Heterochromatic matchings in edge-colored graphs

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Abstract

Let G be an (edge-)colored graph. A heterochromatic matching of G is a matching in which no two edges have the same color. For a vertex v, let $d^c(v)$ be the color degree of v. We show that if $d^c(v) \geq k$ for every vertex v of G, then G has a heterochromatic matching of size $\lceil \frac{5k-3}{12} \rceil$. For a colored bipartite graph with bipartition (X, Y), we prove that if it satisfies a Hall-like condition, then it has a heterochromatic matching of cardinality $\lceil \frac{|X|}{2} \rceil$, and we show that this bound is best possible.

1 Introduction and notation

We consider simple undirected graphs. Let $G = (V, E)$ be a graph. An *edge coloring* of G is a function $C : E \to \{0, 1, 2, \cdots\}$. If G is assigned such a coloring C, then we say that G is an *edge colored graph*, or simply *colored graph*. Denote by $C(e)$ the *color* of the edge $e \in E$. For a subgraph H of G, let $C(H) = \{C(e) : e \in E(H)\}$.

We study heterochromatic matchings, the case H is a matching. Unlike uncolored matchings for which the maximum matching problem is solvable in polynomial time (see [13]), the maximum heterochromatic matching problem is NP-complete, even for bipartite graphs (see [9]).

The heterochromatic subgraphs have received increasing attention in the last decade as mentioned below. Albert, Frieze and Reed [2] proved that the colored complete graph K_n has a heterochromatic Hamiltonian cycle if n is sufficiently large and no color appears more than $\lceil cn \rceil$ times, where $c < 1/32$. Suzuki [17] gave a sufficient and necessary condition for the existence of a heterochromatic spanning tree in a colored connected graph. For more references, see [3, 6, 7, 8, 10].

Theorem 1.1 [16] *Every $n \times n$ Latin square has a partial transversal of length at least $n - 5.53(\log n)^2$, namely every properly edge-colored complete bipartite graph $K_{n,n}$ with n colors has a heterochromatic matching with at least $n - 5.53(\log n)^2$ edges.*

For colored complete graphs, Kaneko and Suzuki gave the following result.

Theorem 1.2 [12] *For $n \geq 3$, each proper edge coloring of K_{2n} has a heterochromatic perfect matching.*

Let G be a colored graph. For a vertex set S, a *color neighborhood* of S is defined as a set $T \subseteq N(S)$ such that there are $|T|$ edges between S and T that are incident at distinct vertices of T and have distinct colors. A *maximum color neighborhood* $N^c(S)$ is a color neighborhood of S with maximum size. In particular, if $S = \{v\}$, then let $d^c(v) = |N^c(v)|$ and call it the *color degree* of v. Given a set S and a color neighborhood T of S, denote by $C(S,T)$ a set of $|T|$ distinct colors on some such set of $|T|$ edges between S and distinct vertices of T.

In [15], we obtained the following result concerning heterochromatic matchings in colored bipartite graphs meeting a color degree condition.

Theorem 1.3 [15] *For a colored bipartite graph G, if $d^c(v) \geq k \geq 3$ for each vertex $v \in V(G)$, then G has a heterochromatic matching of cardinality $\lceil \frac{2k}{3} \rceil$.*

In this paper, we study heterochromatic matchings in general graphs and obtain the following result.

Theorem 1.4 *Let G be a colored graph. If $d^c(v) \geq k$ for each vertex $v \in V(G)$, then G has a heterochromatic matching of cardinality $\lceil \frac{5k-3}{12} \rceil$.*

We propose the following strengthening of Theorem 1.4.

Conjecture 1.1 *Let G be a colored graph. Suppose that $d^c(v) \geq k \geq 4$ for each vertex v of G, then there exists a heterochromatic matching with $\lceil \frac{k}{2} \rceil$ edges.*

The complete graph K_{k+1} with a proper edge coloring satisfies $d^c(v) = k$ for each vertex v, and K_{k+1} contains no heterochromatic matching of cardinality more than $\lceil \frac{k}{2} \rceil$. Thus if the above conjecture holds, it would be best possible.

In [14], large heterochromatic matchings under some color neighborhood conditions in colored bipartite graphs were studied and the following result was obtained.

Theorem 1.5 [14] *Let G be a colored bipartite graph with bipartition (X, Y) and $|X| = |Y| = n$. If $|N^c(S)| \geq |S|$ for all $S \subseteq X$ or $S \subseteq Y$, then G has a heterochromatic matching of cardinality $\lceil \frac{3n}{8} \rceil$.*

In the case of 3-partite 3-uniform hypergraphs, Aharoni [1] verified a conjecture of Ryser. Using this result, we improve the bound in Theorem 1.5 as follows.

Theorem 1.6 *Let G be a colored bipartite graph with bipartition (X, Y). If $|N^c(S)| \geq |S|$ for all $S \subseteq X$, then G has a heterochromatic matching of cardinality $\lceil \frac{|X|}{2} \rceil$.*

Moreover, we show that the bound in Theorem 1.6 is sharp.

2 Proof of Theorem 1.4

Before the proof of Theorem 1.4, we give some notations and a proposition. For a heterochromatic matching M of G, let V_M denote $V(M)$. For a vertex $v \in V(G - V_M)$, let $b_M(v)$ denote $C(M) \cap C(\{vx : \ x \in V(G - V_M)\})$. For a subset V_1 of $V(G - V_M)$, let $b_M(V_1)$ denote $\{b_M(v) : \ v \in V_1\}$. For simplicity, let $b_M = b_M(V(G - V_M))$.

Relative a heterochromatic matching M, an *alternating* 3-path AP_M in G is a path $x'yxy'$ such that $C(xy') = C(x'y) \notin C(M)$, in which $xy \in E(M)$ and $x', y' \in V(G - V_M)$. Given two alternating 3-paths $AP_M^1 = x_1'y_1x_1y_1'$ and $AP_M^2 = x_2'y_2x_2y_2'$, AP_M^1 is *different* from AP_M^2, by the phrase we mean that $C(x_1'y_1) \neq C(x_2'y_2)$ and $x_1y_1 \neq x_2y_2$.

Easily, we can get the following proposition by Theorem 1.2.

Proposition 2.1 *For $m \geq 5$, each proper edge coloring of K_m has a heterochromatic matching of cardinality $\lceil \frac{m-1}{2} \rceil$.*

<u>*Proof of Theorem 1.4*</u>

For $k \leq 3$, Theorem 1.4 holds clearly. So we assume that $k \geq 4$. Suppose the conclusion is false, then we choose a heterochromatic matching M such that
(R_1) $|M| = t$ is maximum;
(R_2) subject to (R_1), $|b_M|$ is maximum.

Let $C(M) = \{c_1, c_2, \cdots, c_t\}$. Since for each vertex v, $d^c(v) \geq k \geq 4$ and $t \leq \lceil \frac{5k-3}{12} \rceil - 1$, it holds that $|V(G - V_M)| \geq 2$. Choose $v_x, v_y \in V(G - V_M)$. Let $N^c(v_x), N^c(v_y)$ be maximum color neighborhoods of v_x, v_y, respectively. Let $N^c(v_x) = S_1 \cup S_2$ $(S_1 \cap S_2 = \emptyset)$, where $C(v_x, S_1) \cap C(M) = \emptyset$ and $C(v_x, S_2) \subseteq C(M)$. Further let $N^c(v_y) = S_3 \cup S_4$ $(S_3 \cap S_4 = \emptyset)$, in which $C(v_y, S_3) \cap C(M) = \emptyset$ and $C(v_y, S_4) \subseteq C(M)$. Clearly $|S_2|, |S_4| \leq t$.

Claim 2.1 $S_1, S_3 \subseteq V_M$.

Proof. Otherwise, there exists a vertex $v \in V(G - V_M)$ such that $C(v_xv)$(or $C(v_yv)$) $\notin C(M)$, then $M \cup \{v_xv\}$(or $\{v_yv\}$) is a heterochromatic matching of cardinality $t + 1$, a contradiction. □

Claim 2.2 *There exists an AP_M in G.*

Proof. Since $|N^c(v_x)| = |S_1| + |S_2| \geq k$, it follows that $|S_1| \geq k - |S_2| \geq k - t$. Similarly $|S_3| \geq k - |S_4| \geq k - t$. Hence $|S_1| + |S_3| \geq 2(k-t) = 2k - 2t > 2t = |V_M|$. Then there exists an edge $xy \in M$ such that x is adjacent with v_y and y is adjacent with v_x, moreover $C(xv_y), C(v_xy) \notin C(M)$. If $C(xv_y) \neq C(v_xy)$, letting $M' = M \cup \{xv_y, v_xy\} - \{xy\}$, we see that M' is a heterochromatic matching and $|M'| = t+1$, a contradiction. Thus $C(xv_y) = C(v_xy)$, and it follows that v_xyxv_y is an AP_M. $\square$

Let l be the maximum number of the vertex-disjoint AP_Ms in G satisfying that every pair of AP_Ms are different. Clearly $1 \leq l \leq t$. For $1 \leq i \leq l$, assume that AP_M^i has edges $\{x_i'y_i, x_iy_i, x_iy_i'\}$, where $x_iy_i \in E(M)$, $x_i', y_i' \in V(G - V_M)$ and $C(x_iy_i') = C(x_i'y_i) = c_i'$.

Let V_L denote $\{x_1', x_2', \cdots, x_l'\} \cup \{y_1', y_2', \cdots, y_l'\}$ and let V_{M_l} denote $\{x_1, x_2, \cdots, x_l\} \cup \{y_1, y_2, \cdots, y_l\}$, where $\{x_1y_1, x_2y_2, \cdots, x_ly_l\} = E(M_l) \subseteq E(M)$. We abbreviate $C_l = C(M_l) = \{c_1, c_2, \cdots, c_l\}$ and $C_L = \{c_1', c_2', \cdots, c_l'\}$. Clearly $C(M) - C(M_l) = C(M - M_l)$. Let $S_0 = V - V_M - V_L$, and we have the following claim.

Claim 2.3 $|S_0| \geq 2$.

Proof. Otherwise, we have that $|S_0| \leq 1$. If $|S_0| = 1$, then assume that $S_0 = \{u\}$. Since for each vertex v of G, $d^c(v) \geq k$, then $2(t+l)+1 \geq k+1$. If $2(t+l)+1 = k+1$, then G is a colored complete graph such that $|V(G)| = k+1$ and $d^c(v) = k \geq 4$ for each vertex v of G. That is, G is an proper-edge-colored complete graph of order at least 5. Thus, by Proposition 2.1, G has a heterochromatic matching of size $\lceil \frac{k}{2} \rceil \geq \lceil \frac{5k-3}{12} \rceil > t$, a contradiction. So we conclude that $2(t+l) \geq k+1$, then $l \geq \frac{k+1}{2} - t$. Now consider the vertices x_1', y_1' and we have the following facts.

Fact 2.1 *Suppose y_1' has a neighbor $v \in V(M_l)\backslash\{x_1\}$ and $C(vy_1') \notin C(M - M_l)$, where without loss of generality, let $v = x_i$ $(2 \leq i \leq l)$. Then*
(1) $C(x_iy_1') = c_i'$.
(2) $|b_M(x_i')| \geq 1$.
(3) $C_l \cap b_M(x_i') = \emptyset$.

Proof. Suppose, to the contrary, $C(x_iy_1') \neq c_i'$, then let

$$M' = \begin{cases} M \cup \{x_iy_1', x_i'y_i\} - \{x_iy_i\} & C(x_iy_1') \notin C_l \text{ or } C(x_iy_1') = c_i; \\ M \cup \{x_iy_1', x_i'y_i, x_j'y_j\} - \{x_iy_i, x_jy_j\} & C(x_iy_1') = c_j, 1 \leq j \leq l,\ j \neq i. \end{cases}$$

Then M' is a heterochromatic matching of cardinality $t+1$, which is a contradiction. Thus it holds that $C(x_iy_1') = c_i'$.

If there is an edge $e \in E(G - V_M)$ such that $C(e) = c_i$, then $e = x_i'y_i'$. Otherwise, assume that e is not incident with x_i', then $M' = M \cup \{x_i'y_i, e\} - \{x_iy_i\}$ is a heterochromatic matching such that $|M'| > t$, a contradiction. If $|b_M(x_i')| = 0$, letting $M' = M \cup \{x_i'y_i\} - \{x_iy_i\}$, we see that M' is a heterochromatic matching such that $|M'| = t$ and $|b_{M'}| \geq |b_M| + |b_{M'}(x_i)| \geq |b_M| + 1$, a contradiction with the choice of M. Thus $|b_M(x_i')| \geq 1$.

If $C_l \cap b_M(x'_i) \neq \emptyset$, we assume that $c_j \in b_M(x'_i)$, $1 \leq j \leq l$. There exists an edge $x'_i z \in E(G - V_M)$ such that $C(x'_i z) = c_j$. Then let

$$M' = \begin{cases} M \cup \{x'_i z, x_i y'_i\} - \{x_i y_i\} & j = i, z = y'_1; \\ M \cup \{x'_i z, x_i y'_1\} - \{x_i y_i\} & j = i, z \neq y'_1; \\ M \cup \{x'_i z, x'_j y_j\} - \{x_j y_j\} & j \neq i, z = y'_j; \\ M \cup \{x'_i z, x_j y'_j\} - \{x_j y_j\} & j \neq i, z \neq y'_j. \end{cases}$$

Clearly, M' is a heterochromatic matching and $|M'| > t$, a contradiction. □

Similarly to Fact 2.1, we can prove the following fact, for simplicity, we omit the proof.

Fact 2.1′. *Suppose x'_1 has a neighbor $v \in V(M_l) \backslash \{y_1\}$ and $C(x'_1 v) \notin C(M - M_l)$, where without loss of generality, let $v = y_i$ $(2 \leq i \leq l)$. Then*
(1) $C(x'_1 y_i) = c'_i$.
(2) $|b_M(y'_i)| \geq 1$.
(3) $C_l \cap b_M(y'_i) = \emptyset$.

Let $N^c(y'_1)$ be a maximum color neighborhood of y'_1 such that $x_1 \in N^c(y'_1)$. Assume that $N^c(y'_1) = P_1 \cup P_2$ $(P_1 \cap P_2 = \emptyset)$, where $C(y'_1, P_1) \cap (C(M - M_l) \cup \{c_1\}) = \emptyset$ and $C(y'_1, P_2) \subseteq C(M - M_l) \cup \{c_1\}$. Further let $P_1^1 = P_1 \cap (V(M_l) \backslash \{y_1\})$, $|P_1^1| = p_1$ and $P_1^2 = P_1 \backslash P_1^1$. Clearly $|P_2| \leq t - l + 1$.

Let $N^c(x'_1)$ be a maximum color neighborhood of x'_1 such that $y_1 \in N^c(x'_1)$. Assume that $N^c(x'_1) = P_3 \cup P_4$ $(P_3 \cap P_4 = \emptyset)$, where $C(x'_1, P_3) \cap (C(M - M_l) \cup \{c_1\}) = \emptyset$ and $C(x'_1, P_4) \subseteq C(M - M_l) \cup \{c_1\}$. Further let $P_3^1 = P_3 \cap (V(M_l) \backslash \{x_1\})$, $|P_3^1| = p_3$ and $P_3^2 = P_3 \backslash P_3^1$. Clearly $|P_4| \leq t - l + 1$.

By symmetry and without loss of generality, we assume that $P_1^1 = \{x_{k_1}(x_{k_1} = x_1), x_{k_2} \cdots, x_{k_{p_1}}\}$ and let $P_1^{1'}$ denote $\{x'_{k_2}, \cdots x'_{k_{p_1}}\}$. Similarly we assume that $P_3^1 = \{y_{j_1}(y_{j_1} = y_1), y_{j_2}, \cdots, y_{j_{p_3}}\}$ and let $P_3^{1'}$ denote $\{y'_{j_2}, \cdots y'_{j_{p_3}}\}$. Firstly, we assume that $P_1^{1'}, P_3^{1'} \neq \emptyset$.

Fact 2.2 $|b_M(P_1^{1'})| \geq p_1 - 1$ *and* $|b_M(P_3^{1'})| \geq p_3 - 1$.

Proof. If $|b_M(P_1^{1'})| < p_1 - 1$ then $M' = M \cup \{x'_{k_2} y_{k_2}, \cdots, x'_{k_{p_1}} y_{k_{p_1}}\} - \{x_{k_2} y_{k_2}, \cdots, x_{k_{p_1}} y_{k_{p_1}}\}$ is a heterochromatic matching such that $|M'| = t$ and $|b_{M'}| \geq |b_M| + p_1 - 1 - |b_M(P_1^{1'})| > |b_M|$, a contradiction. Thus $|b_M(P_1^{1'})| \geq p_1 - 1$. Similarly, we can prove that $|b_M(P_3^{1'})| \geq p_3 - 1$. □

Without loss of generality, we assume that $b_M(P_1^{1'}) = \{c_{l+1}, c_{l+2}, \cdots, c_{l+p_2}\}$. Let $V(M_{p_2}) = \{x_{l+1}, x_{l+2}, \cdots, x_{l+p_2}\} \cup \{y_{l+1}, y_{l+2}, \cdots, y_{l+p_2}\}$. Similarly, we assume that $b_M(P_3^{1'}) = \{c_{i_1}, c_{i_2}, \cdots, c_{i_{p_4}}\}$. Let $V(M_{p_4})$ denote $\{x_{i_1}, x_{i_2}, \cdots, x_{i_{p_4}}\} \cup \{y_{i_1}, y_{i_2}, \cdots, y_{i_{p_4}}\}$.

Fact 2.3 *Suppose x'_1 and y'_1 have a common neighbor $v \in V(M_{p_4}) \cup V(M_{p_2})$, then $C(vx'_1) \in C(M - M_l) \cup \{c_1\}$ or $C(vy'_1) \in C(M - M_l) \cup \{c_1\}$.*

Proof. By contradiction. Otherwise, $C(vx'_1), C(vy'_1) \notin C(M-M_l) \cup \{c_1\}$. Without loss of generality, assume that $v = x_{i_1} \in V(M_{p_4})$ and since $c_{i_1} \in b_M(P_3^{1'})$, moreover we can assume that $c_{i_1} \in b_M(y'_{j_2})$. By the definition of the $b_M(y'_{j_2})$, we conclude that there is an edge $y'_{j_2}z \in E(G - V_M)$ such that $C(y'_{j_2}z) = c_{i_1}$. We distinguish the following cases.

Case 1. $z = x'_1$. Then let

$$M' = \begin{cases} M \cup \{x_{i_1}y'_1, y'_{j_2}z\} - \{x_{i_1}y_{i_1}\} & C(x_{i_1}y'_1) \notin C_l; \\ M \cup \{x_{i_1}y'_1, y'_{j_2}z, x'_{j_2}y_{j_2}\} - \{x_{i_1}y_{i_1}, x_{j_2}y_{j_2}\} & C(x_{i_1}y'_1) = c_{j_2}; \\ M \cup \{x_{i_1}y'_1, x'_jy_j, y'_{j_2}z\} - \{x_{i_1}y_{i_1}, x_jy_j\} & C(x_{i_1}y'_1) = c_j, 2 \le j \le l \text{ and } j \ne j_2. \end{cases}$$

Case 2. $z = y'_1$. Then let

$$M' = \begin{cases} M \cup \{x_{i_1}x'_1, y'_{j_2}z\} - \{x_{i_1}y_{i_1}\} & C(x_{i_1}x'_1) \notin C_l; \\ M \cup \{x_{i_1}x'_1, y'_{j_2}z, x'_{j_2}y_{j_2}\} - \{x_{i_1}y_{i_1}, x_{j_2}y_{j_2}\} & C(x_{i_1}x'_1) = c_{j_2}; \\ M \cup \{x_{i_1}x'_1, x'_jy_j, y'_{j_2}z\} - \{x_{i_1}y_{i_1}, x_jy_j\} & C(x_{i_1}x'_1) = c_j, 2 \le j \le l \text{ and } j \ne j_2. \end{cases}$$

Case 3. $z \notin \{x'_1, y'_1\}$. Then let

$$M' = \begin{cases} M \cup \{x_{i_1}y'_1, y'_{j_2}z\} - \{x_{i_1}y_{i_1}\} & C(x_{i_1}y'_1) \notin C_l; \\ M \cup \{x_{i_1}y'_1, y'_{j_2}z, x'_{j_2}y_{j_2}\} - \{x_{i_1}y_{i_1}, x_{j_2}y_{j_2}\} & C(x_{i_1}y'_1) = c_{j_2}; \\ M \cup \{x_{i_1}y'_1, x_jy'_j, y'_{j_2}z\} - \{x_{i_1}y_{i_1}, x_jy_j\} & C(x_{i_1}y'_1) = c_j, 2 \le j \le l, j \ne j_2, z \ne y'_j; \\ M \cup \{x_{i_1}y'_1, x'_jy_j, y'_{j_2}z\} - \{x_{i_1}y_{i_1}, x_jy_j\} & C(x_{i_1}y'_1) = c_j, 2 \le j \le l, j \ne j_2, z = y'_j. \end{cases}$$

In any case, M' is a heterochromatic matching and $|M'| > t$, which is a contradiction. □

For simplicity, let V' denote $V(M - M_l)$.

Fact 2.4 *$P_1^2 \subseteq V' \cup \{y_1\}$ and $P_3^2 \subseteq V' \cup \{x_1\}$.*

Proof. Suppose, to the contrary, there is a vertex $z \in P_1^2$ and $z \notin V(M - M_l) \cup \{y_1\}$. Since $P_1^2 = P_1 \backslash P_1^1$, then $z \notin V(M_l)$ and $C(y'_1z) \notin C(M - M_l) \cup \{c_1\}$. We distinguish the following two cases.

Case 1. $z \in V(G - V_M - V_L)$. In fact, if $V(G - V_M - V_L) \neq \emptyset$, then $z = u$. Then let

$$M' = \begin{cases} M \cup \{y'_1z\} & C(y'_1z) \notin C_l; \\ M \cup \{y'_1z, x'_jy_j\} - \{x_jy_j\} & C(y'_1z) = c_j, 2 \le j \le l. \end{cases}$$

Case 2. $z \in V_L$. Assume that $z = x'_i (1 \le i \le l)$, then let

$$M' = \begin{cases} M \cup \{y'_1z\} & C(y'_1z) \notin C_l; \\ M \cup \{y'_1z, x_jy'_j\} - \{x_jy_j\} & C(y'_1z) = c_j, 2 \le j \le l. \end{cases}$$

In both cases, M' is a heterochromatic matching and $|M'| > t$, which is a contradiction. Thus it holds that $P_1^2 \subseteq V' \cup \{y_1\}$. Similarly, we have that $P_3^2 \subseteq V' \cup \{x_1\}$. □

By Facts 2.3 and 2.4, we conclude that

$$\begin{aligned} |P_1^2 \cap V'| + |P_3^2 \cap V'| &\le 2|V'| - |M_{p_2}| - |M_{p_4}| \\ &\le 4(t - l) - p_2 - p_4 \end{aligned}$$

On the other hand, $|P_1^2 \cap V'| \geq k - |P_2| - |P_1^1| - |P_1^2 \cap \{y_1\}| \geq k - (t - l + 1) - p_1 - 1$ and $|P_3^2| \geq k - |P_4| - |P_3^1 \cap \{x_1\}| \geq k - t + l - p_3 - 2$. Since $t \leq \lceil \frac{5k-3}{12} \rceil - 1$, $l \geq \frac{k+1}{2} - t$ and, by Fact 2.2, $p_2 \geq p_1 - 1$, $p_4 \geq p_3 - 1$, it follows that

$$\begin{aligned} |P_1^2 \cap V'| + |P_3^2 \cap V'| - [4(t-l) - p_2 - p_4] \\ &\geq 2k - 2t + 2l - p_1 - p_3 - 4 - 4t + 4l + p_1 + p_3 - 2 \\ &\geq 2k - 6t + 6l - 6 \\ &\geq 5k - 12t - 3 \\ &> 0. \end{aligned}$$

Note that if $P_1^{1'} = \emptyset$ or $P_3^{1'} = \emptyset$, the above two inequalities also hold, which is a contradiction.

So we have that $|S_0| \geq 2$, which completes the proof of Claim 2.3. $\square$

Now let $w_1, w_2 \in S_0$. Choose a maximum color neighborhood $N^c(w_1)$ of w_1. Assume that $N^c(w_1) = T_1 \cup T_2$ ($T_1 \cap T_2 = \emptyset$), where $C(w_1, T_1) \cap C(M - M_l) = \emptyset$ and $C(w_1, T_2) \subseteq C(M - M_l)$. Further let $T_1^1 = T_1 \cap V(M_l)$, $|T_1^1| = t_1$ and $T_1^2 = T_1 \backslash T_1^2$. Clearly $|T_2| \leq t - l$.

Similarly, choose a maximum color neighborhood $N^c(w_2)$ of w_2. And let $N^c(w_2) = T_3 \cup T_4$ ($T_3 \cap T_4 = \emptyset$), where $C(w_2, T_3) \cap C(M - M_l) = \emptyset$ and $C(w_2, T_4) \subseteq C(M - M_l)$. Further let $T_3^1 = T_3 \cap V(M_l)$, $|T_3^1| = t_2$ and $T_3^2 = T_3 \backslash T_3^1$. Clearly $|T_4| \leq t - l$.

Claim 2.4 *Suppose $w (w \in \{w_1, w_2\})$ has a neighbor $v \in V(M_l)$ and $C(wv) \notin C(M - M_l)$, where without loss of generality, let $v = x_i$ $(1 \leq i \leq l)$. Then $C(wx_i) = c_i'$.*

Proof. Otherwise, if $C(wx_i) \notin C(M - M_l)$ and $C(wx_i) \neq c_i'$. Then let

$$M' = \begin{cases} M \cup \{wx_i, x_i'y_i\} - \{x_iy_i\} & C(wx_i) \notin C_l \text{ or } C(wx_i) = c_i; \\ M \cup \{wx_i, x_i'y_i, x_j'y_j\} - \{x_iy_i, x_jy_j\} & C(wx_i) = c_j, 1 \leq j \leq l,\ j \neq i. \end{cases}$$

Then M' is a heterochromatic matching of cardinality $t + 1$, which is a contradiction. $\square$

Claim 2.5 $T_1^2 \subseteq V(M - M_l)$ *and* $T_3^2 \subseteq V(M - M_l)$.

Proof. By symmetry, we only prove that $T_1^2 \subseteq V(M - M_l)$. Otherwise, there is an edge w_1z such that $C(w_1z) \notin C(M - M_l)$, in which $z \in T_1^2$ and $z \notin V(M - M_l)$. Since $T_1^2 = T_1 \backslash T_1^1$, $z \notin V(M_l)$. We distinguish the following two cases.

Case 1. $z \in V(G - V_M - V_L)$. Then let

$$M' = \begin{cases} M \cup \{w_1z\} & C(w_1z) \notin C_l; \\ M \cup \{w_1z, x_j'y_j\} - \{x_jy_j\} & C(w_1z) = c_j, 1 \leq j \leq l. \end{cases}$$

Case 2. $z \in V_L$. Without loss of generality, assume that $z = x_1'$, then let

$$M' = \begin{cases} M \cup \{w_1z\} & C(w_1z) \notin C_l; \\ M \cup \{w_1z, x_jy_j'\} - \{x_jy_j\} & C(w_1z) = c_j, 1 \leq j \leq l. \end{cases}$$

In both cases, M' is a heterochromatic matching and $|M'| > t$, which is a contradiction. $\square$

Since $|N^c(w_1)| = |T_1| + |T_2| \geq k$, it follows that $|T_1^2| \geq k - |T_2| - |T_1^1| \geq k - (t-l) - t_1$. Similarly it holds that $|T_3^2| \geq k - (t-l) - t_2$. Then

$$\begin{aligned} |T_1^2| + |T_3^2| - |V'| &\geq 2k - 2t + 2l - t_1 - t_2 - 2(t-l) \\ &\geq 2k - 4t + 4l - t_1 - t_2 \\ &\geq 2l + 1. \end{aligned}$$

So there exists an edge $x_0y_0 \in E(M - M_l)$, where $x_0 \in T_1^2$, $y_0 \in T_3^2$ and $C(w_1x_0) \notin C_l \cup C_L$. Note that $C(w_1x_0), C(w_2y_0) \notin C(M - M_l)$.

If $C(w_2y_0) \in C_l$, suppose $C(w_2y_0) = c_i$, $1 \leq i \leq l$. Let $M' = M \cup \{w_1x_0, w_2y_0, x_iy_i'\} - \{x_iy_i, x_0y_0\}$, then M' is a heterochromatic matching and $|M'| > t$, a contradiction.

If $C(w_2y_0) \notin C_l$ and $C(w_2y_0) \neq C(w_1x_0)$, then let $M' = M \cup \{w_1x_0, w_2y_0\} - \{x_0y_0\}$. Thus M' is a heterochromatic matching and $|M'| > t$, a contradiction.

If $C(w_2y_0) = C(w_1x_0)$, then we obtain an $AP_M = w_2y_0x_0w_1$, where $C(w_2y_0) = C(w_1x_0) \notin C(M) \cup C_L$, $x_0y_0 \in E(M - M_l)$ and $w_1, w_2 \in V(G - V_M)$. So there exists $(l+1)$ vertex-disjoint AP_Ms, in which every pair of AP_Ms are different, which is a contradiction.

The proof of Theorem 1.4 is complete. $\square$

3 Proof of Theorem 1.6

Firstly, we give some preliminaries. A *hypergraph* is a set of subsets, called *hyperedges*, of some ground set, whose elements are called *vertices*. A hypergraph H is called r-*uniform* (or an r-*graph*) if all its hyperedges are of the same size, r. An r-uniform hypergraph is called r-*partite* if its vertex set $V(H)$ can be partitioned into sets $V_1, \cdots, V_r$ in such a way that each hyperedge meets each V_i in precisely one vertex.

A *matching* in a hypergraph is a set of disjoint hyperedges. The *matching number*, $\nu(H)$, of a hypergraph H is the maximal size of a matching in H.

A *cover* of a hypergraph H is a subset of $V(H)$ meeting all hyperedges of H. The *covering number*, $\tau(H)$, of H is the minimal size of a cover of H. Obviously, $\tau \geq \nu$ for all hypergraphs. In a r-uniform hypergraph $\tau \leq r\nu$, since the union of the hyperedges of a maximal matching forms a cover.

Ryser gave a conjecture as follows.

Conjecture 3.1 *In a r-partite r-uniform hypergraph (where $r > 1$), $\tau \leq (r-1)\nu$.*

This conjecture appeared in the Ph.D thesis of Henderson, a student of Ryser. For small values of r, only the case $r = 3$ was studied for general ν. The bounds for this case were improved successively: $\tau \leq \frac{25}{9}\nu$ [11], $\tau \leq \frac{8}{3}\nu$ [18], $\tau \leq \frac{5}{2}\nu$ [19]. Finally, it was proved by Aharoni [1].

Theorem 3.1 [1] *In a tripartite 3-graph,* $\tau \leq 2\nu$.

Proof of Theorem 1.6

Construct a 3-partite 3-uniform hypergraph H as follows. Let $V_1 = X, V_2 = Y$ and $V_3 = C(G)$. A hyperedge $e = \{x, y, c\} \in E(H)$ if and only if in graph G, $x \in X, y \in Y$ and $C(xy) = c$. Clearly, a matching of a hypergraph H is a heterochromatic matching of G. Let M be a maximum heterochromatic matching. Then $|M| = \nu(H)$.

We conclude that $\tau(H) \geq |X|$. Otherwise, assume that $D = D_1 \cup D_2 \cup D_3$ is a cover of H with $|D| \leq |X| - 1$, in which $D_1 \in V_1, D_2 \in V_2$ and $D_3 \in V_3$. Now consider $F = X \backslash D_1$ in graph G, then there exists a maximum color neighborhood $N^c(F)$ such that $|N^c(F)| \geq |F| = |X| - |D_1|$. Thus in the hypergraph H, there exists a hyperedge set E_1 with $|E_1| \geq |F|$ such that
(i) for each hyperedge $e = \{x, y, c\} \in E_1$, it holds that $x \in F$;
(ii) for two hyperedges $e = \{x, y, c\}$, $e' = \{x', y', c'\}$, it holds that $y \neq y'$ and $c \neq c'$.

By (i), D_1 does not meet any hyperedge of E_1. And $D = D_1 \cup D_2 \cup D_3$ is a cover of H, so $D_2 \cup D_3$ meets each hyperedge of E_1. Thus by (ii) and since $D_2 \cap D_3 = \emptyset$, we conclude that $|D_2| + |D_3| \geq |E_1| \geq |F| = |X| - |D_1|$. Therefore, $|D_1| + |D_2| + |D_3| \geq |X|$, a contradiction. So $\tau(H) \geq |X|$ and by Theorem 3.1, $|X| = \tau(H) \leq 2\nu(H) = 2|M|$. That is $|M| \geq \frac{|X|}{2}$, which completes the proof. $\square$

Let $G = sC_4$, a graph with s components, each a C_4. Let C be a proper edge coloring of G with $2s$ colors so that each color appears exactly twice, both times in the same C_4. Any bipartition (X, Y) for G meets the condition in Theorem 1.6. Yet the largest heterochromatic matching has cardinality $s = \frac{|X|}{2}$. Thus this example shows that the bound in Theorem 1.6 is best possible.

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