Global alliances and independent domination in some classes of graphs

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Abstract

A dominating set S of a graph G is a global (strong) defensive alliance if for every vertex $v \in S$, the number of neighbors v has in S plus one is at least (greater than) the number of neighbors it has in $V \setminus S$. The dominating set S is a global (strong) offensive alliance if for every vertex $v \in V \setminus S$, the number of neighbors v has in S is at least (greater than) the number of neighbors it has in $V \setminus S$ plus one. The minimum cardinality of a global defensive (strong defensive, offensive, strong offensive) alliance is denoted by $\gamma_a(G)$ ($\gamma_{\hat{a}}(G)$, $\gamma_o(G)$, $\gamma_{\hat{o}}(G)$).

We compare each of the four parameters $\gamma_a, \gamma_{\hat{a}}, \gamma_o, \gamma_{\hat{o}}$ to the independent domination number i. We show that

$i(G) \le \gamma_a^2(G) - \gamma_a(G) + 1$ and $i(G) \le \gamma_{\hat{a}}^2(G) - 2\gamma_{\hat{a}}(G) + 2$ for every graph

$i(G) \le \gamma_a^2(G)/4 + \gamma_a(G)$ and $i(G) \le \gamma_{\hat{a}}^2(G)/4 + \gamma_{\hat{a}}(G)/2$ for every bipartite graph

$i(G) \le 2\gamma_a(G) - 1$ and $i(G) = 3\gamma_{\hat{a}}(G)/2 - 1$ for every tree

and describe the extremal graphs,

and that $\gamma_o(T) \le 2i(T) - 1$ and $i(T) \le \gamma_{\hat{o}}(T) - 1$ for every tree.

We use a lemma stating that $\beta(T) + 2i(T) \ge n + 1$ in every tree T of order n and independence number $\beta(T)$.
Keywords: independence, domination, alliance, bipartite graph, tree.

1 Introduction

We consider simple graphs $G = (V(G), E(G))$ with vertex set $V(G)$, edge set $E(G)$, order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$ (V, E, n, m when no ambiguity is possible). The degree in G of a vertex v is denoted by $d_G(v)$, or simply $d(v)$, and the number of neighbors of v in a subset S of V by $d_S(v)$.

A subset S of vertices is *dominating* if every vertex of $V \setminus S$ has at least one neighbor in S, and *independent* if no two vertices of S are adjacent. It is well known that a dominating set is independent if and only if it is a maximal independent set and that in every graph, $\gamma(G) \leq i(G) \leq \beta(G)$ where $\gamma(G)$ and $i(G)$ are respectively the minimum cardinality of a dominating set and of an independent dominating set and $\beta(G)$ is the maximum cardinality of an independent set. Alliances are defined in [6] as follows. A subset $S \subseteq V$ is a *defensive alliance* (respectively *strong defensive alliance*) if $d_{V \setminus S}(v) \leq d_S(v) + 1$ (respectively $d_{V \setminus S}(v) < d_S(v) + 1$) for every $v \in S$. In other words, every vertex of S together with its neighbors in S is as strong as (respectively stronger than) the coalition of its neighbors out of S. The subset S is an *offensive alliance* (respectively a *strong offensive alliance*) if $d_S(v) \geq d_{V \setminus S}(v) + 1$ (respectively $d_S(v) > d_{V \setminus S}(v) + 1$) for every vertex $v \in V \setminus S$ dominated by S. In other words, every vertex out of S and dominated by S together with its neighbors out of S is not stronger (respectively weaker) than the coalition of its neighbors in S. Alliances of any sort are *global* if they dominate G. The minimal cardinality of a global defensive (respectively strong defensive, offensive, strong offensive) alliance of G is denoted by $\gamma_a(G)$ (respectively $\gamma_{\hat{a}}(G)$, $\gamma_o(G)$, $\gamma_{\hat{o}}(G)$). Clearly $\gamma(G) \leq \gamma_a(G) \leq \gamma_{\hat{a}}(G)$ and $\gamma(G) \leq \gamma_o(G) \leq \gamma_{\hat{o}}(G)$ for every graph G. Similar notions exist under the name of coalitions or monopolies. In particular a monopoly is a global defensive and offensive alliance [7].

Properties of global alliances can be found in several papers, some of them are referenced below [1, 2, 3, 4, 5, 8, 9], in particular relationships between alliance parameters and other graph parameters valid for all graphs or in some classes of graphs. In [3], Chellali and Haynes compared in trees the independence number β to the four parameters $\gamma_a, \gamma_{\hat{a}}, \gamma_o, \gamma_{\hat{o}}$ by establishing some inequalities between them. They also noticed that for trees T, the independence domination number i is "incomparable" to some global alliance parameters in that sense that $i(T)$ can be smaller than $\gamma_a(T)$ or $\gamma_o(T)$, or greater than $\gamma_{\hat{a}}(T)$. Our purpose is to replace in the comparisons β by i and to refine the notion of incomparability by asking for instance if $i(G)$, even when greater than $\gamma_a(G)$, cannot be bounded by a function of $\gamma_a(G)$. Moreover, we do not limit ourselves to trees.

The principe of the study is to determine for each value of μ among $\gamma_a, \gamma_{\hat{a}}, \gamma_o, \gamma_{\hat{o}}$ and for a class $\mathcal{C}$ of graphs whether a function f such that $i(G) \leq f(\mu(G))$ or $\mu(G) \leq f(i(G))$ for every G in $\mathcal{C}$ can exist, and when the answer is positive, to determine such a function. We consider the classes of all graphs, bipartite graphs and trees. Each of the following four sections is devoted to the comparison of $i(G)$ with one of the four alliance parameters.

We give first some more precisions on the notation. The neighborhood $N(v)$ of a vertex is the set of vertices adjacent with it and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. If $A \subseteq V$, $N_A(v) = N(v) \cap A$. The subgraph induced by A in G is denoted by $G[A]$ and its size by $m(A)$. The graph $G - A$ is obtained from G by deleting the vertices of A and the edges incident with them. If F a subset of edges of G, then $G - F$ is the graph obtained by deleting all the edges of F from G. In several places we consider a graph G constructed from a graph S by adding some new vertices and edges. To lighten the writing, we often use in this case the notation $|S|$ for $n(S)$ or $|V(S)|$. The *corona* of a graph is obtained by attaching a pendant edge at each vertex of G.

2 Global defensive alliances

For the star G of order n, $i(G) = 1$, $\gamma_a(G) = \lceil \frac{n}{2} \rceil$ and $\gamma_{\hat{a}}(G) = \lceil \frac{n+1}{2} \rceil$. Therefore no general bound of the type $\gamma_a(G) \leq f(i(G))$ or $\gamma_{\hat{a}}(G) \leq g(i(G))$ can be satisfied by every graph, even if we reduce ourselves to the class of trees.

We study now the existence of a function f such that $i(G) \leq f(\gamma_a(G))$ for every general graph, bipartite graph or tree.

Definitions 1

(1) $\mathcal{F}_1$ is the family of graphs obtained from a clique $S \sim K_k$ by attaching $k = d_S(u)+1$ leaves at each vertex u of S.

(2) $\mathcal{F}_2$ is the family of bipartite graphs obtained from a balanced complete bipartite graph $S \sim K_{k,k}$ by attaching $k+1 = d_S(u)+1$ leaves at each vertex u of S.

(3) $\mathcal{F}_3$ is the family of trees obtained from a tree S by attaching a set L_u of $d_S(u)+1$ leaves at each vertex u of S.

Proposition 1 (1) If $G \in \mathcal{F}_1$ then $i(G) = \gamma_a^2(G) - \gamma_a(G) + 1$.
(2) If $G \in \mathcal{F}_2$ then $i(G) = \gamma_a^2(G)/4 + \gamma_a(G)$.
(3) If $G \in \mathcal{F}_3$ then $i(G) = 2\gamma_a(G) - 1$.

Proof: If $G \in \mathcal{F}_i$ with $1 \leq i \leq 3$, then $V(S)$ is a minimum dominating set and a defensive alliance of G. Therefore $\gamma(G) \leq \gamma_a(G) \leq |S| = \gamma(G)$ and thus $\gamma_a(G) = |S|$.

(1) If $G \in \mathcal{F}_1$, i.e., $S \sim K_k$, then $i(G) = 1 + (k-1)k = |S|^2 - |S| + 1$.

(2) If $G \in \mathcal{F}_2$, i.e., $S \sim K_{k,k}$, then $|S| = 2k$ and $i(G) = k + k(k+1) = |S|^2/4 + |S|$.

(3) Let $T \in \mathcal{F}_3$ be constructed from a tree S with bipartition classes X and Y. Every maximal independent set I of T can be written as $I = (I \cap V(S)) \cup (\cup_{u \in V(S) \setminus I} L(u))$. Therefore

$$|I| = |I \cap V(S)| + \sum_{u \in V(S) \setminus I} (d_S(u)+1) = |V(S)| + \sum_{u \in V(S) \setminus I} d_S(u).$$

In the sum $\sum_{u \in V(S) \setminus I} d_S(u)$, the edges of S between $V(S) \setminus I$ and I are counted once and the $m(S-I)$ edges joining two vertices in $V(S) \setminus I$ are counted twice. Hence

$$\sum_{u \in V(S) \setminus I} d_S(u) = m(S) + m(S-I) \geq m(S), \quad \text{and}$$

$$|I| \geq |V(S)| + m(S) = 2n(S) - 1.$$

For the particular sets $I = X \cup (\cup_{u \in Y} L(u))$ and $I = Y \cup (\cup_{u \in X} L(u))$, $m(V(S) \setminus I) = \emptyset$ and $|I| = 2n(S) - 1$. Therefore, $i(T) = 2n(S) - 1 = 2\gamma_a(T) - 1$. □

Theorem 1 (1) Every graph G satisfies $i(G) \leq \gamma_a^2(G) - \gamma_a(G) + 1$ with equality if and only if $G \in \mathcal{F}_1$.
(2) Every bipartite graph G satisfies $i(G) \leq \gamma_a^2(G)/4 + \gamma_a(G)$ with equality if and only if $G \in \mathcal{F}_2$.

(3) Every tree G satisfies $i(G) \leq 2\gamma_a(G) - 1$ with equality if and only if $G \in \mathcal{F}_3$.

Proof Let S be a $\gamma_a(G)$-set, W a maximal independent set of $G[S]$, and B a maximal independent set of $G[N_{V\setminus S}(S) \setminus N_{V\setminus S}(W)]$. Then $W \cup B$ is a maximal independent set of G and $i(G) \leq |W| + |B|$. For each $v \in S$, let $L(v) = N_{V\setminus S}(v)$. Since S is a defensive alliance, $|L(v)| \leq d_S(v)+1$ for every $v \in S$, and since the defensive alliance is dominating,

$$\begin{aligned} |B| &\leq |N_{V\setminus S}(S \setminus W)| \leq \textstyle\sum_{v\in S\setminus W} |L(v)| \leq \sum_{v\in S\setminus W}(d_S(v)+1) \\ &\leq |S| - |W| + \textstyle\sum_{v\in S\setminus W} d_S(v). \end{aligned} \tag{1}$$

Therefore

$$i(G) \leq |S| + \sum_{v\in S\setminus W} d_S(v). \tag{2}$$

(1) In every graph, $d_S(v) \leq |S| - 1$. Therefore $i(G) \leq |S| + (|S| - |W|)(|S| - 1)$ with $|W| \geq 1$. Hence

$$i(G) \leq |S|^2 - |S| + 1 = \gamma_a^2(G) - \gamma_a(G) + 1.$$

If $i(G) = |S|^2 - |S| + 1$ then $|W| = 1$ and $d_S(v) = |S| - 1$ for every $v \in S \setminus W$, i. e., S is a clique and W consists of any vertex w of S. Moreover, for any $w \in S$, equality in (1) gives $|B| = |N_{V\setminus S}(S \setminus \{w\})|$, i. e., $|N_{V\setminus S}(S \setminus \{w\})$ is independent, and $|N_{V\setminus S}(S \setminus \{w\})| = \sum_{S\setminus\{w\}} |L(v)| = \sum_{S\setminus\{w\}}(d_S(v) + 1)$, i. e., all the sets $L(v)$ for $v \in S$ are disjoint, independent and of order $d_S(v) + 1$. Therefore $G \in \mathcal{F}_1$. The converse is true by Proposition 1(1).

(2) Suppose now G bipartite. Let U be the set of isolated vertices of $G[S]$ and $X \cup Y$ a bipartition of $G[S \setminus U]$. If we take $W = X \cup U$ then we get by (2),

$$i(G) \leq |S| + \sum_{v\in Y} d_S(v) = |S| + m(S). \tag{3}$$

Since $G[S]$ is bipartite, $m(S) \leq |S|^2/4$ and thus

$$i(G) \leq |S|^2/4 + |S| = \gamma_a^2(G)/4 + \gamma_a(G).$$

If $i(G) = |S|^2/4 + |S|$, then $m(S) = |S|^2/4$, i.e., $U = \emptyset$ and $G[S]$ is a complete balanced bipartite graph. Moreover, equality in (1) implies that all the sets $L(v)$ for $v \in Y$ are disjoint and of respective orders $d_S(v) + 1$. By symmetry between X and Y, the same property holds for all $v \in X$. Hence $G \in \mathcal{F}_2$. The converse is true by Proposition 1(2).

(3) If the bipartite graph G is a tree, then $G[S]$ is a forest. By (3), $i(G) \leq |S| + m(S)$ with $m(S) \leq |S| - 1$. Therefore

$$i(G) \leq 2|S| - 1 = 2\gamma_a(S) - 1.$$

If $i(G) = 2|S| - 1$, then $m(S) = |S| - 1$, i. e., $G[S]$ is a tree, the sets $L(v)$ are all disjoint for $v \in Y$ and of respective order $d_S(v)+1$, and the same holds for all $v \in X$ by symmetry between X and Y. Therefore $G \in \mathcal{F}_3$. The converse is true by Proposition 1(3). □

3 Global strong defensive alliances

As shown by the example of stars in the previous section, we have only to look for bounds on the type $i(G) \leq g(\gamma_{\hat{a}}(G))$ valid for every graph, bipartite graph or tree. Since $\gamma_a(G) \leq \gamma_{\hat{a}}(G)$ for every graph, the increasing functions f such that $i(G) \leq f(\gamma_a(G))$ which were defined inTheorem 1 are convenient but possibly too large. We are looking for sharp bounds.

Definitions 2

(1) $\mathcal{G}_1$ is the family of graphs obtained from a clique $S \sim K_k$ by attaching $k-1 = d_S(u)$ leaves at each vertex u of S.

(2) $\mathcal{G}_2$ is the family of bipartite graphs obtained from a complete balanced bipartite graph $S \sim K_{k,k}$ by attaching $k = d_S(u)$ leaves at each vertex u of S.

(3) $\mathcal{S}$ is the family of trees S such that for every maximal independent set J of S, the number of components of the forest $S - J$ is at most $|S|/2$.
$\mathcal{G}_3$ is the family of trees obtained from a tree S of $\mathcal{S}$ by attaching a set $L(u)$ of $d_S(u)$ leaves at each vertex u of S.

Observation Every tree S in $\mathcal{S}$ is balanced since if X and Y are the two classes of the bipartition of S with $|X| \leq |Y|$, then $S - X$ has $|Y|$ components. Every tree T in $\mathcal{G}_3$ constructed from $S \in \mathcal{S}$ is balanced of order $|T| = |S| + \sum_{u \in V(S)} d_S(u) = |S| + 2m(S) = 3|S| - 2$.

Lemma 1 Let T be a tree constructed from a balanced tree S by attaching a set $L(u)$ of $d_S(u)$ leaves at each vertex u of S. Let I be a maximal independent set of T and q the number of components of the forest induced in T by $V(S) \setminus I$. Then $|I| = 2|S| - q - 1$.

Proof Every maximal independent set of T has the form $I = (V(S) \cap I) \cup (\cup_{u \in V(S) \setminus I} L(u))$. Hence $|I| = |I \cap V(S)| + \sum_{u \in V(S) \setminus I} d_S(u)$. As in the proof of Proposition 1(3), $\sum_{u \in V(S) \setminus I} d_S(u)$ $= m(S) + m(S - I)$ and thus $|I| = |I \cap V(S)| + m(S) + m(S - I)$. Since S is a tree and $S - I$ a forest with q components, $m(S) = |S| - 1$ and $m(S - I) = |V(S) \setminus I| - q$. Therefore $|I| = |I \cap V(S)| + (|S| - 1) + (|S| - |I \cap V(S)| - q) = 2|S| - q - 1$. □

Proposition 2 (1) Every graph G of $\mathcal{G}_1$ satisfies $i(G) = \gamma_{\hat{a}}^2(G) - 2\gamma_{\hat{a}}(G) + 2$.

(2) Every graph G of $\mathcal{G}_2$ satisfies $i(G) = \gamma_{\hat{a}}^2(G)/4 + \gamma_{\hat{a}}(G)/2$.

(3) Every tree G of $\mathcal{G}_3$ satisfies $i(G) = 3\gamma_{\hat{a}}(G)/2 - 1$.

Proof If G is a graph of $\mathcal{G}_i$, $1 \leq i \leq 3$, constructed from a graph S by attaching $d_S(u)$ leaves at each vertex u of S, then $V(S)$ is a global strong defensive alliance and a minimum dominating set of G. Therefore $\gamma(G) \leq \gamma_{\hat{a}}(G) \leq |S| = \gamma(G)$ and thus $\gamma_{\hat{a}}(G) = |S|$.

(1) If S is a clique K_k, then $\gamma_{\hat{a}}(G) = k$ and $i(G) = (k-1)^2 + 1 = \gamma_{\hat{a}}^2(G) - 2\gamma_{\hat{a}}(G) + 2$.

(2) If S is a complete balanced bipartite graph $K_{k,k}$, then $\gamma_{\hat{a}}(G) = 2k$ and $i(G) = k(k+1) = \gamma_{\hat{a}}^2(G)/4 + \gamma_{\hat{a}}(G)/2$.

(3) Let S be a tree of $\mathcal{S}$ of bipartition $X \cup Y$ with $|X| = |Y|$ and let $I = (V(S) \cap I) \cup (\cup_{u \in V(S)\setminus I} L(u)$ be a $i(G)$-set such that $|I \cap V(S)|$ is maximum. By Lemma 1, $|I| = 2|S| - q - 1$ where q is the number of components of the forest induced by $V(S) \setminus I$. If the independent set $I \cap V(S)$ is not maximal in S, let u be a vertex of S not dominated by $I \cap V(S)$. Then I contains the set $L(u)$ and the maximal independent set $(I \setminus L(u)) \cup \{u\}$ of G is smaller than I if $|L(u)| \geq 2$ or contradicts the choice of I if $|L(u)| = 1$. Therefore $I \cap V(S)$ is a maximal independent set J of S. Since $S \in \mathcal{S}$, $q \leq |S|/2$. Therefore $i(G) = |I| \geq 3|S|/2 - 1$. Now the set $X \cup_{y \in Y} L(u)$ is a maximal independent set of G of order $|G|/2 = 3|S|/2 - 1$. Hence $i(G) = 3|S|/2 - 1 = 3\gamma_{\hat{a}}(G)/2 - 1$. □

Theorem 2 (1) Every graph G satisfies $i(G) \leq \gamma_{\hat{a}}^2(G) - 2\gamma_{\hat{a}}(G) + 2$ with equality if and only if $G \in \mathcal{G}_1$.
(2) Every bipartite graph G without isolated vertices satisfies $i(G) \leq \gamma_{\hat{a}}^2(G)/4 + \gamma_{\hat{a}}(G)/2$ with equality if and only if $G \in \mathcal{G}_2$.
(3) Every tree G of order $n \geq 2$ satisfies $i(G) = 3\gamma_{\hat{a}}(G)/2 - 1$ with equality if and only if $G \in \mathcal{G}_3$.

Proof We follow the same idea as in the proof of Theorem 1. Let G be a graph, S a $\gamma_{\hat{a}}(G)$-set, W a maximal independent set of G and B a maximal independent set of $N_{V\setminus S}(S) \setminus N_{V\setminus S}(W)$. Then $W \cup B$ is a maximal independent set of G and $i(G) \leq |W| + |B|$. Moreover since S is a strong defensive alliance, the set $L(v) = N_{V\setminus S}(v)$ has order at most $d_S(v)$ for every vertex v in S. Therefore

$$|B| \leq |N_{V\setminus S}(S \setminus W)| \leq \sum_{v \in S\setminus W} |L(v)| \leq \sum_{S\setminus W} d_S(v) \tag{4}$$

and

$$i(G) \leq |W| + \sum_{v \in S\setminus W} d_S(v). \tag{5}$$

(1) In every graph, $d_S(v) \leq |S| - 1$. Hence by (5),

$$i(G) \leq |W| + (|S| - |W|)(|S| - 1) = |S|(|S| - 1) - |W|(|S| - 2) \text{ with } |W| \geq 1.$$

Therefore

$$i(G) \leq |S|^2 - 2|S| + 2 = \gamma_{\hat{a}}^2(G) - 2\gamma_{\hat{a}}(G) + 2.$$

If $i(G) = \gamma_{\hat{a}}^2(G) - 2\gamma_{\hat{a}}(G) + 2$, then $|W| = 1$ and $d_S(v) = |S| - 1$ for every $v \in S$, i. e., S is a clique and W consists of any unique vertex w of S. Moreover equality everywhere in (4) shows that all the sets $L(v)$ for $v \in S$ are independent and disjoint. Therefore $G \in \mathcal{G}_1$. The converse is true by Proposition 2(1).

(2) Suppose now G bipartite without isolated vertices. Since S is a strong defensive alliance, $G[S]$ has no isolated vertices. Consider the unique bipartition $X_i \cup Y_i$ of each

component S_i of $G[S]$, $1 \le i \le p$, with $|X_i| \le |Y_i|$ and let $X = \cup_{1\le i\le p} X_i$, $Y = \cup_{1\le i\le p} Y_i$. Then $|X| \le |S|/2 \le |Y|$. By taking $W = X$, we get by (5)

$$i(G) \le |X| + \sum_{v\in Y} d_S(v) \le |S|/2 + m(S). \tag{6}$$

Since $G[S]$ is bipartite, $m(S) \le |S|^2/4$. Therefore

$$i(G) \le |S|^2/4 + |S|/2 = \gamma_{\hat{a}}^2(G)/4 + \gamma_{\hat{a}}(G)/2.$$

If $i(G) = \gamma_{\hat{a}}^2(G)/4+\gamma_{\hat{a}}(G)/2$, then $|X| = |S|/2$ and $m(S) = |S|^2/4$, i. e., $G[S]$ is a complete balanced bipartite graph. Moreover by equality in (4), the sets $L(v)$ have respective order $d_S(v)$ and are all disjoint. By symmetry between X and Y, the same property holds for all $v \in X$. Therefore $G \in \mathcal{G}_2$. The converse is true by Proposition 2(2).

(3) If the bipartite graph G is a tree, then $G[S]$ is a forest and $m(S) \le |S| - 1$. By (6),

$$i(G) \le 3|S|/2 - 1 = 3\gamma_{\hat{a}}(G)/2 - 1.$$

If $i(G) = 3\gamma_{\hat{a}}(G)/2-1$, then $|X| = |S|/2$ and $m(S) = |S|-1$, i. e., $G[S]$ is a balanced tree. Moreover the sets $L(v)$ have respective orders $d_S(v)$ and are all disjoint. Let J be any maximal independent set of $G[S]$ and q the number of components of the forest induced by $S \setminus J$. The set $I = J \bigcup_{v\in S\setminus J} L(v)$ is a maximal independent set of G. By Lemma 1, $|I| = 2|S| - q - 1$. Therefore $3|S|/2-1 = i(G) \le 2|S| - q - 1$. Hence $q \le |S|/2$, $G[S] \in \mathcal{S}$ and $G \in \mathcal{G}_3$. The converse is true by Proposition 2(3). □

4 Global offensive alliances

The double star T obtained by adding an edge between the centers of two stars $K_{1,p}$ satisfes $i(T) = 1 + n/2$ and $\gamma_o(T) = 2$. Therefore no general bound of the type $i(G) \le f(\gamma_o(G))$ can exist, even if we limit ourselves to the class of trees.

We are now interested in the existence of bounds of the type $\gamma_o(G) \le f(i(G))$. The bipartite graph G obtained by deleting one edge from a complete bipartite graph $K_{p,p}$ satisfies $i(G) = 2$ and $\gamma_o(G) = n/2$. Therefore no general bound $\gamma_o(G) \le f(i(G))$ can exist, even in the class of bipartite graphs. To study the possibility of such a bound valid for all trees, we first give a result relating $\beta(G)$ and $i(G)$ in this class.

Lemma 2 For every tree T of order n, $\beta(T) + 2i(T) \ge n + 1$ and the bound is sharp.

Proof Let $T = (V, E)$ be a tree of order $n \ge 2$, I a $i(T)$-set and F the set of edges of $T[V \setminus I]$. Then $T - F$ is a forest with $q \le i(T)$ components and since T is a tree, $|F| = q - 1 \le i(T) - 1$. Let A be a set of vertices of $V \setminus I$ containing at least one extremity of each edge in F and such that $|A| \le |F|$. Each vertex of $V \setminus (A \cup I)$ has all its neighbors in $A \cup I$. Hence $V \setminus (A \cup I)$ is an independent set of order $n - (|I| + |A|) \ge$

$n - (|I| + |F|) \geq n - (2i(T) - 1)$. Therefore $\beta(T) + 2i(T) \geq n + 1$. The result is clearly true for $n = 1$.

The star $T \sim K_{1,n-1}$ satisfies $\beta(T) + 2i(T) = n + 1$. More generally, let T be the trees obtained from paths $P_{3k+1} = u_1 u_2 \cdots u_{3k+1}$ by attaching at each vertex u_{3i+1}, $0 \leq i \leq k$, a non-empty set L_i of new leaves. For these trees, $I = \{u_1, u_4, \cdots, u_{3k+1}\}$ is a $i(T)$-set and $B = (\cup_{0 \leq i \leq k} L_i) \cup \{u_2, u_5, \cdots, u_{3k-1}\}$ is a $\beta(T)$-set of order $n - |I| - |\{u_3, u_6, \cdots, u_{3k}\}| = n - |I| - k$. Hence $i(T) = k + 1$, $\beta(T) = n - 2k - 1$ and $\beta(T) + 2i(T) = n + 1$. □

Theorem 3 For every tree T, $\gamma_o(T) \leq 2i(T) - 1$ and the bound is sharp.

Proof As already observed in [3], for every independent set of a connected graph G of order $n \geq 2$, the set $V \setminus S$ is a global offensive alliance of G. Hence $\gamma_o(G) \leq n - \beta(G)$. If the graph is a tree T then, by Lemma 2, $\gamma_o(T) \leq 2i(T) - 1$ and this result remains clearly true for $n = 1$. For the trees satisfying $\beta(T) + 2i(T) = n + 1$ which are described above, $I \cup \{u_3, u_6, \cdots, u_{3k}\}$ is a $\gamma_o(G)$-set. Therefore they also satisfy $\gamma_o(T) = 2i(T) - 1$. □

Remark The inequality $\gamma_o(G) \leq n - \beta(G)$ in the proof of Theorem 3 shows that $\gamma_o(G) \leq \beta(G)$ for every graph without isolates such that $\beta(G) \geq n/2$, and in particular for bipartite graphs. This property was proved in [3] for trees.

5 Global strong offensive alliances

Since all the leaves of any graph G belong to every $\gamma_{\hat{o}}(G)$-set, every star T satisfies $\gamma_{\hat{o}}(T) = n - 1$ while $i(T) = 1$. Therefore no general bound $\gamma_{\hat{o}}(G) \leq f(i(G))$ can exist, even in the class of trees.

We are now interested in the existence of bounds of the type $i(G) \leq f(\gamma_{\hat{o}}(G))$. The bipartite graph G constructed from a cycle $C_4 = xyztx$ by adding an independent set $\{u_1, \cdots, u_p, v_1, \cdots, v_p\}$ of $2p \geq 4$ vertices and the edges $u_i x, u_i z, v_i y, v_i t$ for $1 \leq i \leq p$ satisfies $n = 2p + 4$, $i(G) = n/2$ and $\gamma_{\hat{o}}(G) = 4$. Therefore no general bound $i(G) \leq f(\gamma_{\hat{o}}(G))$ can exist, even in the class of bipartite graphs. The following theorem establishes such a bound in the class of trees.

Theorem 4 For every tree T of order $n \geq 2$, $i(T) \leq \gamma_{\hat{o}}(T) - 1$ and the bound is sharp.

Proof It is proved in [3] that every tree satisfies $\beta(T) \leq \gamma_{\hat{o}}(T)$. Hence $i(T) \leq \gamma_{\hat{o}}(T)$. We prove that the equality is impossible. If $i(T) = \gamma_{\hat{o}}(T)$ then $i(T) = \beta(T)$ and T is a well-covered tree. Therefore $\beta(T) = n/2$ and T is the corona of a tree of vertex set W. Let A be a $\gamma_{\hat{o}}(T)$-set. Then A contains the set L of leaves of T and a dominating set of W since every vertex of $V(T) \setminus A$ must have at least two neighbors in A. Therefore $|A| \geq 1 + n/2$ which contradicts $\beta(T) = \gamma_{\hat{o}}(T)$. Hence $i(T) \leq \gamma_{\hat{o}}(T) - 1$.

Equality occurs if $i(T) = \beta(T) = \gamma_{\hat{o}}(T) - 1$, or if $i(T) = \beta(T) - 1$ and $\beta(T) = \gamma_{\hat{o}}(T)$. The coronas of stars, for which $\gamma(W) = 1$, are the only trees satisfying the first equalities. The subdivided stars, obtained by subdividing once each edge of a star, are examples of graphs satisfying the second ones. □

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