Ear decompositions in combed graphs*

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Abstract

We introduce the concept of combed graphs and present an ear decomposition theorem for this class of graphs. This theorem includes the well known ear decomposition theorem for matching covered graphs proved by Lovász and Plummer. Then we use the ear decomposition theorem to show that any two edges of a 2-connected combed graph lie in a balanced circuit of an equivalent combed graph. This result generalises the theorem that any two edges in a matching covered graph with at least four vertices belong to an alternating circuit.

1 Introduction

Let G be a graph and T a subset of EG. We view circuits as sets of edges. A circuit C in G is T-*conservative* if at most half its edges are in T. We say that T is *conservative* if every circuit in G is T-conservative. In this case we also say that G is *conservative* with respect to T.

Now let G be a bipartite graph with bipartition (A, B). We have written (A, B), rather than $\{A, B\}$, to emphasise that we are imposing an ordering on the members of the bipartition. Let T be a subset of EG. A circuit C is T-*balanced* (or balanced with respect to T) if each vertex of C in B is incident with a unique edge of $T \cap C$. We say that T is *balanced* if every edge of G lies in a T-balanced circuit. In this case, we refer to G as a T-balanced graph.

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We describe T as a *comb* if it is balanced and conservative. We also refer to G as *combed* by T. Combs generalise the "basic combs" of Padayachee [5], where each vertex of B is incident with a unique edge of T. Figure 1 gives an example of a comb that is not basic.

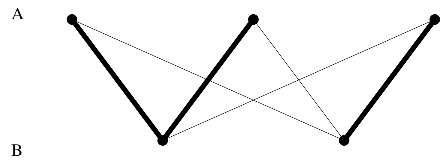

Figure 1: A comb that is not basic. The thick edges are those in the comb.

Note that the properties of being balanced and being conservative are independent: Figure 2 gives examples of a conservative set that is not balanced and of a balanced set that is not conservative.

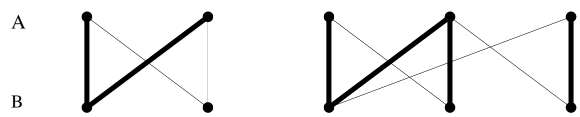

Figure 2: The set of thick edges on the left is conservative but not balanced. The set on the right is balanced but not conservative.

Let G be a T-balanced graph. A connected subgraph H of G is *T-conformal* (or simply *conformal* if T is understood) if $T \cap EH$ is balanced in H.

Suppose that H is a T-conformal subgraph of G. Let C be a T-balanced circuit in G which includes $EG \backslash EH$. Then an *ear* of H (also called an *$C\overline{H}$-arc*) is a subpath of $EG \backslash EH$, of maximal length, whose internal vertices are in $VG \backslash VH$. If there are n such arcs, then we say that G is obtained from H by an *n-ear adjunction*. An *ear decomposition* of G is a sequence $G_1, G_2, \ldots, G_t$ of T-conformal subgraphs of G such that G_1 is induced by a circuit, $G_t = G$ and, for each $i > 0$, G_i is obtained from G_{i-1} by an n-ear adjunction with $n = 1$ or $n = 2$. Our first goal is to show that such an ear decomposition exists in a 2-connected balanced graph. We then use this result to prove that any two edges of a 2-connected comb lie in a common balanced circuit.

2 Ear decompositions in combed graphs

The aim of this section is to prove that any 2-connected balanced graph may be constructed from a circuit by a process of adjoining ears no more than two at a time, as defined in the introduction. A path P is said to be *T-balanced* if each internal vertex of P in B is incident with a unique edge of $T \cap P$.

Lemma 2.1 *Let G be a connected bipartite T-balanced graph with bipartition (A, B). Let v and w be two vertices of G. Then v and w are joined by a T-balanced path P. Moreover,* *if $v \in B$ then P may be chosen so that its edge incident on v is in T or so that its edge incident on v is in $\overline{T}$.*

Proof: We may assume without loss of generality that if $v \in B$ then the edge of P incident on v is to be in T. Let S be the set consisting of v and the vertices joined to v by a balanced path that contains an edge of T incident on v if $v \in B$. It suffices to show that $S = VG$. Accordingly we assume that $S \subset VG$ and look for a contradiction.

Since G is connected and $\emptyset \subset S \subset VG$, there is an edge $f \in \partial S$, and f must belong to a balanced circuit C. Clearly, C has a vertex in S and a vertex in $VG \backslash S$ (for instance, the ends of f). Thus there are T-balanced paths joining v to vertices of C in S that contain an edge of T incident on v if $v \in B$. Let P be a shortest such path (take $P = \emptyset$ if $v \in VC$). Then $C \cup P$ includes a T-balanced path joining v to a vertex of $VG\backslash S$ that contains an edge of T incident on v if $v \in B$. We now have a contradiction to the definition of S. $\square$

A circuit C is said to be *quasibalanced* if it passes through a unique vertex $v \in B$ such that C contains two edges of T incident on v or two edges of $\overline{T}$ incident on v. We refer to v as the *special* vertex of C.

Lemma 2.2 *Let G be a bipartite T-balanced graph with bipartition (A, B). Let H be a connected T-balanced subgraph of G. Suppose that G has a quasibalanced circuit C such that the special vertex v is in VH and there are at least two $C\overline{H}$-arcs. Then $EH \cup C$ includes a balanced circuit D such that $D\backslash EH \neq \emptyset$.*

Proof: Let P_1 and P_2 be $C\overline{H}$-arcs. For each $i \in \{1, 2\}$ let P_i join vertices u_i and v_i, and assume these vertices and v appear on C in the cyclic order u_1, v_1, v, u_2, v_2. (Possibly $v \in \{v_1, u_2\}$.) Let $Q_i = C\backslash P_i$ for both i.

By Lemma 2.1, there is a T-balanced path R in H, of minimal length, joining v to a vertex $w \in VQ_1[u_1, v_2]$. We may assume R to have been chosen so that its edge incident on v is in T if and only if the edges of C incident on v are in $\overline{T}$. We use induction on the number n of $R\overline{C}$-arcs.

Suppose first that $n = 1$. If $w \in A$ then either $C_1 = R \cup Q_1[v, w]$ or $R \cup Q_2[v, w]$ is the required balanced circuit. If $w \in B$ then let e and f be the edges of R and C, respectively, incident on w such that exactly one of them is in T. Without loss of generality let $f \in C_1$. Then C_1 is the required balanced circuit.

We may now suppose that $n > 1$. Let u be the vertex of $VC \cap (VR\backslash\{v\})$ that minimises $|R[v, u]|$. Clearly, $u \in VQ_1[v_1, u_2]$. We may assume without loss of generality that $u \in VQ_1[v, u_2]$. If $u \in A$ then

$$C_2 = Q_2[u_1, v] \cup R[v, u] \cup Q_1[u, u_1]$$

is the required balanced circuit. Suppose $u \in B$. Let a and b be the edges of R and C, respectively, incident on u such that just one of them is in T. If $b \in Q_1[u, u_1]$, then once again C_2 is the required circuit. Suppose therefore that $b \in Q_2[u, v]$. Then C_2 is quasibalanced with special vertex u. Let $R' = R[u, w]$. Then the number of $R'\overline{C_2}$-arcs is less than n. Accordingly we may apply the inductive hypothesis to deduce the existence of the required balanced circuit. $\square$

Theorem 2.3 *Let G be a connected bipartite T-balanced graph with bipartition (A, B). Let H be a proper T-conformal subgraph of G. Then G has a balanced circuit C such that $VC \cap VH \neq \emptyset$ and $C \backslash EH \neq \emptyset$ but there are no more than two $C\overline{H}$-arcs.*

Proof: Since H is a proper subgraph of the connected graph G, there is an edge of $EG \backslash EH$ incident on a vertex of H. This edge must belong to a balanced circuit C in G. If $|VC \cap VH| = 1$ then there are no $C\overline{H}$-arcs. We therefore assume that $|VC \cap VH| > 1$, in which case there is at least one $C\overline{H}$-arc.

We now assume that C is chosen as a balanced circuit which has a $C\overline{H}$-arc but as few $C\overline{H}$-arcs as possible subject to this requirement. If C has no more than two $C\overline{H}$-arcs, then the theorem holds, and so we suppose that C has at least three.

Let P_1, P_2, P_3 be $C\overline{H}$-arcs, and let P_i join vertices u_i and v_i for each i. We may assume that these vertices occur on C in the cyclic order $u_1, v_1, u_2, v_2, u_3, v_3$. For each i we let $P_i' = C \backslash P_i$.

By Lemma 2.1 there is a balanced path Q_0 in H joining vertices in distinct components of $G[C \backslash (P_1 \cup P_2 \cup P_3)]$. Without loss of generality we can therefore assume the existence of a subpath Q of Q_0 joining a vertex $q_1 \in VP_3'[v_1, u_2]$ to a vertex $q_2 \in VP_1'[v_2, u_3]$ such that $Q \cap C = \emptyset$ and $VQ \cap VC = \{q_1, q_2\}$.

We now entertain various possibilities concerning q_1 and q_2. First, if both are in A then the choice of C is contradicted by both the circuits $C_1 = P_1'[q_1, q_2] \cup Q$ and $C_2 = P_2'[q_1, q_2] \cup Q$. If just one of q_1 and q_2 is in A, then one of C_1 and C_2 contradicts the choice of C. We may therefore assume that q_1 and q_2 are in B. Let a_1 and a_2 be the edges of Q incident on q_1 and q_2 respectively. Let b_1 be the edge of C incident on q_1 with the property that exactly one of a_1 and b_1 is in T. Define b_2 similarly with respect to q_2 and a_2. We may assume without loss of generality that $b_1 \in P_1'[q_1, q_2]$ and $b_2 \in P_2'[q_1, q_2]$, as the other possibilities are disposed of by symmetry or the observation that C_1 or C_2 contradicts the choice of C. Now we may apply Lemma 2.2 to the quasibalanced circuit C_2, which has q_1 as its special vertex. We deduce that $C_2 \cup EH$ includes a balanced circuit D such that $D \backslash EH \neq \emptyset$. This circuit must include either P_1 or P_3 but not P_2 and thereby contradicts the choice of C. $\square$

Let G be a connected bipartite T-balanced graph. Theorem 2.3 shows that there is a sequence $G_1, G_2, \ldots, G_n$ of T-conformal subgraphs of G such that G_1 is induced by a circuit, $G_n = G$ and, for all $i > 0$, G_i is obtained from G_{i-1} by the adjunction of one or two not necessarily vertex disjoint ears. If G is 2-connected and combed by T, then the next theorem shows that the vertex disjoint property can also be achieved. We begin with the following lemma.

Lemma 2.4 *Let G be a bipartite graph with bipartition (A, B) and combed by T. Let H be a T-conformal subgraph of G. Suppose there is a balanced circuit C in G such that there are two $C\overline{H}$-arcs P and Q having at least one end in common. Then either $G[EH \cup P]$ or $G[EH \cup Q]$ is T-conformal.*

Proof: Let x be a common end of P and Q. Let y be the other end of P. Let p and q be the edges of P and Q, respectively, incident on x. By Lemma 2.1, there is a T-balanced path R in H joining x and y. Let r be its edge incident on x. If $\{x, y\} \subseteq A$, then $P \cup R$ is a balanced circuit, and it follows immediately that $G[EH \cup P]$ is T-conformal. If $x \in B$ and $y \in A$, then R may be chosen so that $r \in T$ if and only if $p \notin T$. Then once again $P \cup R$ is a balanced circuit and we reach the same conclusion. The argument is similar if $x \in A$ and $y \in B$.

Suppose therefore that $\{x, y\} \subseteq B$. Since $P \cup Q \subseteq C$ and C is balanced, exactly one of p and q is in T. Assume without loss of generality that $p \in T$. Choose R so that its edge incident on y is also in T. If both terminal edges of R or both terminal edges of P are in T, then we have the contradiction that $P \cup R$ is not conservative. Hence neither r nor the edge of P incident on y is in T. Consequently $P \cup R$ is balanced and the proof is complete. □

Theorem 2.5 *Let G be a 2-connected bipartite graph with bipartition (A, B) and combed by T. Let H be a proper T-conformal subgraph of G. Then G has a balanced circuit C such that $C \backslash EH$ is either a non-empty path or the union of two vertex disjoint non-empty paths.*

Proof: Suppose first that $|VC \cap VH| \leq 1$ for any balanced circuit C meeting $EG \backslash EH$. Since G is balanced, it follows that $G[EG \backslash EH]$ is also balanced. Let K be a component of this graph. Since G is 2-connected, subgraphs H and K must have at least two vertices, v and w, in common. By Lemma 2.1 there is a balanced path P in K joining v to w such that if $v \in B$ then the edge of P incident on v is in T. We may assume w chosen so that no internal vertex of P is in H. Similarly there is a balanced path Q in H joining v to w. The choice of w guarantees that $P \cup Q$ is a circuit, X. If v and w are both in A, then X is balanced, contrary to hypothesis. If $v \in B$ but $w \in A$, then we may assume Q to be chosen so that its edge incident on v is not in T. Again we have the contradiction that Q is balanced. Similarly if $v \in A$ and $w \in B$ then we may assume Q chosen so that just one of the edges of P and Q incident on w is in T, and we reach the same contradiction. Assume therefore that $v \in B$ and $w \in B$. Assume Q chosen so that its edge incident on w is in T. If both P and Q have a terminal edge in $\overline{T}$, then once again X is balanced. In the remaining case we have the contradiction that X has more than half its edges in T.

Therefore $|VC \cap VH| \geq 2$ for any balanced circuit C meeting $EG \backslash EH$. We conclude from Theorem 2.3 that there is such a circuit C having just one or two $C\overline{H}$-arcs. It now suffices to show that if there are two such arcs P and Q then either they are vertex disjoint paths or there is a balanced circuit in $G[EH \cup P \cup Q]$ that includes one but not the other. But this fact is an immediate consequence of Lemma 2.4. The proof is complete. □

Double ear adjunctions are sometimes needed even though balanced graphs are bipartite. For example, consider the graph G in Figure 3, where the solid vertices are those in B and the thick edges are those in T. If H is the subgraph spanned by the edge set $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ then a 2-ear adjunction is required to produce G.

We finish this section by showing that the well known ear decomposition theorem for matching covered graphs can be deduced from Theorem 2.5. *Matching covered graphs* are connected graphs in which every edge lies in a perfect matching. We shall assume

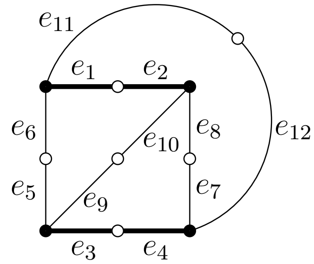

Figure 3: Double ear adjunctions are sometimes necessary.

familiarity of the reader with this theory. Lovász and Plummer proved a fundamental ear decomposition theorem for matching covered graphs which plays an important role in matching theory.

An *ear decomposition* of a matching covered graph G is a sequence $G_0, G_1, \ldots, G_t$ of matching covered subgraphs of G such that $G_0 = K_2$, $G_t = G$ and, for each $i > 0$, G_i is obtained from G_{i-1} by an n-ear adjunction with $n = 1$ or $n = 2$.

Theorem 2.6 (Lovász and Plummer [4]) *Every matching covered graph has an ear decomposition.*

Proof: Let G be a matching covered graph and T a perfect matching of G. We may assume that $|EG| > 1$ and therefore that every edge lies in an alternating circuit. Let H be the bipartite graph obtained from G by subdividing every edge e once so that the two new edges are both in T or both in $\overline{T}$ according to whether or not $e \in T$. Note that H is conservative. Let the bipartition of H be (A, B), where $B = VG$. Thus, each vertex of A is of degree 2 in H. Note that every T-alternating circuit in G corresponds naturally to a T-balanced circuit in H. Thus, H is T-balanced. We can now apply Theorem 2.5 and note that there is a natural correspondence between an ear decomposition of H and an ear decomposition of G. □

3 A generalisation of a theorem of Padayachee

Padayachee [5] also generalises to basic combs the theorem that, in a matching covered graph with at least four vertices, any two edges belong to an alternating circuit. Here we extend the theorem to combed graphs. We start by proving some useful tools.

3.1 Equivalence of combs

Let G be a bipartite graph with bipartition (A, B) and combed by T. Let C be a T-balanced circuit. We shall show that $T \oplus C$ is also a comb in G, where $\oplus$ denotes the symmetric difference. We begin by recording the following theorem of Guan.

Theorem 3.1 (Guan [2]) *Let G be a graph and T a conservative subset of EG. Let C be a circuit with exactly half its edges in T. Then $T \oplus C$ is also conservative in G.*

Note that every T-balanced circuit has exactly half its edges in T. By Theorem 3.1, $T \oplus C$ is conservative for every balanced circuit C in G. Thus, in order to show that $T \oplus C$ is a comb we need only show that $T \oplus C$ is balanced.

Theorem 3.2 *Let G be a bipartite Eulerian graph with bipartition (A, B) and T a conservative subset of EG. Suppose that for every vertex $v \in B$ exactly half the edges incident on v are in T. Then EG is a union of (edge-)disjoint T-balanced circuits. (Thus, T is a comb in G.)*

Proof: By induction on $|EG|$. As the theorem is certainly true if $EG = \emptyset$, we may assume that EG is non-empty. First we construct a balanced circuit C in G and then apply induction to $G\backslash C$.

The hypotheses of the theorem imply that exactly half the edges of G belong to T and no circuit of G has more than half its edges in T. As G is Eulerian, any circuit D constitutes a cell of some partition of EG into disjoint circuits. We conclude that exactly half the edges of D are in T, so that D cannot be quasibalanced. Thus G has no quasibalanced circuits.

Let P be a balanced path, of maximal length, with an end in A. Since EG is non-empty, the maximality of P shows that $P \neq \emptyset$. Therefore P has distinct ends x and y, where $x \in A$. Let e an edge of $EG\backslash P$ incident on y. The maximality of $|P|$ shows that e joins y to another vertex of P. Hence $P \cup \{e\}$ includes a unique circuit, and this circuit must be balanced since it cannot be quasibalanced.

We conclude that G has a balanced circuit C. Note that $G\backslash C$ is Eulerian, bipartite and conservative, with bipartition (A, B). Moreover each vertex of B in $G\backslash C$ has exactly half its incident edges in T, since C is balanced. We may therefore partition $EG\backslash C$ into disjoint balanced circuits, by the inductive hypothesis. As EG is the union of these circuits and C, we have established the existence of a partition of EG into disjoint balanced circuits. □

Lemma 3.3 *Let G be a bipartite graph with bipartition (A, B) and combed by T. Suppose that G is formed by the union of two balanced circuits C and D. Let $T_1 = T \oplus C$ and $H = C \oplus D$. Then T_1 is balanced in G and H is a T_1-conformal subgraph of G.*

Proof: Let e be an edge of G, and let us show that there is a T_1-balanced circuit in G that contains e. Certainly such a balanced circuit exists if $e \in C$, since C is balanced with respect to both T and T_1. We may thus assume that $e \in D\backslash C$. Therefore $e \in EH$. We show now that H and $T_1 \cap EH$ satisfy the hypotheses of Theorem 3.2, thereby proving the lemma.

Graph H is certainly bipartite and Eulerian, and T_1 is conservative in H. Choose $v \in B$ and let ∇v be the set of edges of H incident on v. If $EH \cap \nabla v = \emptyset$, then v is of degree 0 in H. On the other hand, if $|EH \cap \nabla v| = 4$, then v is incident in H with just 2 edges of T_1 and just 2 edges of $\overline{T_1}$. Suppose therefore that $|EH \cap \nabla v| = 2$. Then v is incident in H with either two edges of C, two edges of D or an edge $a \in C \cap D$, an edge $b \in C\backslash D$ and an edge $c \in D\backslash C$. In the first two cases, v is incident in H with just one edge of T_1 and one edge of $\overline{T_1}$. In the last case, if $a \in T$ then $b \notin T$ and $c \notin T$; hence $b \in T_1$ and $c \notin T_1$. If $a \notin T$ then $b \in T$ and $c \in T$; thus $b \notin T_1$ and $c \in T_1$. In any of these subcases v is incident in H with just one edge of T_1 and one edge of $\overline{T_1}$.

We conclude that v is incident in H with equal numbers of edges in T_1 and $\overline{T_1}$. Theorem 3.2 therefore shows that e belongs to a T_1-balanced circuit C' in H. Necessarily C' is also T_1-balanced in G, whence T_1 is balanced in G. This argument also shows that H is a T_1-conformal subgraph of G. □

Theorem 3.4 *Let G be a bipartite graph with bipartition (A, B) and T a comb in G. Let C be a T-balanced circuit in G. Then $T \oplus C$ is balanced in G.*

Proof: Let $T_1 = T \oplus C$. Choose an edge e in G and let us find a T_1-balanced circuit in G that contains e. Certainly such a balanced circuit exists if $e \in C$, since C is balanced with respect to both T and T_1. Assume therefore that $e \notin C$. Since T is balanced, e belongs to a T-balanced circuit D in G.

Let $H = G[C \cup D]$ and $T_H = T \cap EH$. Then H is formed by the union of two T_H-balanced circuits C and D. Clearly, $T_H = T \cap EH$ is conservative in H, as T is conservative in G. Thus, T_H is a comb in H. By Lemma 3.3, $T_H \oplus C$ is balanced in H. Thus, e belongs to a $(T_H \oplus C)$-balanced circuit X in H. But

$$T_H \oplus C = (T \cap EH) \oplus (C \cap EH) = (T \oplus C) \cap EH = T_1 \cap EH.$$

It follows that X is also T_1-balanced in G, whence T_1 is balanced in G, as required. □

Summarising the above results, we have the following corollary.

Corollary 3.5 *Let G be a bipartite graph with bipartition (A, B) and T a comb in G. Let C be a T-balanced circuit in G. Then $T \oplus C$ is also a comb in G.*

The set $T \oplus C$ is said to be obtained from T by a *rotation* about a T-balanced circuit C. A set T' of edges is said to be *equivalent* to T if T' can be obtained from T by a sequence of rotations. Clearly an equivalence relation is herein defined. Corollary 3.5 shows that a set equivalent to a comb is also a comb. We can use our results to determine the conditions under which combs are equivalent.

Theorem 3.6 *Let G be a bipartite graph with bipartition (A, B) and T a comb in G. Then $T' \subseteq EG$ is equivalent to T if and only if*

$$|T \cap \nabla v| \equiv |T' \cap \nabla v| \pmod 2 \tag{1}$$

for each $v \in A$ and

$$|T \cap \nabla v| = |T' \cap \nabla v| \tag{2}$$

for each $v \in B$.

Proof: Suppose first that $T' = T \oplus C$ for some balanced circuit C. In this case congruence (1) is immediate from the fact that $|C \cap \nabla v| \in \{0, 2\}$ for each vertex v. Similarly equation (2) holds since C contains just one edge of each of T and $\overline{T}$ incident on any vertex $v \in B$ through which it passes. It follows inductively that (1) and (2) hold also in the general case.

Conversely suppose that (1) holds for each vertex v of G and that (2) holds for each vertex $v \in B$. Let $H = G[T \oplus T']$. Then H is bipartite, Eulerian and conservative, and exactly half the edges of H incident on a given vertex $v \in B$ are in T. By Theorem 3.2, EH is a union of disjoint balanced circuits. Sets T and T' are obtained from each other by rotations about these circuits. □

3.2 The main theorem

Theorem 3.7 *Let e and f be any two edges of a 2-connected bipartite graph G with bipartition (A, B) and combed by T. Then there is a comb T', equivalent to T, that has a balanced circuit containing e and f.*

Proof: If EG is a circuit, then T itself is the required comb. We may therefore proceed by induction on $|EG|$. Let e and f be two edges of G.

Case 1 *G has a 2-connected T'-conformal proper subgraph H containing e and f, where T' is equivalent to T.*

Certainly T', being equivalent to the comb T, is a comb. Moreover subsets of conservative sets are conservative. If we set $T'_H = T' \cap EH$, it follows that T'_H is a comb in H since H is T'-conformal. By the inductive hypothesis, there is a comb T''_H in H, equivalent to T'_H, with respect to which there is a balanced circuit C containing e and f. Let $T'' = T''_H \cup (T' \backslash EH)$. Then T'' is equivalent to the comb T' and is therefore itself a comb equivalent to T in G. Moreover C is a T''-balanced circuit containing e and f.

Case 2 *EG is the union of two T-balanced circuits with at least one edge in common.*

Let C and D be the two T-balanced circuits such that $G = C \cup D$. If e and f are both in C or both in D the theorem is immediate. Suppose therefore that $e \in C \backslash D$ and $f \in D \backslash C$.

Let $T_1 = T \oplus C$ and $H = C \oplus D$. Then, H is a proper subgraph of G, as the common edge of C and D is not in H. Moreover, H contains e and f. By Lemma 3.3, T_1 is balanced in G and H is a T_1-conformal subgraph of G. Then H has T_1-balanced circuits C' and D' containing e and f, respectively. If C' and D' have an edge in common then $C' \cup D'$ is a 2-connected T_1-conformal proper subgraph of G containing e and f, and we finish by Case 1.

We may thus assume that C' and D' have no edge in common. In this case, let $J = C \cup D'$. Observe that $e \in C$ and $f \in D'$. Moreover, C and D' are both T_1-balanced. Also observe that $G \backslash C$ is a proper subgraph of D, since C and D have an edge in common.

That is, $G\backslash C$ is acyclic. It follows that every circuit of G contains at least one edge of C. Thus, D' contains at least one edge of C. Consequently, J is 2-connected. On the other hand, not all edges of C' are in C, for otherwise we would have $C' = C$ in contradiction to the fact that C' does not meet D' but C does. Moreover, no edge of C' is in D'. Thus, there are edges of C' which are neither in C nor in D'. It follows that J is a proper subgraph of G.

Summarising, J is a 2-connected T_1-conformal proper subgraph of G containing e and f. We now finish by case 1.

Case 3 *The previous cases do not apply.*

Let Y be a T-balanced circuit of G containing e. If Y contains f, we are done. Therefore we may assume that $Y \subset EG$. By Theorem 2.5 we may construct an ear decomposition of G starting with Y, that is, a sequence $G_1, G_2, \ldots, G_n$ of 2-connected T-conformal subgraphs of G such that $G_1 = Y$, $G_n = G$ and, for all $i > 1$, G_i is obtained from G_{i-1} by the adjunction of one or two ears. As $T_i = T \cap EG_i$ is conservative for all $i \geq 1$, T_i is a comb in G_i.

Let P be the ear in G containing f. Let j be the smallest integer such that G_j contains both ends of P. If $j = 1$ then both ends of P are of degree 2 in G_j. If $j > 1$ then at least one end of P is not a vertex of G_{j-1}, and this end is of degree 2 in G_j. Now $P \subset X$ for some balanced circuit X in G. Let $G' = G[EG_j \cup X]$. Then G' is 2-connected (since $|VX \cap VG_j| > 1$) and T-balanced. As Case 1 does not apply, $G = G'$. Since X is balanced and $|VX \cap VG_j| > 1$, G is obtained from G_j by the adjunction of (possibly more than two) ears, one of which is P.

Suppose first that P does not share an end with any other ear of X. Then P has an end, say x, of degree 3 in G. Let g be an edge of G_j incident with x. By the inductive hypothesis, G_j has a comb T'_j, equivalent to T_j, that has a balanced circuit containing e and g. It follows that G has a comb T', equivalent to T, and a T'-balanced circuit C containing e and g, where $T' = T'_j \cup (T\backslash EG_j)$. Let D be a T'-balanced circuit in G containing f. Clearly, D includes P and contains one of the edges of C incident with x. Then $G[C \cup D]$ is a 2-connected T'-conformal subgraph of G containing e and f. As Case 1 does not apply, we conclude that $G = G[C \cup D]$.

Now suppose that there is another ear Q sharing an end with P. Then Lemma 2.4 shows that either $G[EG_j \cup P]$ or $G[EG_j \cup Q]$ is T-balanced. As Case 1 does not apply, we conclude that $H = G[EG_j \cup Q]$ is T-conformal and every balanced circuit containing P also contains all the other ears of X, including Q. By the inductive hypothesis, H has a comb T'_H, equivalent to $T \cap EH$, that has a balanced circuit containing e and the edges of Q. Then G has a comb T', equivalent to T, and a T'-balanced circuit C containing e and Q, where $T' = T'_H \cup (T\backslash EH)$. Let D be a T'-balanced circuit in G containing f. Since T' is a comb, D also includes Q and thus $G[C \cup D]$ is a 2-connected T'-conformal subgraph of G containing e and f. As Case 1 does not apply, $G = G[C \cup D]$.

In any case, EG is the union of two balanced circuits C and D, where $e \in C$ and $f \in D$. We now finish by case 2. This completes the proof. $\square$

Arguing as we did in the proof of Theorem 2.6, we obtain the following corollary, which was proved by Little [3].

Corollary 3.8 *Any two distinct edges of a matching covered graph lie in an alternating circuit.*

4 Final remarks

Theorem 3.7 generalises a result of Padayachee [5]. In order to establish this fact, we need to introduce some new concepts. Let G be a graph. A function t from VG into $\{0,1\}$ is called a *join function* if there are an even number of vertices v for which $t(v) = 1$. In this case, (G,t) is called a *join pair*. A *t-join* in (G,t) is the edge set of a subgraph of G in which the parity of the degree of any vertex v is that of $t(v)$. (See [1] and [6] for earlier papers that study t-joins.) A *minimum* t-join is one of minimum cardinality. This cardinality is denoted by $\tau(G,t)$. A join pair (G,t) is a *join covered pair* if each edge of G lies in a minimum t-join. Note that if T is a t-join in a join pair (G,t) and C is a circuit with more than half its edges in T, then $T \oplus C$ is a t-join with fewer edges than T. Consequently G is conservative with respect to a minimum t-join.

Suppose now that (G,t) is a join covered pair, where G is connected and bipartite with bipartition (A,B), and that $t(b) = 1$ for each $b \in B$. Then (G,t) is said to be a *basic comb* if $\tau(G,t) = |B|$. This equation holds if and only if there is a t-join T, necessarily a minimum one, such that each vertex in B is incident with a unique edge of T. Moreover, since the pair (G,t) is join covered, every edge of G must belong to such a t-join. Fix such a minimum t-join T and, if possible, choose an edge $e \notin T$. Then e belongs to another minimum T-join T'. Furthermore, $T \oplus T'$ is a cycle and therefore a union of disjoint circuits, one of which contains e. Since each vertex of B is incident with just one edge of each of T and T', this circuit must be balanced. Thus e belongs to a balanced circuit. If G is 2-connected, then the same must be true even if $e \in T$: if v is the end of e in B, then there must be an edge $f \notin T$ incident on v, and f must belong to a balanced circuit which necessarily contains e. We conclude that if (G,t) is a basic comb and G is 2-connected, then G is balanced with respect to a minimum t-join T.

A closely related idea is the concept of a conservative graph as that terminology is used in [5]. Let G be a graph. A function w from EG into $\{-1,1\}$ is called a *weight function*. For any edge e we call $w(e)$ the *weight* of e. The *weight*, $w(T)$, of a subset T of EG is the sum of the weights of the elements of T. We say that T is *conservative* if its weight is non-negative. Thus T is conservative if and only if no more than half its edges are of negative weight. The weight function w is *conservative* if every circuit is conservative. In this case the pair (G,w) is called a *conservative graph* in [5].

The concepts of a join pair and a conservative graph are related by a theorem of Guan [2]. Let (G,t) be a join pair where G is connected, and let T be a t-join. Let w be the weight function for which the negative edges are those in T. For example, the weight of any balanced circuit in a bipartite graph is then 0. Guan's theorem asserts that T is minimum if and only if (G,w) is a conservative graph. Let us refer to w as the weight function *associated* with T. Then T is minimum if and only if its associated weight function w is conservative, and therefore if and only if every circuit is conservative. This condition is equivalent to the property that no more than half the edges of any circuit are of negative weight. In other words, T is minimum if and only if no circuit has more than half its edges in T. In our terminology, T is therefore minimum if and only if T is conservative.

Now consider a basic comb (G, t) where G is 2-connected. Let T be a minimum t-join. Then G is conservative and balanced and hence combed by T. Let e and f be two edges of G. By Theorem 3.7 there is a comb T', equivalent to T, with respect to which there is a balanced circuit C containing e and f. Let w be the weight function associated with T. Since T is minimum, w is conservative. Now consider a balanced circuit D. Then D has exactly half its edges in T. Consequently by Theorem 3.1 we find that $D \oplus T$ is conservative. As T' is equivalent to T, we may proceed inductively and conclude that T' is conservative. Hence T' is a minimum t-join and so its associated weight function w' is conservative. Moreover $w'(C) = 0$ since C is balanced. In summary, if (G, t) is a basic comb and G is 2-connected, then for any two edges there is a circuit that contains them both and is of weight 0 with respect to some conservative weight function. This is the theorem of Padayachee for which Theorem 3.7 is a generalisation.

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