Extremal problems for t-partite and t-colorable hypergraphs

Dhruv Mubayi [*] John Talbot [†]

Submitted: Aug 23, 2007; Accepted: Jan 20, 2008; Published: Feb 4, 2008
Mathematics Subject Classification: 05D05

Abstract

Fix integers $t \geq r \geq 2$ and an r-uniform hypergraph F. We prove that the maximum number of edges in a t-partite r-uniform hypergraph on n vertices that contains no copy of F is $c_{t,F}\binom{n}{r} + o(n^r)$, where $c_{t,F}$ can be determined by a finite computation.

We explicitly define a sequence $F_1, F_2, \ldots$ of r-uniform hypergraphs, and prove that the maximum number of edges in a t-chromatic r-uniform hypergraph on n vertices containing no copy of F_i is $\alpha_{t,r,i}\binom{n}{r} + o(n^r)$, where $\alpha_{t,r,i}$ can be determined by a finite computation for each $i \geq 1$. In several cases, $\alpha_{t,r,i}$ is irrational. The main tool used in the proofs is the Lagrangian of a hypergraph.

1 Introduction

An *r-uniform hypergraph* or *r-graph* is a pair $G = (V, E)$ of *vertices*, V, and *edges* $E \subseteq \binom{V}{r}$, in particular a 2-graph is a graph. We denote an edge $\{v_1, v_2, \ldots, v_r\}$ by $v_1 v_2 \cdots v_r$. Given r-graphs F and G we say that G is *F-free* if G does not contain a copy of F. The maximum number of edges in an F-free r-graph of order n is $\mathrm{ex}(n, F)$. For $r = 2$ and $F = K_s$ $(s \geq 3)$ this number was determined by Turán [T41] (earlier Mantel [M07] found $\mathrm{ex}(n, K_3)$). However in general (even for $r = 2$) the problem of determining the exact value of $\mathrm{ex}(n, F)$ is beyond current methods. The corresponding asymptotic problem is to determine the *Turán density* of F, defined by $\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,F)}{\binom{n}{r}}$ (this always exists by a simple averaging argument due to Katona et al. [KNS64]). For 2-graphs the Turán

[*]Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, IL 60607, and Department of Mathematical Sciences, Carnegie-Mellon University, Pittsburgh, PA 15213. Email: mubayi@math.uic.edu. Research supported in part by NSF grants DMS-0400812 and 0653946 and an Alfred P. Sloan Research Fellowship.

[†]Department of Mathematics, UCL, London, WC1E 6BT, UK. Email: talbot@math.ucl.ac.uk. This author is a Royal Society University Research Fellow.

density is determined by the chromatic number of the forbidden subgraph F. The explicit relationship is given by the following fundamental result.

Theorem 1 (Erdős–Stone–Simonovits [ES46], [ES66]). *If F is a 2-graph then* $\pi(F) = 1 - \frac{1}{\chi(F)-1}$.

When $r \geq 3$, determining the Turán density is difficult, and there are only a few exact results. Here we consider some closely related hypergraph extremal problems. Call a hypergraph H t-partite if its vertex set can be partitioned into t classes, such that every edge has at most one vertex in each class. Call H t-colorable, if its vertex set can be partitioned into t classes so that no edge is entirely contained within a class.

Definition 2. *Fix $t, r \geq 2$ and an r-graph F. Let $ex^*_t(n, F)$ ($ex_t(n, F)$) denote the maximum number of edges in a t-partite (t-colorable) r-graph on n vertices that contains no copy of F. The t-partite Turán density of F is $\pi^*_t(F) = \lim_{n\to\infty} ex^*_t(n, F)/\binom{n}{r}$ and the t-chromatic Turán density of F is $\pi_t(F) = \lim_{n\to\infty} ex_t(n, F)/\binom{n}{r}$.*

Note that it is easy to show that these limits exist. In this paper, we determine $\pi^*_t(F)$ for all r-graphs F and determine $\pi_t(F)$ for an infinite family of r-graphs (previously no nontrivial value of $\pi_t(F)$ was known). In many cases our examples yield irrational values of $\pi_t(F)$. For the usual Turán density, $\pi(F)$ has not been proved to be irrational for any F, although there are several conjectures stating irrational values.

In order to describe our results, we need the concept of G-colorings which we introduce now. If F and G are hypergraphs (not necessarily uniform) then F is *G-colorable* if there exists $c : V(F) \to V(G)$ such that $c(e) \in E(G)$ whenever $e \in E(F)$. In other words, F is G-colorable if there is a homomorphism from F to G.

Let $K^{(r)}_t$ denote the complete r-graph of order t. Then an r-graph F is *t-partite* if F is $K^{(r)}_t$-colorable, and F is *t-colorable* if it is $H^{(r)}_t$-colorable where $H^{(r)}_t$ is the (in general non-uniform) hypergraph consisting of all subsets $A \subseteq \{1, 2, \dots, t\}$ satisfying $2 \leq |A| \leq r$). The *chromatic number* of F is $\chi(F) = \min\{t \geq 1 : F \text{ is } t\text{-colorable}\}$. Note that while a 2-graph is t-colorable iff it is t-partite this is no longer true for $r \geq 3$, for example $K^{(3)}_4$ is 2-colorable but not 2-partite or 3-partite.

Let $\mathcal{G}^{(r)}_t$ denote the collection of all t-vertex r-graphs with vertex $\{1, 2, \dots, t\}$. A tool which has proved very useful in extremal graph theory and which we will use later is the Lagrangian of an r-graph. Let

$$\mathbb{S}_t = \{\vec{x} \in \mathbb{R}^t : \sum_{i=1}^{t} x_i = 1, x_i \geq 0 \text{ for } 1 \leq i \leq t\}.$$

If $G \in \mathcal{G}^{(r)}_t$ and $\vec{x} \in \mathbb{S}_t$ then we define

$$\lambda(G, \vec{x}) = \sum_{v_1 v_2 \cdots v_r \in E(G)} x_{v_1} x_{v_2} \cdots x_{v_t}.$$

The *Lagrangian* of G is $\max_{\vec{x}\in\mathbb{S}_t} \lambda(G, \vec{x})$. The first application of the Lagrangian to extremal graph theory was due to Motzkin and Strauss who gave a new proof of Turán's theorem. We are now ready to state our main result.

Theorem 3. *If F is an r-graph and $t \geq r \geq 2$ then*

$$\pi_t^*(F) = \max\{r!\lambda(G) : G \in \mathcal{G}_t^{(r)} \text{ and } F \text{ is not } G\text{-colorable}\}.$$

As an example of Theorem 3, suppose that $t = 4, r = 3$, and $F = K_4^{(3)}$. Let H denote the unique 3-graph with four vertices and three edges. Now F is F-colorable, but it is not H-colorable, and the Lagrangian $\lambda(H)$ of H is 4/81, achieved by assigning the degree three vertex a weight of 1/3 and the other three vertices a weight of 2/9. Consequently, Theorem 3 says that the maximum number of edges in an n-vertex 4-partite 3-graph containing no copy of $K_4^{(3)}$ is $(8/27)\binom{n}{3} + o(n^3)$. This is clearly achievable, by the 4-partite 3-graph with part sizes $n/3, 2n/9, 2n/9, 2n/9$, with all possible triples between three parts that include the largest (of size $n/3$), and no triples between the three small parts.

Chromatic Turán densities were previously considered in [T07] where they were used to give an improved upper bound on $\pi(H)$, where H is defined in the previous paragraph. However no non-trivial chromatic Turán densities have previously been determined. For each $r \geq t \geq 2$ we are able to give an infinite sequence of r-graphs whose t-chromatic Turán densities are determined exactly.

For $l \geq t \geq 2$ and $r \geq 2$ define

$$\beta_{r,t,l} := \max\{\lambda(G) : G \text{ is a } t\text{-colorable } r\text{-graph on } l \text{ vertices}\}.$$

It seems obvious that $\beta_{r,t,l}$ is achieved by the t-chromatic r-graph of order l with all color classes of size $\lfloor l/t \rfloor$ or $\lceil l/t \rceil$ and all edges present except those within the classes. Note that if $t|l$ then this would give

$$\beta_{r,t,l} = \left(\binom{l}{r} - t\binom{l/t}{r} \right) \frac{1}{l^r}.$$

However, we are only able to prove this for $r = 2, 3$. If the above statement is true, then $\beta_{r,t,l}$ can be computed by calculating the maximum of an explicit polynomial in one variable over the unit interval. In any case it can be obtained by a finite computation (for fixed r, t, l). Let $\alpha_{r,t,l} = r!\beta_{r,t,l}$.

Theorem 4. *Fix $l \geq r \geq 2$. Let $L_{l+1}^{(r)}$ be the r-graph obtained from the complete graph K_{l+1} by enlarging each edge with a set of $r-2$ new vertices. If $t \geq 2$ then*

$$\pi_t(L_{l+1}^{(r)}) = \alpha_{r,t,l}$$

where $\alpha_{r,t,l}$ is defined above.

The remainder of the paper is arranged as follows. In the next section we prove Theorem 3 and in the last section we prove Theorem 4 and the statements about computing $\beta_{r,t,l}$, for $r = 2, 3$.

2 Proof of Theorem 3

If $G \in \mathcal{G}_t^{(r)}$ and $\vec{x} = (x_1, \ldots, x_t) \in \mathbb{Z}_+^t$ then the *$\vec{x}$-blow-up* of G is the r-graph $G(\vec{x})$ constructed from G by replacing each vertex v by a class of vertices of size x_v and taking all edges between any r classes corresponding to an edge of G. More precisely we have $V(G(\vec{x})) = X_1 \dot\cup \cdots \dot\cup X_t$, $|X_i| = x_i$ and

$$E(G(\vec{x})) = \{\{v_{i_1} v_{i_2} \cdots v_{i_r}\} : v_{i_j} \in X_{i_j}, \{i_1 i_2 \cdots i_r\} \in E(G)\}.$$

If $\vec{x} = (s, s, \ldots, s)$ and $G = K_t^{(r)}$ then $G(\vec{x})$ is the complete t-partite r-graph with class size s, denoted by $K_t^{(r)}(s)$. Note that if F and G are both r-graphs then F is G-colorable iff there exists $\vec{x} \in \mathbb{Z}_+^t$ such that $F \subseteq G(\vec{x})$.

An r-graph G is said to be *covering* if each pair of vertices in $V(G)$ is contained in a common edge. If $W \subset V$ and G is an r-graph with vertex V then $G[W]$ is the induced subgraph of G formed by deleting all vertices not in W and removing all edges containing these vertices.

Lemma 5 (Frankl and Rödl [FR84]). *If G is an r-graph of order n then there exists $\vec{y} \in \mathbb{S}_n$ with $\lambda(G) = \lambda(G, \vec{y})$, such that if $P = \{v \in V(G) : y_v > 0\}$ then $G[P]$ is covering.*

Supersaturation for ordinary Turán densities was shown by Erdős [E71]. The proof for G-chromatic Turán densities is essentially identical but for completeness we give it. We require the following classical result.

Theorem 6 (Erdős [E64]). *If $r \geq 2$ and $t \geq 1$ then $ex(n, K_r^{(r)}(t)) = O(n^{r-\lambda_{r,t}})$, with $\lambda_{r,t} > 0$.*

Lemma 7 (Supersaturation). *Fix $t \geq r \geq 2$. If G is an r-graph, $\mathcal{H}$ is a finite family of r-graphs, $s \geq 1$ and $\vec{s} = (s, s, \ldots, s)$ then $\pi_t^*(\mathcal{H}(\vec{s})) = \pi_t^*(\mathcal{H})$ (where $\mathcal{H}(\vec{s}) = \{H(\vec{s}) : H \in \mathcal{H}\}$).*

Proof: Let $p = \max\{|V(H)| : H \in \mathcal{H}\}$. By adding isolated vertices if necessary we may suppose that every $H \in \mathcal{H}$ has exactly p vertices.

First we claim that if F is an n-vertex r-graph with density at least $\alpha + 2\epsilon$, where $\alpha, \epsilon > 0$, and $r \leq m \leq n$ then at least $\epsilon\binom{n}{m}$ of the m-vertex induced subgraphs of F have density at least $\alpha + \epsilon$. To see this note that if it fails to hold then

$$\binom{n-r}{m-r}(\alpha + 2\epsilon)\binom{n}{r} \leq \sum_{W \in \binom{V(F)}{m}} e(F[W]) < \epsilon\binom{n}{m}\binom{m}{r} + (1-\epsilon)\binom{n}{m}(\alpha + \epsilon)\binom{m}{r},$$

which is impossible.

Let $\epsilon > 0$ and suppose that F is a t-partite n-vertex r-graph with density at least $\pi_t^*(\mathcal{H}) + 2\epsilon$. We need to show that if n is sufficiently large then F contains a copy of $\mathcal{H}(\vec{s})$. Let $m \geq m(\epsilon)$ be sufficiently large that any t-partite m-vertex r-graph with density at least $\pi_t^*(\mathcal{H}) + \epsilon$ contains a copy of some $H \in \mathcal{H}$. We say that $W \in \binom{V(F)}{m}$ is *good* if

$F[W]$ contains a copy of some $H \in \mathcal{H}$. By the claim at least $\epsilon\binom{n}{m}$ m-sets are good, so if $\delta = \epsilon/|\mathcal{H}|$ then at least $\delta\binom{n}{m}$ m-sets contain a fixed $H^* \in \mathcal{H}$.

Thus the number of p-sets $U \subset V(F)$ such that $F[U] \simeq H^*$ is at least

$$\frac{\delta\binom{n}{m}}{\binom{n-p}{m-p}} = \frac{\delta\binom{n}{p}}{\binom{m}{p}}. \tag{1}$$

Let J be the p-graph with vertex set $V(F)$ and edge set consisting of those p-sets $U \subset V(F)$ such that $F[U] \simeq H^*$. Now, by Theorem 6, $\mathrm{ex}_t^*(n, K_p^{(p)}(l)) \leq \mathrm{ex}(n, K_p^{(p)}(l)) = O(n^{p-\lambda_{p,l}})$, where $\lambda_{p,l} > 0$. Hence (1) implies that for any $l \geq p$ if n is sufficiently large then $K_p^{(p)}(l) \subset J$.

Finally consider a coloring of the edges of $K_p^{(p)}(l)$ with $p!$ different colors, where the color of the edge is given by the order in which the vertices of H^* are embedded in it. By Ramsey's theorem if l is sufficiently large then there is a copy of $K_p^{(p)}(s)$ with all edges the same color. This yields a copy of $H^*(\vec{s})$ in F as required. □

Proof of Theorem 3. Let $\alpha_{r,t} = \max\{r!\lambda(G) : G \in \mathcal{G}_t^{(r)}$ and F is not G-colorable$\}$. (This is well-defined since $|\mathcal{G}_t^{(r)}| \leq 2^{\binom{t}{r}}$ is finite.)

If $G \in \mathcal{G}_t^{(r)}$ and F is not G-colorable then for any $\vec{x} \in \mathbb{Z}_+^t$ we have $F \not\subseteq G(\vec{x})$. Let $\vec{y} \in \mathbb{S}_t$ satisfy $\lambda(G, \vec{y}) = \lambda(G)$. For $n \geq 1$ let $\vec{x}_n = (\lfloor y_1 n \rfloor, \ldots, \lfloor y_t n \rfloor) \in \mathbb{Z}_+^t$. If $G_n = G(\vec{x}_n)$ then

$$\lim_{n\to\infty} \frac{e(G_n)}{\binom{n}{r}} = r!\lambda(G).$$

Moreover since each G_n is F-free, t-partite and of order at most n we have $\pi_t^*(F) \geq r!\lambda(G)$. Hence $\pi_t^*(F) \geq \alpha_{r,t}$.

Let $\mathcal{H}(F) = \{H \in \mathcal{G}_t^{(r)} : F$ is H-colorable$\}$.

It is sufficient to show that

$$\pi_t^*(\mathcal{H}(F)) \leq \alpha_{r,t}. \tag{2}$$

Indeed, if we assume that (2) holds, then let $s \geq 1$ be minimal such that every $H \in \mathcal{H}(F)$ satisfies $F \subseteq H(\vec{s})$, where $\vec{s} = (s, s, \ldots, s)$. (Note that s exists since F is H-colorable for every $H \in \mathcal{H}(F)$). Now by supersaturation (Lemma 7) if $\epsilon > 0$, then any t-partite r-graph G_n with $n \geq n_0(s, \epsilon)$ vertices and density at least $\alpha_{r,t} + \epsilon$ will contain a copy of $H(\vec{s})$ for some $H \in \mathcal{H}(F)$. In particular G_n contains F and so $\pi_t^*(F) \leq \alpha_{r,t}$.

Let $\pi_t^*(\mathcal{H}(F)) = \gamma$ and $\epsilon > 0$. If n is sufficiently large there exists an $\mathcal{H}(F)$-free, t-partite r-graph G_n of order n satisfying

$$\frac{r!e(G_n)}{n^r} \geq \gamma - \epsilon.$$

Taking $\vec{y} = (1/n, 1/n, \ldots, 1/n) \in \mathbb{S}_n$ we have

$$r!\lambda(G_n) \geq r!\lambda(G_n, \vec{y}) = \frac{r!e(G_n)}{n^r} \geq \gamma - \epsilon.$$

Now Lemma 5 implies that there exists $\vec{z} \in \mathbb{S}_n$ satisfying

• $\lambda(G_n) = \lambda(G_n, \vec{z})$ and

• $G_n[P]$ is covering where $P = \{v \in V(G) : z_v > 0\}$.

Since G_n is t-partite, we conclude that $G_n[P]$ has at most t vertices. Moreover, G_n is $\mathcal{H}(F)$-free and so $G_n[P] \notin \mathcal{H}(F)$. Thus F is not $G_n[P]$-colorable, and we have $\gamma - \epsilon \leq r!\lambda(G_n[P]) \leq \alpha_{r,t}$. Thus $\pi_t^*(\mathcal{H}(F)) \leq \alpha_{r,t} + \epsilon$ for all $\epsilon > 0$. Hence (2) holds and the proof is complete. □

3 Infinitely many chromatic Turán densities

For $l, r \geq 2$ let $\mathcal{K}_l^{(r)}$ be the family of r-graphs with at most $\binom{l}{2}$ edges that contain a set S, called the *core*, of l vertices, with each pair of vertices from S contained in an edge. Note that $L_{l+1}^{(r)} \in \mathcal{K}_{l+1}^{(r)}$. We need the following Lemma that was proved in [M06]. For completeness, we repeat the proof below.

Lemma 8. *If $K \in \mathcal{K}_{l+1}^{(r)}$, $s = \binom{l+1}{2} + 1$ and $\vec{s} = (s, s, \ldots, s)$ then $L_{l+1}^{(r)} \subseteq K(\vec{s})$.*

Proof. We first show that $L_{l+1}^{(r)} \subset L(\binom{l+1}{2} + 1)$ for every $L \in \mathcal{K}_{l+1}^{(r)}$. Pick $L \in \mathcal{K}_{l+1}^{(r)}$, and let $L' = L(\binom{l+1}{2} + 1)$. For each vertex $v \in V(L)$, suppose that the clones of v are $v = v^1, v^2, \ldots, v^{\binom{l+1}{2}+1}$. In particular, identify the first clone of v with v.

Let $S = \{w_1, \ldots, w_{l+1}\} \subset V(L)$ be the core of L. For every $1 \leq i < j \leq l+1$, let $E_{ij} \in L$ with $E_{ij} \supset \{w_i, w_j\}$. Replace each vertex z of $E_{ij} - \{w_i, w_j\}$ by z^q where $q > 1$, to obtain an edge $E'_{ij} \in L'$. Continue this procedure for every i, j, making sure that whenever we encounter a new edge it intersects the previously encountered edges only in L. Since the number of clones is $\binom{l+1}{2} + 1$, this procedure can be carried out successfully and results in a copy of $L_{l+1}^{(r)}$ with core S. Therefore $L_{l+1}^{(r)} \subset L' = L(\binom{l+1}{2} + 1)$. Consequently, Lemma 7 implies that $\pi(L_{l+1}^{(r)}) \leq \pi(\mathcal{K}_{l+1}^{(r)})$. □

Proof of Theorem 4. Let $l \geq r \geq 2$ and $t \geq 2$. We will prove that

$$\pi_t(\mathcal{K}_{l+1}^{(r)}) = \alpha_{r,t,l}. \tag{3}$$

The theorem will then follow immediately from Lemmas 7 and 8. Let

$$\mathcal{B}_{r,t,l} = \{G : G \text{ is a } t\text{-colorable } \mathcal{K}_{l+1}^{(r)}\text{-free } r\text{-graph}\}.$$

Claim. $\max\{\lambda(G) : G \in \mathcal{B}_{r,t,l}\} = \beta_{r,t,l} = \alpha_{r,t,l}/r!$.

Proof of Claim. If $G \in \mathcal{B}_{r,t,l}$ has order n then Lemma 5 implies that there is $\vec{y} \in \mathbb{S}_n$ such that $\lambda(G) = \lambda(G, \vec{y})$ with $G[P]$ covering, where $P = \{v \in V(G) : y_v > 0\}$. Since G is $\mathcal{K}_{l+1}^{(r)}$-free, we conclude that $|P| = p \leq l$. Hence there is $H \in \mathcal{B}_{r,t,l}$ such that $\lambda(H) = \lambda(G)$ and H has order at most l. Consequently, $\max\{\lambda(G) : G \in \mathcal{B}_{r,t,l}\} \leq \beta_{r,t,l}$. For the other inequality, we just observe that an l-vertex r-graph must be $\mathcal{K}_{l+1}^{(r)}$-free. □

Now we can quickly complete the proof of the theorem by proving (3). For the upper bound, observe that if $G \in \mathcal{B}_{r,t,l}$ has order n then by the Claim

$$\frac{e(G)}{n^r} \leq \lambda(G) \leq \frac{\alpha_{r,t,l}}{r!}$$

and so $\pi_t(\mathcal{K}_{l+1}^{(r)}) \leq \alpha_{r,t,l}$. For the lower bound, suppose that $G \in \mathcal{B}_{r,t,l}$ has order p and satisfies $\lambda(G) = \beta_{r,t,l}$. Then there exists $\vec{y} \in \mathbb{S}_p$ such that $\lambda(G, \vec{y}) = \lambda(G) = \beta_{r,t,l}$. For $n \geq p$ define $\vec{y}_n = (\lfloor y_1 n \rfloor, \ldots, \lfloor y_p n \rfloor)$. Now $\{G(\vec{y}_n)\}_{n=p}^{\infty}$ is a sequence of t-colorable $\mathcal{K}_{l+1}^{(r)}$-free r-graphs and hence

$$\pi_t(\mathcal{K}_{l+1}^{(r)}) \geq \lim_{n\to\infty} \frac{e(G_n)}{\binom{n}{r}} = r!\lambda(G) = \alpha_{r,t,l}.$$

□

Now we prove that $\beta_{r,t,l}$ can be computed by only considering maximum t-colorable r-graphs with almost equal part sizes when $r = 2, 3$. The case $r = 2$ follows trivially from Lemma 5 so we consider the case $r = 3$.

Theorem 9. *Fix $l \geq t \geq 2$. Then $\beta_{3,t,l}$ is achieved by the t-chromatic 3-graph of order l with all color classes of size $\lfloor l/t \rfloor$ or $\lceil l/t \rceil$ and all edges present except those within the classes.*

Remark: Note that if $t|l$ then this implies that $\beta_{3,t,l} = (\binom{l}{3} - t\binom{l/t}{3})\frac{1}{l^3}$.

Proof. Let G be a t-chromatic 3-graph of order l satisfying $\lambda(G) = \beta_{3,t,l}$. We may suppose (by adding edges as required) that $V(G) = V_1 \cup V_2 \cup \cdots \cup V_t$ and that all edges not contained in any V_i are present. We may also suppose that $|V_1| \geq |V_2| \geq \cdots \geq |V_t|$. Let $\vec{x} \in \mathbb{S}_p$ satisfy $\lambda(G, \vec{x}) = \lambda(G)$.

If $v, w \in V_i$ and $x_v > x_w$ then setting $\delta = (x_v - x_w)/2 > 0$ and defining a new weighting $\vec{x}'$ by $x'_v = x_v - \delta$, $x'_w = x_w + \delta$ and $x'_u = x_u$ for $u \in V \backslash \{v, w\}$ it is easy to check that $\lambda(G, \vec{x}') > \lambda(G, \vec{x})$, contradicting the assumption that $\lambda(G, \vec{x}) = \lambda(G)$. Hence we may suppose that there are $x_1, \ldots, x_t \geq 0$ such that all vertices in V_i receive weight x_i.

In fact we can assume that all the x_i are non-zero. Since $\vec{x} \in \mathbb{S}_p$ there exists k such that $x_k > 0$. Suppose that $x_j = 0$ for some $j \in \{1, 2, \ldots, t\}$. Let $a_k = |V_k|$, $a_j = |V_j|$ and $\epsilon = x_k a_j a_k/(a_j + a_k)$. Define a new weighting $\vec{x}''$ by $x''_v = x_v$ for $v \in V \backslash (V_k \cup V_j)$, $x''_v = \epsilon/a_j$ for $v \in V_j$ and $x''_v = x_k - \epsilon/a_k$ for $v \in V_k$. It is straightforward to check that $\vec{x}'' \in \mathbb{S}_p$ and $\lambda(G, \vec{x}'') > \lambda(G, \vec{x})$, contradicting the maximality of $\lambda(G, \vec{x})$. Hence we may suppose that all the x_i are non-zero.

Let $l = bt + c$, $0 \leq c < t$. To complete the proof we need to show that all of the V_i have order b or $b + 1$. Suppose, for a contradiction, that there exist V_i and V_j with $a_i = |V_i|$, $a_j = |V_j|$ and $a_i \geq a_j + 2$. We will construct a new t-colorable l-vertex 3-graph $\tilde{G}$ with $\lambda(\tilde{G}) > \lambda(G)$.

We construct $\tilde{G}$ from G by moving a vertex v from V_i to V_j and inserting all new allowable edges (i.e. those which contain v and 2 vertices from $V_i \backslash \{v\}$) while deleting any edges which now lie in V_j. By our assumption that $\beta_{3,t,l} = \lambda(G) = \lambda(G, \vec{x})$ we must have $\lambda(\tilde{G}, \vec{x}) \leq \lambda(G, \vec{x})$. Comparing terms in $\lambda(G, \vec{x})$ and $\lambda(\tilde{G}, \vec{x})$ this implies that

$$\binom{a_j}{2} x_i x_j^2 \geq \binom{a_i - 1}{2} x_i^3. \tag{4}$$

In particular, since $x_i, x_j > 0$, we have $x_i < x_j$.

We give a new weighting $\vec{y}$ for $\tilde{G}$ by setting

$$y_v = \begin{cases} a_i x_i/(a_i - 1), & v \in V_i, \\ a_j x_j/(a_j + 1), & v \in V_j, \\ x_k, & v \in V_k \text{ and } k \neq i, j. \end{cases}$$

It is easy to check that $\vec{y} \in \mathbb{S}_l$ is a legal weighting for $\tilde{G}$. We will derive a contradiction by showing that $\lambda(\tilde{G}) \geq \lambda(\tilde{G}, \vec{y}) > \lambda(G, \vec{x}) = \lambda(G)$.

If $w = a_i x_i + a_j x_j = (a_i - 1) y_i + (a_j + 1) y_j$ then

$$\begin{aligned} \lambda(\tilde{G}, \vec{y}) - \lambda(G, \vec{x}) &= (1 - w) \left(\binom{a_i - 1}{2} y_i^2 + \binom{a_j + 1}{2} y_j^2 + (a_i - 1)(a_j + 1) y_i y_j \right. \\ &\quad \left. - \binom{a_i}{2} x_i^2 - \binom{a_j}{2} x_j^2 - a_i a_j x_i x_j \right) + \binom{a_i - 1}{2} (a_j + 1) y_i^2 y_j + \\ &\quad \binom{a_j + 1}{2} (a_i - 1) y_i y_j^2 - \binom{a_i}{2} a_j x_i^2 x_j - \binom{a_j}{2} a_i x_i x_j^2 \\ &= \frac{(1 - w)}{2} \left(\frac{a_j x_j^2}{a_j + 1} - \frac{a_i x_i^2}{a_i - 1} \right) + \frac{a_i a_j x_i x_j}{2} \left(\frac{x_j}{a_j + 1} - \frac{x_i}{a_i - 1} \right). \end{aligned}$$

Using (4) it is easy to check that this is strictly positive. □

Corollary 10. *The t-chromatic Turán density can take irrational values.*

Proof. We consider $\beta_{3,2,2k}$ for $k \geq 3$. In fact, we focus on $\beta_{3,2,6}$, the maximum density of a 2-chromatic 3-graph that contains no copy of $\mathcal{K}_6^{(3)}$. By the previous Theorem, this is 6 times the Lagrangian of the 3-graph with vertex set $\{a, a', a'', b, b'\}$ and all edges present except $\{a, a', a''\}$. Assigning weight x to the a's and weight y to the b's, we must maximize $6(6x^2y + 3xy^2)$ subject to $3x + 2y = 1$ and $0 \leq x \leq 1/3$. A short calculation shows that the choice of x that maximizes this expression is $(\sqrt{13} - 2)/9$, and this results in an irrational value for the Lagrangian. Similar computations hold for larger k as well. □

References

[E64] P. Erdős, *On extremal problems of graphs and generalized graphs*, Israel J. Math. 2 (1964), 183–190.

[E71] P. Erdős, *On some extremal problems on r-graphs*, Disc. Math. 1 (1971), 1–6.

[ES66] P. Erdős and M. Simonovits, *A limit theorem in graph theory*, Studia Sci. Mat. Hung. Acad. 1 (1966), 51–57.

[ES46] P. Erdős and A.H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. 52(1946), 1087–1091.

[FR84] P. Frankl and V. Rödl, *Hypergraphs do not jump*, Combinatorica 4 (1984), 149–159.

[KNS64] G. Katona, T. Nemetz and M. Simonovits, *On a problem of Turan in the theory of graphs* (in Hungarian) Mat. Lapok 15 (1964), 228–238.

[M07] W. Mantel, *Problem 28*, Wiskundige Opgaven, 10 (1907), 60-61.

[MS65] T. Motzkin and E. Strauss. *Maxima for graphs and a new proof of a theorem of Turan*, Canadian Journal of Mathematics, 17 (1965), 533–540.

[M06] D. Mubayi, *A hypergraph extension of Turán's theorem*, J. Combin. Theory, Ser. B 96 (2006), 122–134.

[T07] J. Talbot, *Chromatic Turán problems and a new upper bound for the Turán density of $\mathcal{K}_4^-$*. Europ. J. Comb 28 (2007), 2125–2142.

[T41] P. Turán, *On an extremal problem in graph theory*, Mat. Fiz. Lapok 48 (1941).