Star coloring high girth planar graphs

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Abstract

A *star coloring* of a graph is a proper coloring such that no path on four vertices is 2-colored. We prove that every planar graph with girth at least 9 can be star colored using 5 colors, and that every planar graph with girth at least 14 can be star colored using 4 colors; the figure 4 is best possible. We give an example of a girth 7 planar graph that requires 5 colors to star color.

Keywords: star coloring, planar graph coloring.

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1 Introduction

Recall that a *proper coloring* of a graph is an assignment of colors to the vertices of the graph such that adjacent vertices are assigned different colors. A *star coloring* of a graph G is a proper coloring such that no path on four vertices is 2-colored. A *k-star coloring* of a graph G is a star coloring of G using at most k colors. The smallest k such that G has a k-star coloring is the *star chromatic number* of G.

In 1973 Grünbaum [5] introduced star colorings and acyclic colorings. An *acyclic coloring* is a proper coloring such that no cycle is 2-colored. Every star coloring is an acyclic coloring but star coloring a graph typically requires more colors than acyclically coloring the same graph. In general, many star coloring questions are not as well understood as their acyclic counterparts. For example, Borodin [3] proved that every planar graph can be acyclically 5-colored. This result is best possible and was conjectured by Grünbaum [5]. On the other hand, Albertson, Chappell, Kierstead, Kündgen, and Ramamurthi [1] proved that every planar graph can be star colored using 20 colors, and gave an example of a planar graph that requires 10 colors to star color; but this gap remains open.

Planar graphs of high girth are typically easier to color in the sense that fewer colors are needed. For instance Grötzsch [6] proved that every planar graph of girth at least 4 can be properly colored using 3 colors. Borodin, Kostochka, and Woodall [4] proved that every planar graph of girth at least 5 can be acyclically colored using 4 colors, and every planar graph of girth at least 7 can be acyclically colored using 3 colors; the figure 3 is best possible.

Even under high girth assumptions, the upper bounds for star colorings are not as tight as the corresponding acyclic bounds. A result by Nešetřil and Ossona de Mendez [9] implies that every planar graph of girth at least 4 can be star colored using 18 colors; whereas Kierstead, Kündgen, and Timmons [7] gave an example of a bipartite planar graph that requires 8 colors to star color. Albertson et al. [1] proved that every planar graph of girth at least 5 can be star colored using 16 colors, every planar graph of girth at least 7 can be star colored with 9 colors, and planar graphs of sufficiently large girth can be star colored using 4 colors; but no specific bound on the girth requirement was given. They also gave an example of a planar graph of arbitrarily high girth that requires 4 colors to star color.

This paper improves upon the upper bounds for planar graphs of girth at least 9. In Section 2 we introduce relevant definitions and notation. In Section 3 we prove that every planar graph of girth at least 14 can be star colored using 4 colors. In Section 4 we prove that every planar graph of girth at least 9 can be star colored using 5 colors. In Section 5 we give an example of a planar graph of girth 7 that requires 5 colors to star color. In Section 6 we collect the current best known bounds and present some open problems.

2 Preliminaries

All graphs considered are loopless graphs without multiple edges. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$ respectively. If G is a planar graph with a fixed embedding, we denote the set of faces of G by $F(G)$. The *length* of a face f, denoted $l(f)$, is the number of edges on the boundary walk of f. If v is a vertex with degree d then we say v is a *d-vertex*. We will denote the degree of v by $\deg(v)$. Degree 2 vertices will play a prominent role. If v is a d-vertex adjacent to k 2-vertices, we say v is a *d(k)-vertex*. A 1-vertex is also called a *pendant* vertex.

The *neighborhood* of a vertex v is the set of all vertices in $V(G)$ that are adjacent to v. Vertices in the neighborhood of v are the *neighbors* of v. The *second neighborhood* of a vertex v is the set of all vertices in $V(G) - \{v\}$ that are adjacent to a neighbor of v. A vertex in the second neighborhood of v is a *second neighbor* of v. A set $S \subset V(G)$ is *independent* if no two of its vertices are neighbors, and *2-independent* if no two of its vertices are neighbors or second neighbors. If $S \subset V(G)$, then $G[S]$ is the subgraph of G induced by S.

A path on n vertices will be denoted by P_n. A cycle on n vertices will be denoted by C_n. The graph obtained by adding a pendant vertex to each vertex of C_n will be denoted by C'_n. When n is not divisible by 3, it is easy to see that C'_n requires 4 colors to star color (see Example 5.3 in [1]).

Proposition 2.1 *There exist planar graphs of arbitrarily high girth that require 4 colors to star color.*

3 Girth 14 planar graphs

Albertson et al. [1] use the idea of partitioning the vertices of a graph into a forest and a 2-independent set to obtain a star coloring. We use this idea to show that planar graphs of girth at least 14 can be star colored using 4 colors, matching the construction from Proposition 2.1.

Theorem 3.1 *The vertices of a planar graph of girth at least 14 can be partitioned into two disjoint sets I and F such that $G[F]$ is a forest and I is a 2-independent set in G.*

It is easy to see that $G[F]$ can be 3-star colored (in each component of $G[F]$, fix an arbitrary root and then give each vertex color 1, 2 or 3 according as its distance from the root is 0, 1 or 2 modulo 3). Now using a fourth color for I gives a 4-star coloring of G, so we immediately have:

Corollary 3.2 *If G is a planar graph of girth at least 14 then G is 4-star colorable.*

Proof of Theorem 3.1.

Let G be a minimal counterexample with the smallest number of vertices and give G a fixed embedding in the plane. We may assume G is connected and has minimum degree 2 since pendant vertices may be put in F.

Claim 1: G has no 2(2)-vertex.

Suppose x is a 2(2)-vertex in G with neighbors y and z. Consider a desired partition for $G - \{x, y, z\}$. We extend the partition to G which provides the needed contradiction. If possible, put x into I, and put y and z into F. If x cannot be put into I, then a second neighbor of x must be in I. Put x, y and z into F. $G[F]$ is acyclic as any new cycle must pass through both second neighbors of x, but one of these second neighbors is in I. This extends the desired partition to G, a contradiction.

Claim 2: G has no 3(3)-vertex adjacent to two 2(1)-vertices.

Suppose x is a 3(3)-vertex adjacent to 2(1)-vertices y and z. Label the nearby vertices as indicated in Figure 3.1, where vertices depicted with $\circ$ may have other neighbors. Consider a desired partition for $G - \{x, x_1, y, y_1, z, z_1\}$. If possible, put x into I, and put all other vertices into F. If x cannot be put into I, then it must be that $x_2 \in I$. If $y_2 \in F$ then put y into I, and put all other vertices into F. If $y_2 \in I$ then put all vertices into F. This extends the desired partition to G, a contradiction.

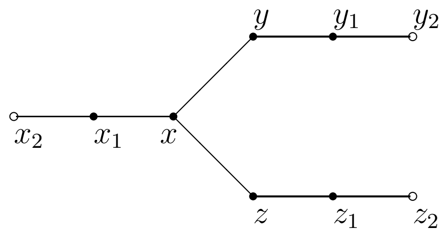

Figure 3.1: Claim 2

The proof is now finished by a simple discharging argument. Euler's Formula can be written in the form

$$(12|E(G)| - 14|V(G)|) + (2|E(G)| - 14|F(G)|) = -28,$$

which implies

$$\sum_{v \in V(G)} (6\deg(v) - 14) + \sum_{f \in F(G)} (l(f) - 14) = -28.$$

Since G has girth 14, $l(f) \geq 14$ for each $f \in F(G)$. This implies that the right sum is non-negative and so the left sum must be negative. For each vertex v in $V(G)$, assign a charge of $6\deg(v) - 14$ to v. The charge is now redistributed according to the following rules:

1. Each 2(1)-vertex receives a charge of 2 from its neighbor of degree greater than 2.

2. Each 2(0)-vertex receives a charge of 1 from each neighbor.

The net charge of $V(G)$ after the redistribution is calculated. Let $v \in V(G)$.

Case 1: v is a 2-vertex

By Claim 1, v is not a 2(2)-vertex. If v is a 2(1)-vertex, then by Rule 1, v receives charge 2. Since v does not send out any charge, the charge of v after redistribution is $6 \cdot 2 - 14 + 2 = 0$.

If v is a 2(0)-vertex, then by Rule 2, v receives charge 1 from each neighbor. Since v does not send out any charge, the charge of v after redistribution is $6 \cdot 2 - 14 + 1 + 1 = 0$.

Case 2: v is a 3-vertex

If v is a 3(3)-vertex, then by Claim 2, v is adjacent to at most one 2(1)-vertex. Then v at most will send out charge 2 to one 2(1)-vertex, and charge 1 to each of its of other two neighbors. The charge of v after redistribution is at least $6 \cdot 3 - 14 - 2 - 1 - 1 = 0$.

If v is a $3(k)$-vertex with $k \leq 2$, then at most v will send out charge $2k$ to k 2(1)-vertices. The charge of v after redistribution is at least $6 \cdot 3 - 14 - 2k \geq 0$ as $k \leq 2$.

Case 3: v has degree greater than 3

At most v sends out charge $2\deg(v)$. The charge of v after redistribution is at least $6\deg(v) - 14 - 2\deg(v) = 4\deg(v) - 14 \geq 0$ as $\deg(v) \geq 4$.

Cases 1–3 show that the charge of each vertex after redistribution is non-negative so that the net charge assigned to $V(G)$ is non-negative. This contradicts the fact that the net charge assigned to $V(G)$ is negative. Thus no such minimal counterexample exists. □

4 Girth 9 planar graphs

To prove that girth 9 planar graphs can be star colored with 5 colors, we use a similar approach as used for girth 14 planar graphs, except that the partition is into three sets.

Theorem 4.1 *The vertices of a planar graph of girth at least 9 can be partitioned into three disjoint sets F, I_1 and I_2 such that $G[F]$ is a forest, I_1 is a 2-independent set in $G[F \cup I_1]$, and I_2 is a 2-independent set in G.*

Corollary 4.2 *If G is a planar graph of girth at least 9 then G is 5-star colorable.*

Proof. Let G be a planar graph with girth at least 9, and consider the partition of G given by Theorem 4.1. Star color the vertices in F using colors 1, 2 and 3. Assign colors 4 and 5 to the vertices in I_1 and I_2 respectively. A potentially 2-colored P_4 cannot use color 5 since I_2 is a 2-independent set in G. Similarly it cannot use color 4 since I_1 is a 2-independent set in $G[F \cup I_1]$; and colors 1, 2 and 3 form a star coloring of $G[F]$. □

Proof of Theorem 4.1.

Let G be a minimal counterexample with the smallest number of vertices and give G a fixed embedding in the plane. We may assume G is connected and has minimum degree 2.

Claim 1: G has no 2(2)-vertex.

This follows as in Claim 1 of Theorem 3.1 by taking $I = I_2$.

Claim 2: G has no 2(1)-vertex adjacent to a 3-vertex.

Suppose x is a 2(1)-vertex adjacent to a 3-vertex y. Let z be the 2-vertex adjacent to x. Consider a desired partition for $G - \{x, z\}$. If $y \in I_1 \cup I_2$, then put x and z into F; so assume $y \in F$. If possible, put x into $I_1 \cup I_2$ and put z into F. Assume x cannot be put into $I_1 \cup I_2$. Then a second neighbor of x must be in I_1, and another second neighbor of x must be in I_2. Then x and z may be put into F as any cycle created by adding vertices to F must pass through two distinct second neighbors of x. This is impossible since x only has three distinct second neighbors, two of which are in $I_1 \cup I_2$. This extends the desired partition to G, a contradiction.

Claim 3: G has no 3(3)-vertex.

Suppose x is a 3(3)-vertex with neighbors y, z and t. Consider a desired partition of the subgraph of G obtained by removing x and its neighbors. If possible, put x into $I_1 \cup I_2$ and put all other vertices into F. Assume x cannot be put into $I_1 \cup I_2$. Then a second neighbor of x must be in I_1, and another second neighbor of x must be in I_2. Then we may put all vertices into F since any cycle created by adding vertices to F must pass through two distinct second neighbors of x.

Claim 4: G has no 3(2)-vertex adjacent to another 3(2)-vertex.

Suppose x and y are adjacent 3(2)-vertices. Label the nearby vertices as indicated in Figure 4.1. Consider a desired partition for $G - \{x, x_1, x_1', y, y_1, y_1'\}$.

Suppose $x_2 \in I_1 \cup I_2$. If possible, put y into $I_1 \cup I_2$ and put all other vertices into F. Assume y cannot be put into $I_1 \cup I_2$. Then $\{y_2, y_2'\} \subset I_1 \cup I_2$ and all vertices may be put into F.

Therefore $x_2 \notin I_1 \cup I_2$ so that $x_2 \in F$. By symmetry, x_2', y_2 and y_2' must also be in F. Then we may put x into I_1, y into I_2, and all other vertices into F.

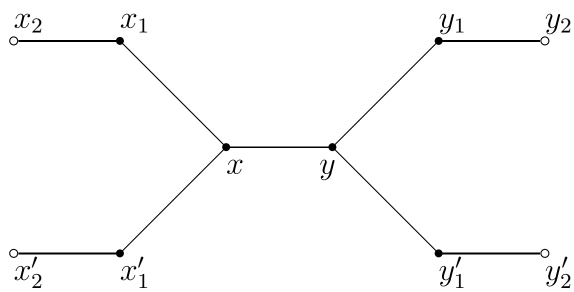

Figure 4.1: Claim 4

Claim 5: *G has no 3(1)-vertex adjacent to two 3(2)-vertices.*

Suppose x is a 3(1)-vertex adjacent to 3(2)-vertices y and z. Label the nearby vertices as indicated in Figure 4.2. Consider a desired partition for $G - \{x, x_1, z, z_1, z_1', y, y_1, y_1'\}$. If possible, put one of y, z into I_1, put the other into I_2, and put all other vertices into F. Suppose this is not possible. Then we may assume $\{y_2, z_2\} \subset I_1 \cup I_2$ or $z_2 \in I_1$, $z_2' \in I_2$.

First suppose $\{y_2, z_2\} \subset I_1 \cup I_2$. Put x into I_1 if $x_2 \in I_2$, and into I_2 otherwise; and put all other vertices into F.

Now suppose $z_2 \in I_1$, $z_2' \in I_2$. If y_2 or y_2' is in $I_1 \cup I_2$, then we are back in the previous case so assume $\{y_2, y_2'\} \subset F$. Put y into I_1, and put all other vertices into F.

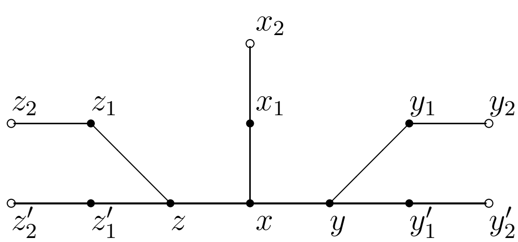

Figure 4.2: Claim 5

Claim 6: *G has no 4(4)-vertex adjacent to a 2(1)-vertex.*

Suppose x is a 4(4)-vertex adjacent to a 2(1)-vertex y. Consider a desired partition for the subgraph obtained by removing x, y, and their neighbors. If possible, put x into $I_1 \cup I_2$, and put all other vertices into F. Assume this is not possible. Then a second neighbor of x must be in I_1 and another second neighbor of x must be in I_2. We can put y into one of I_1, I_2 since only one of y's second neighbors was not removed, and we put all other vertices into F.

Definition 4.3 A *weak d(k)-vertex* is a $d(k)$-vertex all of whose degree 2 neighbors are 2(1)-vertices.

Claim 7: G has no weak 4(3)-vertex adjacent to a 3-vertex.

Suppose x is a weak 4(3)-vertex adjacent to a 3-vertex y. Label the nearby vertices as indicated Figure 4.3. Consider a desired partition for $G - \{x, x_1, x_1', x_1'', x_2, x_2', x_2''\}$. If possible put x into $I_1 \cup I_2$, and put all other vertices into F. Assume this is not possible. Then at least two of y, y_1 and y_1' must be in $I_1 \cup I_2$; so assume $y_1 \in I_1 \cup I_2$.

If $y \in I_1 \cup I_2$, then move y into F. If $y_1' \in F$, then x may be put into one of I_1, I_2, and all other vertices may be put into F. Assume $y_1' \in I_1 \cup I_2$. Then any cycle obtained by adding vertices to F must include at least one of x_1, x_1'. If possible, put x_1 and x_1' into $I_1 \cup I_2$, and put all other vertices into F. Otherwise $\{x_3, x_3'\} \subset I_1 \cup I_2$ and all vertices may be put into F.

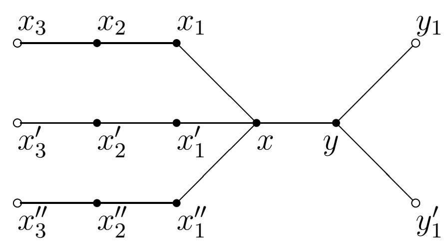

Figure 4.3: Claim 7

Claim 8: G has no 4(3)-vertex adjacent to a 3(2)-vertex.

Suppose x is a 4(3)-vertex adjacent to a 3(2)-vertex y. Label the nearby vertices as indicated in Figure 4.4. Consider a desired partition for $G - \{x, x_1, x_1', x_1'', y, y_1, y_1'\}$. To show that the partition can be extended to G, we consider two cases.

Case 1: x has at most one second neighbor in $I_1 \cup I_2$.

Since x has at most one second neighbor in $I_1 \cup I_2$, x can be put into one of I_1, I_2. If y_2 or y_2' is in $I_1 \cup I_2$, then put all remaining vertices into F. Otherwise, y_2 and y_2' are both in F. Put y into I_1 if $x \in I_2$, and I_2 otherwise; and put all remaining vertices into F.

Case 2: At least two second neighbors of x are in $I_1 \cup I_2$.

If $\{y_2, y_2'\} \subset I_1 \cup I_2$, then put all vertices into F. Otherwise, at least one of y_2, y_2' is in F so that we may put y into $I_1 \cup I_2$, and all other vertices into F.

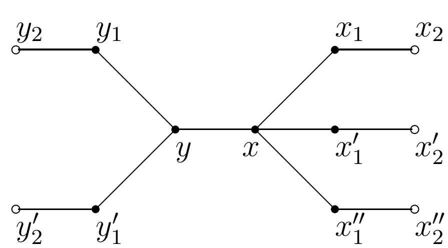

Figure 4.4: Claim 8

Claim 9: G has no 4(3)-vertex adjacent to a weak 4(3)-vertex.

Suppose x is a 4(3)-vertex adjacent to a weak 4(3)-vertex y. Consider a desired partition for the subgraph of G obtained by removing y, and its neighbors and second neighbors. If possible, put x into $I_1 \cup I_2$, put y into I_1 if $x \in I_2$, and I_2 otherwise; and put all other vertices into F. Assume x cannot be put into $I_1 \cup I_2$. Then a second neighbor of x is in I_1, and another second neighbor of x is in I_2. Put y into I_1 and put all other vertices into F.

Claim 10: G has no weak 4(2)-vertex adjacent to two weak 4(3)-vertices.

Suppose x is a weak 4(2)-vertex adjacent to two weak 4(3)-vertices y and z. Consider a desired partition for the subgraph of G obtained by removing x, y and z, and all their neighbors and second neighbors. Put x into I_2, put y and z into I_1, and put all other vertices into F. Note that I_1 is 2-independent in $G[F \cup I_1]$, although it is not 2-independent in G.

Claim 11: G has no weak 4(2)-vertex adjacent to a weak 4(3)-vertex and a 3(2)-vertex.

Suppose x is a weak 4(2)-vertex adjacent to a weak 4(3)-vertex y and a 3(2)-vertex z. Consider a desired partition for the subgraph of G obtained by removing x and y, and all their neighbors and second neighbors. If possible, put z and y into I_1, put x into I_2, and put all other vertices into F. Assume it is not possible to put z into I_1. Then a second neighbor of z must be in I_1. Put x into I_1, put y into I_2, and put all other vertices into F.

Claim 12: G has no 5(5)-vertex adjacent to four 2(1)-vertices.

Suppose x is a 5(5)-vertex adjacent to four 2(1)-vertices and a 2-vertex y. Let z be the neighbor of y where $z \neq x$. Consider a desired partition for the subgraph of G obtained by removing x, all of its neighbors and second neighbors except for z. Since only one second neighbor of x was not removed, x can be put into one of I_1, I_2, and we put all other vertices into F.

Claim 13: G has no weak 5(4)-vertex adjacent to a weak 4(3)-vertex.

Suppose x is a weak 5(4)-vertex adjacent to weak 4(3)-vertex y. Consider a partition for the subgraph of G obtained by removing x and y, and all their neighbors and second neighbors. Put x into I_1, put y into I_2, and put all other vertices into F.

The proof is now finished by a discharging argument. Euler's formula can be written in the form

$$(14|E(G)| - 18|V(G)|) + (4|E(G)| - 18|F(G)|) = -36,$$

which implies

$$\sum_{v \in V(G)} (7\deg(v) - 18) + \sum_{f \in F(G)} (2l(f) - 18) = -36.$$

Since G has girth 9, $l(f) \geq 9$ for each face $f \in F(G)$. This implies that the right sum is non-negative and so the left sum must be negative. For each vertex v in $V(G)$, assign a charge of $7\deg(v) - 18$ to v. The charge is now redistributed according to the following rules:

1. Each 2(0)-vertex receives a charge of 2 from each neighbor.
2. Each 2(1)-vertex receives a charge of 4 from the neighbor of degree greater than two.
3. Each 3(2)-vertex receives a charge of 1 from the neighbor of degree greater than two.
4. Each weak 4(3)-vertex receives a charge of 2 from the neighbor of degree greater than two.

The net charge of $V(G)$ after the redistribution is calculated. Let $v \in V(G)$.

Case 1: v is a 2-vertex

By Claim 1, v is not a 2(2)-vertex. If v is a 2(1)-vertex, then v receives charge 4 from its neighbor of degree greater than two and v does not send out any charge. The charge of v after redistribution is $7 \cdot 2 - 18 + 4 = 0$. If v is a 2(0)-vertex, then v receives charge 2 from each neighbor and v does not send out any charge. The charge of v after redistribution is $7 \cdot 2 - 18 + 2 + 2 = 0$.

Case 2: v is a 3-vertex

By Claim 2, v is not adjacent to a 2(1) vertex. By Claim 7, v is not adjacent to a weak 4(3)-vertex. Thus v will only send charge to 2(0)-vertices and 3(2)-vertices. By Claim 3, v is not a 3(3)-vertex.

If v is a 3(2)-vertex, then v sends out charge 4 to two 2(0)-vertices and receives charge 1 from its neighbor of degree greater than two. By Claim 4, v will not send out any charge to another 3(2)-vertex. The charge of v after redistribution is $7 \cdot 3 - 18 - 4 + 1 = 0$.

If v is a 3(1)-vertex, then v sends out charge 2 to a 2(0)-vertex and by Claim 5, v will send out at most charge 1 to a 3(2)-vertex. The charge of v after redistribution is at least $7 \cdot 3 - 18 - 2 - 1 = 0$.

If v is a 3(0)-vertex then at most v will send out charge 3 to three 3(2)-vertices. The charge of v after redistribution is at least $7 \cdot 3 - 18 - 3 = 0$.

Case 3: v is a 4-vertex

If v is a 4(4)-vertex then by Claim 6, v is not adjacent to a 2(1)-vertex. Therefore v sends out charge 8 to four 2(0)-vertices. The charge of v after redistribution is $7 \cdot 4 - 18 - 8 = 2$.

If v is a 4(3)-vertex then we consider three subcases.

Subcase 3.1a: v is adjacent to three 2(1)-vertices i.e. v is a weak 4(3)-vertex

By Rule 2, v sends charge 12 to three 2(1)-vertices. By Claim 7, v is not adjacent to a 3-vertex so that v does not send any charge to a 3(2)-vertex. By Claim 9, v is not adjacent to a weak 4(3)-vertex so that v does not send any charge to a weak 4(3)-vertex. By Rule 4, v receives charge 2 from its neighbor of degree greater than two. The charge of v after redistribution is $7 \cdot 4 - 18 - 12 + 2 = 0$.

Subcase 3.1b: v is adjacent to two 2(1)-vertices and a 2(0)-vertex

By Claim 8, v is not adjacent to a 3(2)-vertex so that v does not send any charge to a 3(2)-vertex. By Claim 9, v is not adjacent to a weak 4(3)-vertex so that v does not send any charge to a weak 4(3)-vertex. The charge of v after redistribution is $7\cdot4-18-4-4-2=0$

Subcase 3.1c: v is adjacent to at most one 2(1)-vertex

In this case, v will at most send out charge 4 to a 2(1)-vertex and at most charge 2 to each of its remaining neighbors. The charge of v after redistribution is at least $7 \cdot 4 - 18 - 4 - 2 - 2 - 2 = 0$.

If v is a 4(2)-vertex then we consider two subcases.

Subcase 3.2a: v is adjacent to two 2(1)-vertices

By Claim 10, v is not adjacent to two weak 4(3)-vertices. By Claim 11, v is not adjacent to a weak 4(3)-vertex and a 3(2)-vertex. Suppose v is adjacent to a weak 4(3)-vertex. Then v sends charge 8 to two 2(1)-vertices, and charge 2 to a weak 4(3)-vertex. The charge of v after redistribution is $7\cdot4-18-8-2=0$. Now suppose v is not adjacent to a weak 4(3)-vertex. Then v may be adjacent to two 3(2)-vertices. The charge of v after redistribution is at least $7 \cdot 4 - 18 - 8 - 1 - 1 = 0$.

Subcase 3.2b: v is adjacent to at most one 2(1)-vertex

In this case, v will send out charge of at most 4 to a 2(1)-vertex, and at most 2 to each of its other three neighbors. The charge of v after redistribution is at least $7 \cdot 4 - 18 - 4 - 6 = 0$.

If v is a 4(1)-vertex, then v sends out charge of at most 4 to a 2(1)-vertex and at most 2 to each of its other three neighbors. The charge of v after redistribution is at least $7 \cdot 4 - 18 - 4 - 6 = 0$.

If v is a 4(0)-vertex, then v sends out charge of at most 8 to four weak 4(3)-vertices. The charge of v after redistribution is at least $7 \cdot 4 - 18 - 8 = 2$.

Case 4: v is a 5-vertex

If v is a 5(5)-vertex, then by Claim 12, v is adjacent to at most three 2(1)-vertices. Therefore v will send out charge of at most 12 to three 2(1)-vertices, and at most 4 to two other vertices. The charge of v after redistribution is at least $7 \cdot 5 - 18 - 12 - 4 = 1$.

If v is a 5(4)-vertex then we consider two subcases.

Subcase 4.1: v is a weak 5(4)-vertex.

By Claim 13, v is not adjacent to a weak 4(3)-vertex so that v will send out charge of at most 16 to four 2(1)-vertices, and at most 1 to a 3(2)-vertex. The charge of v after redistribution is at least $7 \cdot 5 - 18 - 16 - 1 = 0$.

Subcase 4.2: v is not a weak 5(4)-vertex

By definition, v is adjacent to at most three 2(1)-vertices, and v will send out charge at most 2 to each remaining neighbor. The charge of v after redistribution is at least $7 \cdot 5 - 18 - 12 - 2 - 2 = 1$.

If v is a $5(k)$-vertex with $k \leq 3$, then v sends out charge at most $4k$ to k 2(1)-vertices, and at most $(5-k)\cdot 2$ to its other neighbors. The charge of v after redistribution is at least $7 \cdot 5 - 18 - 4k - (5-k)\cdot 2 \geq 7 \cdot 5 - 18 - 12 - 4 = 1$ as $k \leq 3$.

Case 5: v is a vertex of degree greater than 5

At most v will send out charge $4\deg(v)$. The charge of v after redistribution is at least $7\deg(v) - 18 - 4\deg(v) = 3\deg(v) - 18 \geq 0$ as $\deg(v) \geq 6$.

Cases 1–5 show that the charge of each vertex after redistribution is non-negative so that the net charge assigned to $V(G)$ is non-negative. This contradicts the fact that the net charge assigned to $V(G)$ is negative. Thus no such minimal counterexample exists. $\square$

5 A construction

In this section we give an example of a planar graph of girth 7 that requires 5 colors to star color. We begin with two definitions that play a key role in the construction.

Definition 5.1 A *k-cluster with center v* is a graph C together with a star coloring f such that*:*

1. C has vertex set $\{v, x_1, x_2, \ldots, x_k, x'_1, x'_2, \ldots, x'_k\}$ where the x'_i's need not be distinct;
2. v has k distinct neighbors $x_1, x_2, \ldots, x_k$;
3. each neighbor x_i of v is adjacent to a vertex $x'_i \neq v$ with $f(x'_i) = f(v)$.

Call the k neighbors of v the *special neighbors* of v. The edge $x_i x'_i$ is said to be a *leg* of the k-cluster.

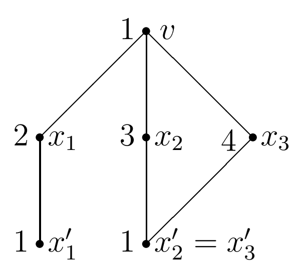

Figure 5.1: A 3-cluster with center v and legs $x_1x'_1$, $x_2x'_2$ and $x_3x'_3$

k-clusters are useful for our purposes as they can forbid colors to appear on neighbors of the center of the cluster that are not contained in the cluster. For example, if v is as in Figure 5.1, then any neighbor of v not in 3-cluster cannot be assigned colors 1, 2, 3 or 4 without creating a 2-colored P_4 or an improper coloring. Thus we make use of k-clusters by attaching them to existing graphs in a specific fashion. The center of the k-cluster is identified with a vertex in the graph, but the other vertices in the k-cluster are new vertices not contained in the vertex set of the graph. We formalize this idea in the next definition.

Definition 5.2 Let v be the center of a k-cluster C with legs $x_1x'_1, \ldots, x_kx'_k$. Let G be a graph and let v be a vertex in G with $x_1, x_2, \ldots, x_k, x'_1, x'_2, \ldots, x'_k$ not in G. *Attaching C to v in G* results in a graph with vertex set $V(G) \cup V(C)$, where $V(G) \cap V(C) = \{v\}$, and edge set $E(G) \cup E(C)$, in which $V(C)$ is colored as in C.

A 1-cluster attached to x and a 1-cluster attached to y in the graph H_1 can be seen in Figure 5.2.

We now proceed to the construction. First we construct a graph G_2 such that any 4-star coloring of G_2 contains a 2-cluster. We then construct a graph G_3 such that if a 2-cluster is attached to G_3, then a coloring of the 2-cluster cannot be extended to a 4-star coloring of G_3. Lastly we construct G using G_2 and $|V(G_2)|$ copies of G_3, where each copy of G_3 is associated with a vertex of G_2. Then in any 4-star coloring of G, the induced copy of G_2 contains a 2-cluster, say with center v, and the copy of G_3 associated with v cannot be colored.

G_2 will be constructed from copies of two smaller graphs H_1 and H_2. We first prove three lemmas regarding these smaller graphs before showing that any 4-star coloring of G_2 contains a 2-cluster.

Let $b_1, b_2, \ldots, b_{10}$ be the vertices of P_{10}, and let x and y be two isolated vertices. For $i \in \{1, 4, 7, 10\}$ add edges xb_i and yb_i, and subdivide xb_i and yb_i with a_i and c_i respectively. Call this graph H_1. Later in the construction of H_2, we will add copies of H_1 between two specific vertices. When we add a copy of H_1 between two vertices, say u and w, we are identifying u and w with x and y respectively.

Lemma 5.3 *Let f be a 4-star coloring of H_1 such that $f(a_i) = f(a_{i+3})$ and $f(c_i) = f(c_{i+3})$ for some $i \in \{1, 4, 7\}$. Then H_1 contains a 2-cluster.*

Proof. If $f(a_i) = f(c_i)$, then a_i is the center of a 2-cluster with legs xa_{i+3} and b_ic_i; so assume $f(a_i) \neq f(c_i)$. If any of b_i, b_{i+1}, b_{i+2}, b_{i+3} receives color $f(a_i)$ or $f(c_i)$, then we have either an improper coloring or a 2-cluster centered at one of a_i, c_i, a_{i+3}, c_{i+3}. However, it is not possible to star color the path induced by b_i, b_{i+1}, b_{i+2} and b_{i+3} with only two colors. □

Lemma 5.4 *Attach a 1-cluster to x and a 1-cluster to y in H_1 with legs x_1x_1' and y_1y_1' respectively. Let f be a 4-star coloring of the clusters such that $f(x) = f(y)$. Then f cannot be extended to a 4-star coloring of H_1 without creating a 2-cluster.*

Proof. Suppose f can be extended to a 4-star coloring of H_1 without creating a 2-cluster. Let $f(x) = f(y) = 1$ and $f(x_1) = 2$ where x_1 is the special neighbor of x (see Figure 5.2). Observe that for each $i \in \{1, 4, 7, 10\}$, $f(a_i) \notin \{1, 2\}$ and $f(c_i) \notin \{1, f(y_1)\}$ where y_1 is the special neighbor of y.

Suppose $f(a_i) = f(a_{i+3}) = 3$ with $i \in \{1, 4, 7\}$. If $f(c_i) = 3$, then a_i is the center of a 2-cluster with legs b_ic_i and xa_{i+3}. Similarly, $f(c_{i+3}) \neq 3$. By Lemma 5.3, $f(c_i) \neq f(c_{i+3})$ so assume $f(c_i) = 2$ and $f(c_{i+3}) = 4$. If $f(b_i) = 1$ or $f(b_{i+3}) = 1$, then x is the center of a 2-cluster. Therefore $f(b_i) = 4$ and $f(b_{i+3}) = 2$. If $f(b_{i+1}) = 2$, then b_{i+1} is the center of a 2-cluster with legs b_ic_i and $b_{i+2}b_{i+3}$. If $f(b_{i+1}) = 3$, then a_i is the center of a 2-cluster with legs b_ib_{i+1} and xa_{i+3}. Thus $f(b_{i+1}) = 1$ and similarly, $f(b_{i+2}) = 1$ and we have an improper coloring.

Hence $f(a_i) \neq f(a_{i+3})$ and so we may assume $f(a_1) = f(a_7) = 3$ and $f(a_4) = f(a_{10}) = 4$. Observe that if $f(a_i) = f(c_i)$ for some $i \in \{1, 4, 7, 10\}$ then a_i is the center of a 2-cluster.

Suppose $f(y_1) = 2$. By a similar argument as used in the previous paragraph, we must have $f(c_1) = f(c_7) = 4$ and $f(c_4) = f(c_{10}) = 3$. This forces $f(b_4) = f(b_7) = 2$. If $f(b_5) = 3$ or $f(b_5) = 4$, then c_4 or a_4, respectively, is the center of a 2-cluster; so $f(b_5) = 1$. Similarly, $f(b_6) = 1$ and we have an improper coloring. Therefore $f(y_1) \neq 2$ and so we may assume $f(y_1) = 3$. Then we must have $f(c_1) = f(c_7) = 4$ and $f(c_4) = f(c_{10}) = 2$, so that $f(b_1) = 2$ and $f(b_4) = 3$. If $f(b_2) = 3$ then b_2 is the center of a 2-cluster, and if $f(b_2) = 4$ then c_1 is the center of a 2-cluster. Thus $f(b_2) = 1$, and similarly $f(b_3) = 1$ and we have an improper coloring. □

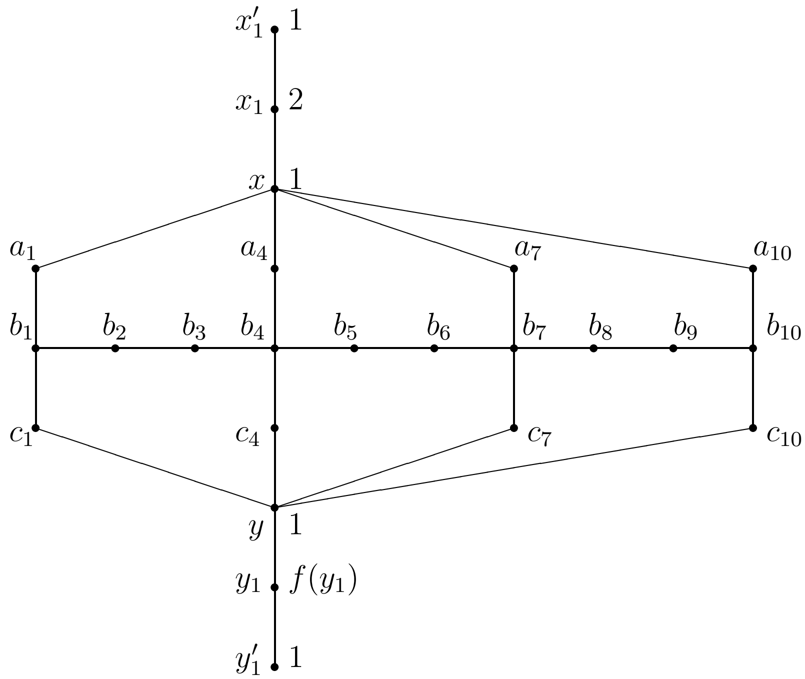

Figure 5.2: Lemma 5.4, 1-clusters attached to x and y in H_1

We now construct H_2. Let $b_1, b_2, \ldots, b_{22}$ be the vertices of P_{22}, and let x and y be two isolated vertices. For $i \equiv 1$ modulo 3, add edges xb_i and yb_i, and subdivide xb_i and yb_i with a_i and c_i respectively. For $i \equiv 0$, 2 modulo 3, add copies of H_1 between x and b_i, and between y and b_i. Call this graph H_2. In the construction of G_2 we will add copies of H_2 between two specific vertices. When we add a copy of H_2 between two vertices, say u and w, we are identifying u and w with x and y respectively.

Lemma 5.5 *Attach a 1-cluster to x and a 1-cluster to y in H_2 with legs x_1x_1' and y_1y_1' respectively. Let f be a 4-star coloring of the clusters such that $f(x_1) = f(y_1)$. Then f cannot be extended to a 4-star coloring of H_2 without creating a 2-cluster.*

Proof. Suppose f can be extended to a 4-star coloring of H_2 without creating a 2-cluster. If $f(x) = f(y)$ then since x and y have an induced copy of H_1 between them, we may apply Lemma 5.4 to this copy of H_1. Assume $f(x) \neq f(y)$, say $f(x) = 1$, $f(y) = 3$, and $f(x_1) = f(y_1) = 2$.

Suppose $f(a_i) = f(a_j) = 4$ for some $i < j$ with $\{i, j\} \subset \{4, 7, 10, 13, 16, 19\}$. If $f(b_i) = 1$ or $f(b_i) = 3$ then x or y, respectively, is the center of a 2-cluster; so $f(b_i) = 2$. Similarly, $f(b_j) = 2$. If $f(c_i) = 4$ then a_i is the center of a 2-cluster with legs b_ic_i and xa_j; so $f(c_i) = 1$, and similarly $f(c_j) = 1$. If $f(b_{i-1}) = 1$ or $f(b_{i+1}) = 1$, then c_i is the center of a 2-cluster, and if $f(b_{i-1}) = 4$ or $f(b_{i+1}) = 4$ then a_i is the center of a 2-cluster; so $f(b_{i-1}) = 3$ and $f(b_{i+1}) = 3$. Since b_{i+1} and y are centers of 1-clusters with legs b_ib_{i-1} and y_1y_1', respectively, and $f(b_{i+1}) = f(y)$, we can apply Lemma 5.4 to the induced copy of H_1 between b_{i+1} and y.

Hence we may assume that all but at most one a_i with $i \in \{4, 7, 10, 13, 16, 19\}$ receives color 3. Similarly, all but at most one c_i with $i \in \{4, 7, 10, 13, 16, 19\}$ receives color 1. But there are six vertices a_i and six vertices c_i with $i \in \{4, 7, 10, 13, 16, 19\}$. So there is a j such that $f(a_j) = f(a_{j+3}) = 3$ and $f(c_j) = f(c_{j+3}) = 1$ and we may apply Lemma 5.3 to the copy of H_1 induced by x, y, a_{j+k}, c_{j+k}, b_{j+l} where $k \in \{-3, 0, 3, 6\}$ and $l \in \{-3, -2, \ldots, 5, 6\}$. $\square$

We now construct G_2. Let $u_1, \ldots, u_7$ be the vertices on the 7-cycle in C_7' (the graph obtained by adding a pendant vertex to each vertex on C_7). For $1 \le i \le 7$, let u_{i+7} be the pendant vertex adjacent to u_i. Add an isolated vertex x. For $1 \le i \le 14$, add five pendant vertices $u_i^1, \ldots, u_i^5$ to u_i, and add five pendant vertices $x_i^1, \ldots, x_i^5$ to x (At this point $\deg(u_i) = 8$ for $1 \le i \le 7$, $\deg(u_i) = 6$ for $8 \le i \le 14$, and $\deg(x) = 70$). For each i, j, add a copy of H_2 between u_i^j and x_i^j. This completes the construction of G_2. It is easy to see that G_2 is planar.

Lemma 5.6 *Any 4-star coloring of G_2 must contain a 2-cluster.*

Proof. Let f be a 4-star coloring of G_2 and assume $f(x) = 1$. Since C_7' requires 4 colors to star color, there is an $i \in \{1, 2, \ldots, 14\}$ such that $f(u_i) = 1$. Observe $\{f(u_i^j), f(x_i^j)\} \subset \{2, 3, 4\}$ for $1 \le j \le 5$.

If there exists a $k \ne j$ such that $f(u_i^j) = f(u_i^k)$, then u_i^j is the center of a 1-cluster with leg $u_iu_i^k$; and there are at most two values of j for which there does not exist such a k. Similarly, there are at most two values of j for which x_i^j is not the center of a 1-cluster with special neighbor x. Thus there is at least one j for which both u_i^j and x_i^j are centers of 1-clusters with special neighbors u_i and x respectively. By Lemma 5.5, the copy of H_2 between the centers of 1-clusters u_i^j and x_i^j must contain a 2-cluster. $\square$

G_3 is constructed in a similar fashion as G_2. Before constructing G_3, we need to construct a smaller graph H_3, and prove a technical lemma.

Let $b_1, b_2, \ldots, b_{10}$ be the vertices of P_{10}, and let x and y be two isolated vertices. For each $i \in \{1, \ldots, 10\}$, add edges xb_i and yb_i. For each $i \in \{1, \ldots, 10\}$, subdivide xb_i twice with a_i and s_i so that s_i is adjacent to x, and subdivide yb_i twice with c_i and t_i so that t_i is adjacent to y. Call this graph H_3. Later in the construction of G_3, we will add copies of H_3 between two specific vertices. When we add a copy of H_3 between two vertices, say u and w, we are identifying u and w with x and y respectively.

Lemma 5.7 *Attach a 2-cluster to x in H_3 with legs $x_1x'_1$, $x_2x'_2$ and attach a 2-cluster to y in H_3 with legs $y_1y'_1$, $y_2y'_2$ where $y_1 = x$. Let f be a 4-star coloring of the clusters such that $f(y) \notin \{f(x), f(x_1), f(x_2)\}$. Then f cannot be extended to a 4-star coloring of H_3.*

Proof. Suppose f can be extended to a 4-star coloring of H_3. Let $f(x) = 1$, $f(x_1) = f(y_2) = 2$, $f(x_2) = 3$, and $f(y) = 4$. Since x and y are both 2-clusters, we must have $f(s_i) = 4$ and $f(t_i) = 3$ for all i. An important observation is that only colors 2 and 3 are available for each a_i, and only colors 1 and 2 are available for each c_i.

First we show $f(b_i) \neq 2$ for $i \in \{4, 5, 6, 7\}$. Suppose $f(b_i) = 2$ and $f(b_{i+1}) = 1$ for some $i \in \{4, 5, 6, 7\}$. Then $f(c_i) = 1$ and $f(c_{i+1}) = 2$; but then $c_ib_ib_{i+1}c_{i+1}$ is 2-colored. Similarly, if $f(b_i) = 2$ and $f(b_{i+1}) = 3$, then $f(a_i) = 3$, $f(a_{i+1}) = 2$ and $a_ib_ib_{i+1}a_{i+1}$ is 2-colored. Therefore, if $f(b_i) = 2$, we must have $f(b_{i-1}) = f(b_{i+1}) = 4$. This forces $f(a_{i+1}) = 3$ and $f(c_{i+1}) = 1$, $f(b_{i+2}) = 1$, and $f(c_{i+2}) = 2$. Clearly $f(b_{i+3}) \neq 1$, and if $f(b_{i+3}) = 4$ then $c_{i+1}b_{i+1}b_{i+2}b_{i+3}$ is 2-colored. If $f(b_{i+3}) = 2$ then $f(c_{i+3}) = 1$ and $c_{i+2}b_{i+2}b_{i+3}c_{i+3}$ is 2-colored; so $f(b_{i+3}) = 3$. This forces $f(a_{i+3}) = 2$ but then c_{i+3} cannot be colored. We conclude that $f(b_i) \neq 2$ whenever $i \in \{4, 5, 6, 7\}$.

The path $b_4b_5b_6b_7$ requires 3 colors to star color so that $f(b_i) = 3$ for some $i \in \{4, 5, 6, 7\}$. This forces $f(a_i) = 2$ and $f(c_i) = 1$. If $f(b_{i+1}) = 2$ then $f(a_{i+1}) = 3$ and $a_ib_ib_{i+1}a_{i+1}$ is 2-colored; so $f(b_{i+1}) = 4$. Similarly, $f(b_{i-1}) = 4$. This forces $f(a_{i+1}) = 2$, $f(c_{i+1}) = 1$, $f(b_{i+2}) = 1$, and $f(c_{i+2}) = 2$. If $f(b_{i+3}) = 2$ then $f(c_{i+3}) = 1$; but then $c_{i+2}b_{i+2}b_{i+3}c_{i+3}$ is 2-colored. So $f(b_{i+3}) = 3$ which forces $f(c_{i+3}) = 2$; but then a_{i+3} cannot be colored. $\square$

We now construct G_3. Let $u_1, \ldots, u_7$ be the vertices on the 7-cycle in C'_7 and let u_{i+7} be the pendant vertex adjacent to u_i for $1 \leq i \leq 7$. Add an isolated vertex x. For $i \in \{8, \ldots, 14\}$, add edges xu_i and subdivide xu_i with w_i. For $i \in \{8, \ldots, 14\}$, add a copy of H_3 between x and w_i. This completes the construction of G_3.

Lemma 5.8 *Attach a 2-cluster to x in G_3 with special neighbors x_1 and x_2 and let f be a 4-star coloring of the 2-cluster. Then f cannot be extended to a 4-star coloring of G_3.*

Proof. Suppose f can be extended to a 4-star coloring of G_3. Assume $f(x) = 1$, $f(x_1) = 2$ and $f(x_2) = 3$. Observe that $f(w_i) = 4$ for $i \in \{8, \ldots, 14\}$. Since C'_7 requires 4 colors to star color, there is an $i \in \{1, \ldots 7\}$ with $f(u_i) = 4$. Then w_{i+7} is the center of a 2-cluster with legs $u_{i+7}u_i$ and xw_{i+8} ($i + 8$ reduced modulo 7 if necessary). Observe $f(w_{i+7}) \notin \{1, 2, 3\}$ and that x is a special neighbor of w_{i+7}. Therefore by Lemma 5.7, the induced copy of H_3 between x and w_{i+7} cannot be 4-star colored. $\square$

We now construct G using G_2 and $|V(G_2)|$ copies of G_3. For each vertex v in G_2, attach a copy of G_3 by identifying x with v, where x is as in the description of the construction of G_3.

Theorem 5.9 *G is a planar graph of girth 7 that is not 4-star colorable.*

Proof. Suppose f is a 4-star coloring of G. By Lemma 5.6, the induced copy of G_2 in G must contain a 2-cluster, say with center v. By Lemma 5.8, the copy of G_3 attached to v cannot be 4-star colored, a contradiction. □

The above construction and argument depends heavily on the idea of a k-cluster. k-clusters are used in [7] to construct a bipartite planar graph that requires 8 colors to star color, and in [10] to construct a planar graph of girth 5 requiring 6 colors to star color.

6 Known bounds and open problems

The table below shows the current best known bounds for the star chromatic number for planar graphs of a given girth. The best known bound is given along with the corresponding reference. Bounds without reference are established in this paper.

Girth	Best known bounds	
g	lower bound	upper bound
3	10 [1]	20 [1]
4	8 [7]	18 [9]
5	6 [10]	16 [1]
6	5	8 [8]
7	5	7 [10]
8	4 [1]	6 [10]
9-13	4 [1]	5
14+	4 [1]	4

Table 6.1: Best known bounds

Problem 1: Determine the smallest girth g such that any planar graph of girth at least g can be partitioned into a forest and a 2-independent set.

Theorem 3.1 shows that all planar graphs of girth at least 14 admit such a partition, while Theorem 5.9 gives an example of girth 7 graph that does not admit such a partition. We believe that girth 14 is too high and that Theorem 3.1 can be improved.

Problem 2: Determine the smallest girth g such that any planar graph of girth at least g can be star colored with 4 colors.

Corollary 3.2 shows that planar graphs of girth at least 14 can be star colored with 4 colors, while Theorem 5.9 shows that there is a girth 7 planar graph that requires 5 colors to star color. We believe that any planar graph with girth at least 8 can be star colored with 4 colors.

Problem 3: Determine the smallest k such that any planar graph has a star coloring with k colors.

While Corollary 4.2, Corollary 3.2, and [10] improve upon the upper bounds for planar graphs of high girth, less is known about planar graphs of low girth. As mentioned in the introduction, Albertson et al. [1] show the star chromatic number for planar graphs is somewhere from 10 to 20. The gap is also wide for bipartite planar graphs. In [7], it is shown that bipartite planar graphs can be star colored using 14 colors, and an example of a bipartite planar graph requiring 8 colors to star color is given.

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