A counterexample to a conjecture of Erdős, Graham and Spencer

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Submitted: Oct 6, 2008; Accepted: Dec 2, 2008; Published: Dec 9, 2008
Mathematics Subject Classification: 11B75

Abstract

It is conjectured by Erdős, Graham and Spencer that if $1 \le a_1 \le a_2 \le \cdots \le a_s$ with $\sum_{i=1}^{s} 1/a_i < n - 1/30$, then this sum can be decomposed into n parts so that all partial sums are ≤ 1. In this note we propose a counterexample which gives a negative answer to this conjecture.

Keywords: Erdős-Graham-Spencer conjecture; Erdős problem; Partition.

1 Introduction

Erdős ([2], p. 41) asked the following question: is it true that if a_i's are positive integers with $1 < a_1 < a_2 < \cdots < a_s$ and $\sum_{i=1}^{s} 1/a_i < 2$, then there exists a subset A of $\{1, 2, \ldots, s\}$ such that

$$\sum_{i \in A} \frac{1}{a_i} < 1, \quad \sum_{i \in \{1,\ldots,s\} \setminus A} \frac{1}{a_i} < 1?$$

Sándor [3] gave a simple construction to show that the answer is negative: let $\{a_i\} = \{$ divisors of 120 with the exception of 1 and 120 $\}$. Furthermore, Sándor[3] proved the following results:

Theorem 1. *For every $n \ge 2$, there exist integers $1 < a_1 < a_2 < \cdots < a_s$ with $\sum_{i=1}^{s} 1/a_i < n$ and this sum cannot be split into n parts so that all partial sums are ≤ 1.*

*This author is supported by Natural Science Research Project of Ordinary Universities in Jiangsu Province (08KJB110002),P.R.China.

Theorem 2. *Let $n \geq 2$. If $1 < a_1 < a_2 < \cdots < a_s$ with $\sum_{i=1}^{s} 1/a_i < n - \frac{n}{e^{n-1}}$, then this sum can be decomposed into n parts so that all partial sums are ≤ 1.*

If we allow repetition of integers, it is conjectured by Erdős, Graham and Spencer ([2],p. 41) that if $1 \leq a_1 \leq a_2 \leq \cdots \leq a_s$ with $\sum_{i=1}^{s} 1/a_i < n - 1/30$, then this sum can be decomposed into n parts so that all partial sums are ≤ 1. This is not true for $\sum_{i=1}^{s} 1/a_i \leq n - 1/30$ as shown by $a_1 = 2, a_2 = a_3 = 3, a_4 = \ldots = a_{5n-3} = 5$. Sándor[3] proved a weaker assertion when the $n - 1/30$ was replaced by $n - 1/2$.

Let $\alpha(n)$ denote the least real number such that: for any integers $1 \leq a_1 \leq a_2 \leq \cdots \leq a_s$ with $n \geq 2$ and $\sum_{i=1}^{s} 1/a_i < n - \alpha(n)$, this sum can be decomposed into n parts so that all partial sums are ≤ 1. Erdős-Graham-Spencer conjecture hoped that $\alpha(n) \leq 1/30$ and Sándor's result stated that $\alpha(n) \leq 1/2$. In [1] Yong-Gao Chen proved that $\alpha(n) \leq 1/3$ and in [4] Jin-Hui Fang and Yong-Gao Chen proved that $\alpha(n) \leq 2/7$.

The purpose of this article is to give a counterexample to Erdős-Graham-Spencer conjecture:

$$a_1 = 2, a_2 = a_3 = 3, a_4 = 4, a_5 = \cdots = a_{11n-12} = 11,$$

which stats that

Theorem 3. $\alpha(n) \geq 5/132$.

2 Proof of Theorem 3

Clearly,

$$\sum_{i=1}^{11n-12} \frac{1}{a_i} = n - \frac{5}{132}.$$

For any partition $\{1, \ldots, 11n-12\} = \cup_{j=1}^{n} A_j$, we will prove that there exists $1 \leq j \leq n$ such that $\sum_{k \in A_j} 1/a_k > 1$. Without loss of generality, we let $1 \in A_1$. Let $l = \big|A_1 \cap \{2, 3, 4\}\big|$. Below we distinguish four cases.

Case 1. $l \geq 2$.

In this case we must have

$$\sum_{k \in A_1} \frac{1}{a_k} \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1$$

and we are done.

Case 2. $l = 1$ and $4 \notin A_1$.

Assume that $t \in \mathbb{N}$ and

$$\sum_{k \in A_1} \frac{1}{a_k} = \frac{1}{2} + \frac{1}{3} + \frac{t}{11}.$$

If $t \geq 2$, we have

$$\sum_{k \in A_1} \frac{1}{a_k} \geq \frac{1}{2} + \frac{1}{3} + \frac{2}{11} = \frac{134}{132} > 1.$$

If $0 \le t \le 1$, we must have

$$\sum_{j=2}^{n} \sum_{k \in A_j} \frac{1}{a_k} \ge n - \frac{5}{132} - (\frac{1}{2} + \frac{1}{3} + \frac{1}{11}) = n - 1 + \frac{5}{132} > n - 1.$$

Thus there exists $2 \le j \le n$ such that $\sum_{k \in A_j} 1/a_k > 1$ and we are done.

Case 3. $l = 1$ and $4 \in A_1$.

Assume that

$$\sum_{k \in A_1} \frac{1}{a_k} = \frac{1}{2} + \frac{1}{4} + \frac{t}{11}.$$

One can see that

$$\sum_{k \in A_1} \frac{1}{a_k} = \frac{1}{2} + \frac{1}{4} + \frac{3}{11} = \frac{135}{132} > 1$$

when $t \ge 3$ and

$$\sum_{j=2}^{n} \sum_{k \in A_j} \frac{1}{a_k} \ge n - \frac{5}{132} - (\frac{1}{2} + \frac{1}{4} + \frac{2}{11}) = n - 1 + \frac{4}{132} > n - 1,$$

hence there exists $2 \le j \le n$ such that $\sum_{k \in A_j} 1/a_k > 1$ when $t \le 2$. So we prove it.

Case 4. $l = 0$.

Assume that

$$\sum_{k \in A_1} \frac{1}{a_k} = \frac{1}{2} + \frac{t}{11} \le 1.$$

Then we have $t \le \frac{11}{2}$ and hence $t \le 5$. Thus we conclude that

$$\sum_{j=2}^{n} \sum_{k \in A_j} \frac{1}{a_k} \ge \frac{2}{3} + \frac{1}{4} + \frac{11n - 21}{11} = n - 1 + \frac{1}{132} > n - 1,$$

and hence there exists $2 \le j \le n$ with $\sum_{k \in A_j} 1/a_k > 1$. Now we complete the proof. $\square$

Acknowledgment. The author acknowledge professor Zhi-wei Sun for introducing this subject and the referee for his/her helpful suggestions.

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