some $u \neq 1$ such that $R_u \cap C = \emptyset$ then $I(1, v) = I(u, v)$ for all v, which contradicts that C is a identifying code. If there is a $[1, b]$-vertex (u, v) then $I(1, v) = I(u, v)$, a contradiction. Therefore, each row R_x with $x \neq 1$ contains at least 2 vertices of C, thus $|C| \geq 2(n-1) \geq \frac{3n}{2}$ if $n \geq 4$.
Moreover, observe that if $n \geq 4$ is odd and $|C| = \lfloor \frac{3n}{2} \rfloor$ then $|R_x \cap C| \geq 1$ for all x.

Now, by symmetry, one may assume that there is no column C_y with $C_y \cap C = \emptyset$. Let us now introduce some notations. A 1-*row* (respectively 1-*column*) is a row R_x (resp. column C_y) such that $|R_x \cap C| = 1$ (resp. $|C_y \cap C| = 1$). A non 1-row (resp. column) will be denoted by 2^+-*row* (resp. 2^+-*column*).

Case 2 : There is no $[1, 1]$-vertex.
Let n_1 be the number of 1-rows and 1-columns and $n_2 = 2n - n_1$.
We claim that $n_1 \leq n_2$. Indeed, associate to each 1-row R_x the column C_y where $(x, y) = R_x \cap C$. Since there is no $[1, 1]$-vertex, C_y is a 2^+-column. By Lemma 1, a column C_y can not be associated to two distinct 1-rows.
Similarly, one can construct an injection from the set of 1-columns to the set of 2^+-rows. Thus $n_1 \leq n_2$. Since $n_1 + n_2 = 2n$, this implies that:

$$n_2 \geq n \tag{1}$$

Moreover, by double counting, we have:

$$|C| \geq \frac{n_1 + 2n_2}{2} \tag{2}$$

By (1) and (2), we obtain that

$$|C| \geq \frac{2n + n_2}{2} \geq \frac{3n}{2} \tag{3}$$

Now, from (3), if n is odd then $|C| > \lfloor \frac{3n}{2} \rfloor$ and we are done, which concludes Case 2.

In order to get the uniqueness result we need the following lemma :

Lemma 2 *Let n be an even integer and C be an identifying code of $K_n \Box K_n$. If there is no $[1,1]$-vertex and $|C| = \frac{3n}{2}$ then, up to row and column permutations, $C = D \cup A$.*

Proof: By (1)–(3), we have that $n_1 = n_2 = n$ and each 2^+-row (resp. column) contains exactly 2 elements of C. Let r_1 (resp. c_1) be the number of 1-rows (resp. 1-columns). Since $r_1 + 2(n - r_1) = c_1 + 2(n - c_1) = |C|$, then $r_1 = c_1$. Since $r_1 + c_1 = n_1 = n$ then $r_1 = c_1 = \frac{n}{2}$. Up to row (resp. column) permutations, one may assume that $R_1, \ldots, R_{\frac{n}{2}}$ (resp. $C_{\frac{n}{2}+1}, \ldots, C_n$) are 1-rows (resp. 1-columns).
Since there is no $[1,1]$-vertex, then for every $x \in \{1, \ldots, \frac{n}{2}\}$ and $y \in \{\frac{n}{2} + 1, \ldots, n\}$ we have

$$(x, y) \notin C \quad (A).$$

This implies that for every $y \in \{\frac{n}{2} + 1, \ldots, n\}$, we have

$$|(C_y \cap (\cup_{x \in \{\frac{n}{2}+1,\ldots,n\}} R_x)) \cap C| = 1 \quad (*).$$

Now, by (A) and Lemma 1, for $x \in \{\frac{n}{2}+1, \ldots, n\}$, there is at most one $y \in \{\frac{n}{2}+1, \ldots, n\}$ such that $(x, y) \in C$. By $(*)$, for every $x \in \{\frac{n}{2} + 1, \ldots, n\}$, we have

$$|(R_x \cap (\cup_{y \in \{\frac{n}{2}+1,\ldots,n\}} C_y)) \cap C| = 1 \quad (**).$$

Thus, by $(*)$ and $(**)$, up to **column** permutations one may assume that (x, x) belongs to C for every $x \in \{\frac{n}{2} + 1, \ldots, n\}$.

Since each row R_x with $x \leq \frac{n}{2}$ is a 1-row, then, by Lemma 1, each column C_y with $y \leq \frac{n}{2}$ contains at most one element in $C \cap \{1, \ldots, \frac{n}{2}\} \times \{y\}$. Moreover, by (A), each element $(x, y) \in C$ with $x \in \{1, \ldots, \frac{n}{2}\}$ satisfies that $y \in \{1, \ldots, \frac{n}{2}\}$. Here, again, up to **row** permutations, one may assume that $(x, x) \in C$ with $x \in \{1, \ldots, \frac{n}{2}\}$.

Now, since each row R_x (resp. column C_y) with $x > \frac{n}{2}$ (resp. $y \leq \frac{n}{2}$) contains two elements of C, up to row (or column) permutations one may assume that $(n - x + 1, x) \in C$ for all $x \leq \frac{n}{2}$.

Now to conclude that, up to rows and columns permutations, $C = D \cup A$ it is enough to **first** make, say, row permutations on $\{R_{\frac{n}{2}+1}, \ldots, R_n\}$ in order to get A. ■

Case 3 : There exists a $[1,1]$-vertex.
Without loss of generality, suppose that $(1,1)$ is a $[1,1]$-vertex. Observe that $C \setminus \{(1,1)\}$ is an identifying code of the subgraph $G_{2,n}$ induced by $\{2,\ldots,n\} \times \{2,\ldots,n\}$.
If there is a $[1,1]$-vertex (x,y) in $G_{2,n}$ then $I(x,1) = I(1,y) = \{(1,1),(x,y)\}$, a contradiction.
If n is odd then by Case 2, $|C| \geq 1 + \frac{3(n-1)}{2} = \lfloor \frac{3n}{2} \rfloor$. This shows the first part of Theorem 1. For the uniqueness when $|C| = \lfloor \frac{3n}{2} \rfloor$, we apply Lemma 2 as shown in Figure 4.

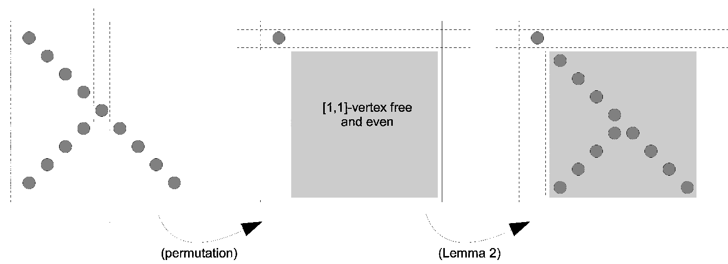

Figure 4: Uniqueness in the case where n is odd.

If n is even then, again, by Case 2, $|C| \geq \lceil 1 + \frac{3(n-1)}{2} \rceil \geq \lfloor \frac{3n}{2} \rfloor$. This terminates the proof of Theorem 1. ■

Note that for the case where n is even, there are at least two distinct minimum identifying codes of $K_n \square K_n$. We exhibited one in Figure 1, in which there is no $[1,1]$-vertex. It is easy to construct another one, having a $[1,1]$-vertex, say $(1,1)$. Indeed, consider the (unique) optimal identifying code C of $G_{2,n}$, and let (x,x) be the unique $[1,1]$-vertex of C. It is easy to see that $C \cup \{(1,1),(x,y)\}$ is a minimum identifying code of $K_n \square K_n$, where y is any coordinate different from x.

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