On unimodality problems in Pascal's triangle *

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Abstract

Many sequences of binomial coefficients share various unimodality properties. In this paper we consider the unimodality problem of a sequence of binomial coefficients located in a ray or a transversal of the Pascal triangle. Our results give in particular an affirmative answer to a conjecture of Belbachir *et al* which asserts that such a sequence of binomial coefficients must be unimodal. We also propose two more general conjectures.

1 Introduction

Let $a_0, a_1, a_2, \ldots$ be a sequence of nonnegative numbers. It is called *unimodal* if $a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots$ for some m (such an integer m is called a mode of the sequence). In particular, a monotone (increasing or decreasing) sequence is known as unimodal. The sequence is called *concave* (resp. *convex*) if for $i \geq 1$, $a_{i-1} + a_{i+1} \leq 2a_i$ (resp. $a_{i-1} + a_{i+1} \geq 2a_i$). The sequence is called *log-concave* (resp. *log-convex*) if for all $i \geq 1$, $a_{i-1}a_{i+1} \leq a_i^2$ (resp. $a_{i-1}a_{i+1} \geq a_i^2$). By the arithmetic-geometric mean inequality, the concavity implies the log-concavity (the log-convexity implies the convexity). For a sequence $\{a_i\}$ of positive numbers, it is log-concave (resp. log-convex) if and only if the sequence $\{a_{i+1}/a_i\}$ is decreasing (resp. increasing), and so the log-concavity implies the unimodality. The unimodality problems, including concavity (convexity) and log-concavity (log-convexity), arise naturally in many branches of mathematics. For details, see [3, 4, 13, 17, 18, 19, 21, 22] about the unimodality and log-concavity and [7, 10] about the log-convexity.

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Many sequences of binomial coefficients share various unimodality properties. For example, the sequence $\left\{\binom{n}{k}\right\}_{k=0}^{n}$ is unimodal and log-concave in k. On the other hand, the sequence $\left\{\binom{n}{k}\right\}_{n=k}^{+\infty}$ is increasing, log-concave and convex in n (see Comtet [5] for example). As usual, let $\binom{n}{k}=0$ unless $0 \le k \le n$. Tanny and Zuker [14, 15] showed the unimodality and log-concavity of the binomial sequences $\left\{\binom{n_0-i}{i}\right\}_i$ and $\left\{\binom{n_0-id}{i}\right\}_i$. Very recently, Belbachir *et al* [1] showed the unimodality and log-concavity of the binomial sequence $\left\{\binom{n_0+i}{id}\right\}_i$. They further proposed the following.

Conjecture 1 ([1, Conjecture 1]). *Let $\binom{n}{k}$ be a fixed element of the Pascal triangle crossed by a ray. The sequence of binomial coefficients located along this ray is unimodal.*

$$\begin{array}{ccccccccccccccc}
 & & & & & & & \binom{0}{0} & & & & & & & \\
 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & \\
 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & \\
 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & \\
 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} & \\
\binom{7}{0} & & \binom{7}{1} & & \binom{7}{2} & & \binom{7}{3} & & \binom{7}{4} & & \binom{7}{5} & & \binom{7}{6} & & \binom{7}{7}
\end{array}$$

Figure 1: a ray with $d = 3$ and $\delta = 2$.

The object of this paper is to study the unimodality problem of a sequence of binomial coefficients located in a ray or a transversal of the Pascal triangle. Let $\left\{\binom{n_i}{k_i}\right\}_{i\ge 0}$ be such a sequence. Then $\{n_i\}_{i\ge 0}$ and $\{k_i\}_{i\ge 0}$ form two arithmetic sequences (see Figure 1). Clearly, we may assume that the common difference of $\{n_i\}_{i\ge 0}$ is nonnegative (by changing the order of the sequence). For example, the sequence $\left\{\binom{n_0-i}{i}\right\}_{i=0}^{\lfloor \frac{n_0}{2}\rfloor}$ coincides with the sequence $\left\{\binom{n_0-\lfloor\frac{n_0}{2}\rfloor+i}{\lfloor\frac{n_0}{2}\rfloor-i}\right\}_{i=0}^{\lfloor \frac{n_0}{2}\rfloor}$ except for the order. On the other hand, the sequence $\left\{\binom{n_i}{k_i}\right\}_{i\ge 0}$ is the same as the sequence $\left\{\binom{n_i}{n_i-k_i}\right\}_{i\ge 0}$ by the symmetry of the binomial coefficients. So we may assume, without loss of generality, that the common difference of $\{k_i\}_{i\ge 0}$ is nonnegative. Thus it suffices to consider the unimodality of the sequence $\{\binom{n_0+id}{k_0+i\delta}\}_{i\ge 0}$ for nonnegative integers d and δ. The following is the main result of this paper, which in particular, gives an affirmative answer to Conjecture 1.

Theorem 1. *Let n_0, k_0, d, δ be four nonnegative integers and $n_0 \ge k_0$. Define the sequence*

$$C_i = \binom{n_0+id}{k_0+i\delta}, \qquad i = 0, 1, 2, \ldots.$$

Then

(i) *if $d = \delta > 0$ or $\delta = 0$, the sequence is increasing, convex and log-concave;*

(ii) *if $d < \delta$, the sequence is log-concave and therefore unimodal;*

(iii) *if $d > \delta > 0$, the sequence is increasing, convex, and asymptotically log-convex (i.e., there exists a nonnegative integer m such that $C_m, C_{m+1}, C_{m+2}, \ldots$ is log-convex).*

This paper is organized as follows. In the next section, we prove Theorem 1. In Section 3, we present a combinatorial proof of the log-concavity in Theorem 1 (ii). In Section 4, we show more precise results about the asymptotically log-convexity for certain particular sequences of binomial coefficients in Theorem 1 (iii). Finally in Section 5, we propose some open problems and conjectures.

Throughout this paper we will denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the largest integer $\le x$ and the smallest integer $\ge x$ respectively.

2 The proof of Theorem 1

The following result is folklore and we include a proof of it for completeness.

Lemma 1. *If a sequence $\{a_i\}_{i\ge 0}$ of positive numbers is unimodal (resp. increasing, decreasing, concave, convex, log-concave, log-convex), then so is its subsequence $\{a_{n_0+id}\}_{i\ge 0}$ for arbitrary fixed nonnegative integers n_0 and d.*

Proof. We only consider the log-concavity case since the others are similar. Let $\{a_i\}_{i\ge 0}$ be a log-concave sequence of positive numbers. Then the sequence $\{a_{i-1}/a_i\}_{i\ge 0}$ is increasing. Hence $a_{j-1}/a_j \le a_k/a_{k+1}$ for $1 \le j \le k$, i.e., $a_{j-1}a_{k+1} \le a_j a_k$. Thus

$$a_{n-d}a_{n+d} \le a_{n-d+1}a_{n+d-1} \le a_{n-d+2}a_{n+d-2} \le \cdots \le a_{n-1}a_{n+1} \le a_n^2,$$

which implies that the sequence $\{a_{n_0+id}\}_{i\ge 0}$ is log-concave. □

The proof of Theorem 1. (i) If $\delta = 0$, then $C_i = \binom{n_0+id}{k_0}$. The sequence $\binom{i}{k_0}$ is increasing, convex and log-concave in i, so is the sequence C_i by Lemma 1. The case $d = \delta$ is similar since $C_i = \binom{n_0+id}{n_0-k_0}$.

(ii) To show the log-concavity of $\{C_i\}$ when $d < \delta$, it suffices to show that

$$\binom{n+d}{k+\delta}\binom{n-d}{k-\delta} \le \binom{n}{k}^2$$

for $n \ge k$. Write

$$\begin{aligned}\binom{n+d}{k+\delta}\binom{n-d}{k-\delta} &= \frac{(n+d)!(n-d)!}{(n-k+d-\delta)!(k+\delta)!(n-k+\delta-d)!(k-\delta)!} \\ &= \binom{n+d}{n-k}\binom{n-d}{n-k}\frac{\binom{n-k}{\delta-d}}{\binom{n-k+\delta-d}{\delta-d}}\frac{\binom{k-d}{\delta-d}}{\binom{k+\delta}{\delta-d}}.\end{aligned}$$

Now $\binom{n-k}{\delta-d} \le \binom{n-k+\delta-d}{\delta-d}$, $\binom{k-d}{\delta-d} \le \binom{k+\delta}{\delta-d}$ and $\binom{n+d}{n-k}\binom{n-d}{n-k} \le \binom{n}{n-k}^2$ by (i). Hence

$$\binom{n+d}{k+\delta}\binom{n-d}{k-\delta} \le \binom{n}{n-k}^2 = \binom{n}{k}^2,$$

as required.

(iii) Assume that $d > \delta > 0$. By Vandermonde's convolution formula, we have

$$\binom{n+d}{k+\delta} = \sum_{r+s=k+\delta} \binom{n}{r}\binom{d}{s} \ge \binom{n}{k}\binom{d}{\delta} \ge 2\binom{n}{k},$$

which implies that $\binom{n+d}{k+\delta} > \binom{n}{k}$ and $\binom{n+d}{k+\delta} + \binom{n-d}{k-\delta} \ge 2\binom{n}{k}$. Hence the sequence $\{C_i\}$ is increasing and convex.

It remains to show that the sequence $\{C_i\}$ is asymptotically log-convex. Denote

$$\Delta(i) := \binom{n_0+(i+1)d}{k_0+(i+1)\delta}\binom{n_0+(i-1)d}{k_0+(i-1)\delta} - \binom{n_0+id}{k_0+i\delta}^2.$$

Then we need to show that $\Delta(i)$ is positive for all sufficiently large i. Write

$$\begin{aligned}
\Delta(i) &= \frac{(n_0+id)![n_0+(i-1)d]!}{(k_0+i\delta)![k_0+(i+1)\delta]![n_0-k_0+i(d-\delta)]![n_0-k_0+(i+1)(d-\delta)]!} \\
&\quad \times \left\{ \prod_{j=1}^{d}(n_0+id+j) \prod_{j=1}^{d-\delta}[n_0-k_0+(i-1)(d-\delta)+j] \prod_{j=1}^{\delta}[k_0+(i-1)\delta+j] \right. \\
&\quad \left. - \prod_{j=1}^{d}[n_0+(i-1)d+j] \prod_{j=1}^{d-\delta}[n_0-k_0+i(d-\delta)+j] \prod_{j=1}^{\delta}(k_0+i\delta+j) \right\} \\
&= \frac{(n_0+id)![n_0+(i-1)d]!d^d\delta^\delta(d-\delta)^{(d-\delta)}}{(k_0+i\delta)![k_0+(i+1)\delta]![n_0-k_0+i(d-\delta)]![n_0-k_0+(i+1)(d-\delta)]!} P(i),
\end{aligned}$$

where

$$\begin{aligned}
P(i) &= \prod_{j=1}^{d}\left(i+\frac{n_0+j}{d}\right) \prod_{j=1}^{d-\delta}\left(i+\frac{n_0-k_0-d+\delta+j}{d-\delta}\right) \prod_{j=1}^{\delta}\left(i+\frac{k_0-\delta+j}{\delta}\right) \\
&\quad - \prod_{j=1}^{d}\left(i+\frac{n_0-d+j}{d}\right) \prod_{j=1}^{d-\delta}\left(i+\frac{n_0-k_0+j}{d-\delta}\right) \prod_{j=1}^{\delta}\left(i+\frac{k_0+j}{\delta}\right).
\end{aligned}$$

Then it suffices to show that $P(i)$ is positive for sufficiently large i. Clearly, $P(i)$ can be viewed as a polynomial in i. So it suffices to show that the leading coefficient of $P(i)$ is positive.

Note that $P(i)$ is the difference of two monic polynomials of degree $2d$. Hence its degree is less than $2d$. Denote

$$P(i) = a_{2d-1}i^{2d-1} + a_{2d-2}i^{2d-2} + \cdots .$$

By Vieta's formula, we have

$$
\begin{aligned}
a_{2d-1} &= -\left(\sum_{j=1}^{d}\frac{n_0+j}{d}+\sum_{j=1}^{d-\delta}\frac{n_0-k_0-d+\delta+j}{d-\delta}+\sum_{j=1}^{\delta}\frac{k_0-\delta+j}{\delta}\right)\\
&\quad+\left(\sum_{j=1}^{d}\frac{n_0-d+j}{d}+\sum_{j=1}^{d-\delta}\frac{n_0-k_0+j}{d-\delta}+\sum_{j=1}^{\delta}\frac{k_0+j}{\delta}\right)\\
&= \sum_{j=1}^{d}\left(\frac{n_0-d+j}{d}-\frac{n_0+j}{d}\right)\\
&\quad+\sum_{j=1}^{d-\delta}\left(\frac{n_0-k_0+j}{d-\delta}-\frac{n_0-k_0-d+\delta+j}{d-\delta}\right)\\
&\quad+\sum_{j=1}^{\delta}\left(\frac{k_0+j}{\delta}-\frac{k_0-\delta+j}{\delta}\right)\\
&= \sum_{j=1}^{d}(-1)+\sum_{j=1}^{d-\delta}1+\sum_{j=1}^{\delta}1\\
&= -d+(d-\delta)+\delta\\
&= 0.
\end{aligned}
$$

Using the identity

$$
\sum_{1\le i<j\le n} x_i x_j = \frac{1}{2}\left[\left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2\right],
$$

we obtain again by Vieta's formula

$$
\begin{aligned}
a_{2d-2} &= \frac{1}{2}\left[\left(\sum_{j=1}^{d}\frac{n_0+j}{d}+\sum_{j=1}^{d-\delta}\frac{n_0-k_0-d+\delta+j}{d-\delta}+\sum_{j=1}^{\delta}\frac{k_0-\delta+j}{\delta}\right)^2\right.\\
&\quad\left.-\left(\sum_{j=1}^{d}\frac{n_0+j}{d}\right)^2-\left(\sum_{j=1}^{d-\delta}\frac{n_0-k_0-d+\delta+j}{d-\delta}\right)^2-\left(\sum_{j=1}^{\delta}\frac{k_0-\delta+j}{\delta}\right)^2\right]\\
&\quad-\frac{1}{2}\left[\left(\sum_{j=1}^{d}\frac{n_0-d+j}{d}+\sum_{j=1}^{d-\delta}\frac{n_0-k_0+j}{d-\delta}+\sum_{j=1}^{\delta}\frac{k_0+j}{\delta}\right)^2\right.\\
&\quad\left.-\left(\sum_{j=1}^{d}\frac{n_0-d+j}{d}\right)^2-\left(\sum_{j=1}^{d-\delta}\frac{n_0-k_0+j}{d-\delta}\right)^2-\left(\sum_{j=1}^{\delta}\frac{k_0+j}{d-\delta}\right)^2\right].
\end{aligned}
$$

But $a_{2d-1}=0$ implies

$$\left(\sum_{j=1}^{d}\frac{n_0+j}{d}+\sum_{j=1}^{d-\delta}\frac{n_0-k_0-d+\delta+j}{d-\delta}+\sum_{j=1}^{\delta}\frac{k_0-\delta+j}{\delta}\right)^2$$
$$=\left(\sum_{j=1}^{d}\frac{n_0-d+j}{d}+\sum_{j=1}^{d-\delta}\frac{n_0-k_0+j}{d-\delta}+\sum_{j=1}^{\delta}\frac{k_0+j}{\delta}\right)^2,$$

so we have

$$\begin{aligned}
a_{2d-2} &= \frac{1}{2}\sum_{j=1}^{d}\left[\left(\frac{n_0-d+j}{d}\right)^2-\left(\frac{n_0+j}{d}\right)^2\right]\\
&\quad+\frac{1}{2}\sum_{j=1}^{d-\delta}\left[\left(\frac{n_0-k_0+j}{d-\delta}\right)^2-\left(\frac{n_0-k_0-d+\delta+j}{d-\delta}\right)^2\right]\\
&\quad+\frac{1}{2}\sum_{j=1}^{\delta}\left[\left(\frac{k_0+j}{\delta}\right)^2-\left(\frac{k_0-\delta+j}{\delta}\right)^2\right]\\
&= -\frac{1}{2}\sum_{j=1}^{d}\frac{2n_0-d+2j}{d}+\frac{1}{2}\sum_{j=1}^{d-\delta}\frac{2(n_0-k_0)-(d-\delta)+2j}{d-\delta}+\frac{1}{2}\sum_{j=1}^{\delta}\frac{2k_0-\delta+2j}{\delta}\\
&= -\frac{1}{2}(2n_0+1)+\frac{1}{2}(2n_0-2k_0+1)+\frac{1}{2}(2k_0+1)\\
&= \frac{1}{2}.
\end{aligned}$$

Thus $P(i)$ is a polynomial of degree $2d-2$ with positive leading coefficient, as desired. This completes the proof of the theorem. □

3 Combinatorial proof of the log-concavity

In Section 2 we have investigated the unimodality of sequences of binomial coefficients by an algebraic approach. It is natural to ask for a combinatorial interpretation. Lattice path techniques have been shown to be useful in solving the unimodality problem. As an example, we present a combinatorial proof of Theorem 1 (ii) following Bóna and Sagan's technique in [2].

Let $\mathbb{Z}^2=\{(x,y):x,y\in\mathbb{Z}\}$ denote the two-dimensional integer lattice. A lattice path is a sequence $P_1,P_2,\ldots,P_\ell$ of lattice points on $\mathbb{Z}^2$. A southeastern lattice path is a lattice path in which each step goes one unit to the south or to the east. Denote by $P(n,k)$ the set of southeastern lattice paths from the point $(0,n-k)$ to the point $(k,0)$. Clearly, the number of such paths is the binomial coefficient $\binom{n}{k}$.

Recall that, to show the log-concavity of $C_i=\binom{n_0+id}{k_0+i\delta}$ where $n_0\geq k_0$ and $d<\delta$, it suffices to show $\binom{n+d}{k+\delta}\binom{n-d}{k-\delta}\leq\binom{n}{k}^2$ for $n\geq k$. Here we do this by constructing an injection

$$\phi: P(n+d,k+\delta)\times P(n-d,k-\delta)\longrightarrow P(n,k)\times P(n,k).$$

Consider a path pair $(p,q) \in P(n+d,k+\delta) \times P(n-d,k-\delta)$. Then p and q must intersect. Let I_1 be the first intersection. For two points $P(a,b)$ and $Q(a,c)$ with the same x-coordinate, define their vertical distance to be $d_v(P,Q) = b-c$. Then the vertical distance from a point of p to a point of q starts at $2(\delta - d)$ for their initial points and ends at 0 for their intersection I_1. Thus there must be a pair of points $P \in p$ and $Q \in q$ before I_1 with $d_v(P,Q) = \delta - d$. Let (P_1, Q_1) be the first such pair of points. Similarly, after the last intersection I_2 there must be a last pair of points $P_2 \in p$ and $Q_2 \in q$ with the horizontal distance $d_h(P_2,Q_2) = -\delta$ (the definition of d_h is analogous to that of d_v). Now p is divided by two points P_1, P_2 into three subpaths p_1, p_2, p_3 and q is divided by Q_1, Q_2 into three subpaths q_1, q_2, q_3. Let p_1' be obtained by moving p_1 down to Q_1 south $\delta - d$ units and p_3' be obtained by moving p_3 right to Q_2 east δ units. Then we obtain a southeastern lattice path $p_1' q_2 p_3'$ in $P(n,k)$. We can similarly obtain the second southeastern lattice path $q_1' p_2 q_3'$ in $P(n,k)$, where q_1' is q_1 moved north $\delta - d$ units and q_3' is q_3 moved west δ units. Define $\phi(p,q) = (p_1' q_2 p_3', q_1' p_2 q_3')$. It is not difficult to verify that ϕ is the required injective. We omit the proof for brevity.

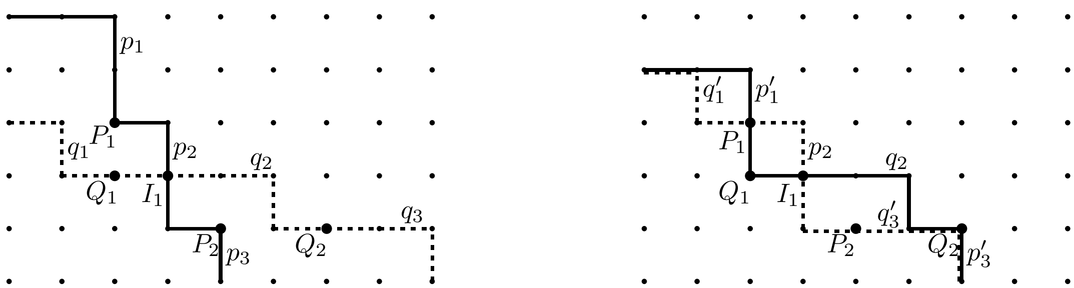

Figure 2: the constructing of ϕ.

4 Asymptotic behavior of the log-convexity

Theorem 1 (iii) tells us that the sequence $C_i = \binom{n_0+id}{k_0+i\delta}$ is asymptotically log-convex when $d > \delta > 0$. We can say more for a certain particular sequence of binomial coefficients. For example, it is easy to verify that the central binomial coefficients $\binom{2i}{i}$ is log-convex for $i \geq 0$ (see Liu and Wang [10] for a proof). In this section we give two generalizations of this result. The first one is that every sequence of binomial coefficients located along a ray with origin $\binom{0}{0}$ is log-convex.

Proposition 1. *Let d and δ be two positive integers and $d > \delta > 0$. Then the sequence $\left\{\binom{id}{i\delta}\right\}_{i\geq 0}$ is log-convex.*

Before showing Proposition 1, we first demonstrate two simple but useful facts.

Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$ be two n-tuples of real numbers. We say that α *alternates left of* β, denoted by $\alpha \preceq \beta$, if

$$a_1^* \leq b_1^* \leq a_2^* \leq b_2^* \cdots \leq a_n^* \leq b_n^*,$$

where a_j^* and b_j^* are the jth smallest elements of α and β, respectively.

Fact 1 Let $f(x)$ be a nondecreasing function. If $(a_1, a_2, \ldots, a_n) \preceq (b_1, b_2, \ldots, b_n)$, then $\prod_{i=1}^n f(a_i) \le \prod_{i=1}^n f(b_i)$.

Fact 2 Let x_1, x_2, y_1, y_2 be four positive numbers and $\frac{x_1}{y_1} \le \frac{x_2}{y_2}$. Then $\frac{x_1}{y_1} \le \frac{x_1+x_2}{y_1+y_2} \le \frac{x_2}{y_2}$.

Proof of Proposition 1. By Lemma 1, we may assume, without loss of generality, that d and δ are coprime. We need to show that

$$\Delta(i) := \binom{(i+1)d}{(i+1)\delta}\binom{(i-1)d}{(i-1)\delta} - \binom{id}{i\delta}^2 \ge 0$$

for all $i \ge 1$. Write

$$\Delta(i) = \frac{(id)![(i-1)d]!d^d\delta^\delta(d-\delta)^{(d-\delta)}\prod_{j=1}^{d}\left(i+\frac{j}{d}\right)\prod_{j=1}^{\delta}\left(i+\frac{j}{\delta}\right)\prod_{j=1}^{d-\delta}\left(i+\frac{j}{d-\delta}\right)}{(i\delta)![(i+1)\delta]![i(d-\delta)]![(i+1)(d-\delta)]!}Q(i),$$

where

$$Q(i) = \prod_{j=1}^{\delta}\left(1-\frac{1}{i+\frac{j}{\delta}}\right)\prod_{j=1}^{d-\delta}\left(1-\frac{1}{i+\frac{j}{d-\delta}}\right) - \prod_{j=1}^{d}\left(1-\frac{1}{i+\frac{j}{d}}\right).$$

Then we only need to show that $Q(i) \ge 0$ for $i \ge 1$. We do this by showing

$$\left(\frac{1}{d}, \ldots, \frac{d-1}{d}, \frac{d}{d}\right) \preceq \left(\frac{1}{\delta}, \ldots, \frac{\delta-1}{\delta}, \frac{\delta}{\delta}, \frac{1}{d-\delta}, \ldots, \frac{d-\delta-1}{d-\delta}, \frac{d-\delta}{d-\delta}\right),$$

or equivalently,

$$\left(\frac{1}{d}, \ldots, \frac{d-1}{d}\right) \preceq \left(\frac{1}{\delta}, \ldots, \frac{\delta-1}{\delta}, \frac{1}{d-\delta}, \ldots, \frac{d-\delta-1}{d-\delta}, 1\right).$$

Note that $(d, \delta) = 1$ implies all fractions $\left\{\frac{j}{d}\right\}_{j=1}^{d-1}$, $\left\{\frac{j}{\delta}\right\}_{j=1}^{\delta-1}$ and $\left\{\frac{j}{d-\delta}\right\}_{j=1}^{d-\delta-1}$ are different. Hence it suffices to show that every term of $\left\{\frac{j}{\delta}\right\}_{j=1}^{\delta-1} \bigcup \left\{\frac{j}{d-\delta}\right\}_{j=1}^{d-\delta-1}$ is precisely in one of $d-2$ open intervals $\left(\frac{k}{d}, \frac{k+1}{d}\right)$, where $k = 1, \ldots, d-2$. Indeed, neither two terms of $\left\{\frac{j}{\delta}\right\}_{j=1}^{\delta-1}$ nor two terms of $\left\{\frac{j}{d-\delta}\right\}_{j=1}^{d-\delta-1}$ are in the same interval since their difference is larger than $\frac{1}{d}$. On the other hand, if $\frac{j}{\delta}$ and $\frac{j'}{d-\delta}$ are in a certain interval $\left(\frac{k}{d}, \frac{k+1}{d}\right)$, then so is $\frac{j+j'}{d}$ by Fact 2, which is impossible. Thus there exists precisely one term of $\left\{\frac{j}{\delta}\right\}_{j=1}^{\delta-1} \bigcup \left\{\frac{j}{d-\delta}\right\}_{j=1}^{d-\delta-1}$ in every open interval $\left(\frac{k}{d}, \frac{k+1}{d}\right)$, as desired. This completes our proof. $\square$

For the second generalization of the log-convexity of the central binomial coefficients, we consider sequences of binomial coefficients located along a vertical ray with origin $\binom{n_0}{0}$ in the Pascal triangle.

Proposition 2. *Let $n_0 \ge 0$ and $V_i(n_0) = \binom{n_0+2i}{i}$. Then $V_0(n_0), V_1(n_0), \ldots, V_m(n_0)$ is log-concave and $V_{m-1}(n_0), V_m(n_0), V_{m+1}(n_0), \ldots$ is log-convex, where $m = n_0^2 - \left\lceil \frac{n_0}{2} \right\rceil$.*

Proof. The sequence $V_i(0) = \binom{2i}{i}$ is just the central binomial coefficients and therefore log-convex for $i \geq 0$. It implies that the sequence $V_i(1) = \binom{1+2i}{i}$ is log-convex for $i \geq 0$ since $V_i(1) = \frac{1}{2}V_{i+1}(0)$. Now let $n_0 \geq 2$ and define $f(i) = V_{i+1}(n_0)/V_i(n_0)$ for $i \geq 0$. Then, to show the statement, it suffices to show that

$$f(0) > f(1) > \cdots > f(m-1) \quad \text{and} \quad f(m-1) < f(m) < f(m+1) < \cdots \tag{1}$$

for $m = n_0^2 - \lceil \frac{n_0}{2} \rceil$.

By the definition we have

$$f(i) = \frac{\binom{n_0+2(i+1)}{i+1}}{\binom{n_0+2i}{i}} = \frac{(n_0+2i+1)(n_0+2i+2)}{(i+1)(n_0+i+1)}. \tag{2}$$

The derivative of $f(i)$ with respect to i is

$$f'(i) = \frac{2i^2 - 2(n_0-2)(n_0+1)i - (n_0+1)(n_0^2-2)}{(i+1)^2(n_0+i+1)^2}.$$

The numerator of $f'(i)$ has the unique positive zero

$$\begin{aligned} r &= \frac{2(n_0-2)(n_0+1) + \sqrt{4(n_0-2)^2(n_0+1)^2 + 8(n_0+1)(n_0^2-2)}}{4} \\ &= \frac{(n_0-2)(n_0+1)}{2} + \frac{n_0\sqrt{n_0^2-1}}{2}. \end{aligned}$$

It implies that $f'(i) < 0$ for $0 \leq i < r$ and $f'(i) > 0$ for $i > r$. Thus we have

$$f(0) > f(1) > \cdots > f(\lfloor r \rfloor) \quad \text{and} \quad f(\lceil r \rceil) < f(\lceil r \rceil + 1) < f(\lceil r \rceil + 2) < \cdots. \tag{3}$$

It remains to compare the values of $f(\lfloor r \rfloor)$ and $f(\lceil r \rceil)$. Note that

$$\frac{n_0^2 - n_0\sqrt{n_0^2-1}}{2} = \frac{n_0}{2(n_0 + \sqrt{n_0^2-1})} < \frac{1}{2}.$$

Hence

$$\left\lceil \frac{n_0\sqrt{n_0^2-1}}{2} \right\rceil = \begin{cases} \frac{n_0^2}{2}, & \text{if } n_0 \text{ is even;} \\ \frac{n_0^2+1}{2}, & \text{if } n_0 \text{ is odd,} \end{cases}$$

and so

$$\lceil r \rceil = \frac{(n_0-2)(n_0+1)}{2} + \left\lceil \frac{n_0\sqrt{n_0^2-1}}{2} \right\rceil = \begin{cases} n_0^2 - \frac{n_0}{2} - 1, & \text{if } n_0 \text{ is even;} \\ n_0^2 - \frac{n_0+1}{2}, & \text{if } n_0 \text{ is odd.} \end{cases}$$

If n_0 is even, then by (2) we have

$$f(\lceil r \rceil) = \frac{16n_0^2 - 8}{4n_0^2 - 1} = 4 - \frac{4}{4n_0^2 - 1}$$

and

$$f(\lfloor r\rfloor) = f(\lceil r\rceil - 1) = \frac{16n_0^4 - 40n_0^2 + 16}{4n_0^4 - 9n_0^2 + 4} = 4 - \frac{4(n_0^2 - 2)}{4n_0^4 - 9n_0^2 + 4}.$$

Thus $f(\lfloor r\rfloor) > f(\lceil r\rceil)$ since $f(\lfloor r\rfloor) - f(\lceil r\rceil) = \frac{8}{(4n_0^2-1)(4n_0^4-9n_0^2+4)} > 0$. Also, $\lceil r\rceil = m - 1$. Combining (3) we obtain (1).

If n_0 is odd, then

$$f(\lceil r\rceil) = 4 - \frac{4(n_0^2 + 1)}{4n_0^4 + 3n_0^2 + 1}$$

and

$$f(\lfloor r\rfloor) = 4 - \frac{4(n_0^2 - 1)}{4n_0^4 - 5n_0^2 + 1}.$$

It is easy to verify that $f(\lfloor r\rfloor) < f(\lceil r\rceil)$. Also, $\lfloor r\rfloor = \lceil r\rceil - 1 = m - 1$. Thus (1) follows. This completes our proof. $\square$

5 Concluding remarks and open problems

In this paper we show that the sequence $C_i = \binom{n_0+id}{k_0+i\delta}$ is unimodal when $d < \delta$. A further problem is to find out the value of i for which C_i is a maximum. Tanny and Zuker [14, 15, 16] considered such a problem for the sequence $\binom{n_0-id}{i}$. For example, it is shown that the sequence $\binom{n_0-i}{i}$ attains the maximum when $i = \left\lfloor (5n_0 + 7 - \sqrt{5n_0^2 + 10n_0 + 9})/10 \right\rfloor$. Let $r(n_0, d)$ be the least integer at which $\binom{n_0-id}{i}$ attains its maximum. They investigated the asymptotic behavior of $r(n_0, d)$ for $d \to \infty$ and concluded with a variety of unsolved problems concerning the numbers $r(n_0, d)$. An interesting problem is to consider analogue for the general binomial sequence $C_i = \binom{n_0+id}{k_0+i\delta}$ when $d < \delta$. It often occurs that unimodality of a sequence is known, yet to determine the exact number and location of modes is a much more difficult task.

A finite sequence of positive numbers $a_0, a_1, \ldots, a_n$ is called a *Pólya frequency sequence* if its generating function $P(x) = \sum_{i=0}^n a_i x^i$ has only real zeros. By the Newton's inequality, if $a_0, a_1, \ldots, a_n$ is a Pólya frequency sequence, then

$$a_i^2 \geq a_{i-1}a_{i+1}\left(1 + \frac{1}{i}\right)\left(1 + \frac{1}{n-i}\right)$$

for $1 \leq i \leq n-1$, and the sequence is therefore log-concave and unimodal with at most two modes (see Hardy, Littlewood and Pólya [9, p. 104]). Darroch [6] further showed that each mode m of the sequence $a_0, a_1, \ldots, a_n$ satisfies

$$\left\lfloor \frac{P'(1)}{P(1)} \right\rfloor \leq m \leq \left\lceil \frac{P'(1)}{P(1)} \right\rceil.$$

We refer the reader to [3, 4, 8, 11, 12, 13, 20] for more information.

For example, the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ is a Pólya frequency sequence with the unique mode $n/2$ for even n and two modes $(n \pm 1)/2$ for odd n. On the other hand, the sequence $\binom{n}{0}, \binom{n-1}{1}, \binom{n-2}{2}, \ldots, \binom{\lceil n/2 \rceil}{\lfloor n/2 \rfloor}$ is a Pólya frequency sequence since its generating function is precisely the matching polynomial of a path on n vertices. Hence we make the more general conjecture that every sequence of binomial coefficients located in a transversal of the Pascal triangle is a Pólya frequency sequence.

Conjecture 2. *Let $C_i = \binom{n_0+id}{k_0+i\delta}$ where $n_0 \geq k_0$ and $\delta > d > 0$. Then the finite sequence $\{C_i\}_i$ is a Pólya frequency sequence.*

In Proposition 2 we have shown that the sequence $V_i(n_0) = \binom{n_0+2i}{i}$ is first log-concave and then log-convex. It is possible that an arbitrary sequence of binomial coefficients located along a ray in the Pascal triangle has the same property as the sequence $V_i(n_0)$. We leave this as a conjecture to end this paper.

Conjecture 3. *Let $C_i = \binom{n_0+id}{k_0+i\delta}$ where $n_0 \geq k_0$ and $d > \delta > 0$. Then there is a nonnegative integer m such that $C_0, C_1, \ldots, C_{m-1}, C_m$ is log-concave and $C_{m-1}, C_m, C_{m+1}, \ldots$ is log-convex.*

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