On the Turán Properties of Infinite Graphs

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Abstract

Let $G^{(\infty)}$ be an infinite graph with the vertex set corresponding to the set of positive integers $\mathbb{N}$. Denote by $G^{(l)}$ a subgraph of $G^{(\infty)}$ which is spanned by the vertices $\{1,\dots,l\}$. As a possible extension of Turán's theorem to infinite graphs, in this paper we will examine how large $\liminf_{l\to\infty}\frac{|E(G^{(l)})|}{l^2}$ can be for an infinite graph $G^{(\infty)}$, which does not contain an increasing path I_k with $k+1$ vertices. We will show that for sufficiently large k there are I_k–free infinite graphs with $\frac{1}{4}+\frac{1}{200}<\liminf_{l\to\infty}\frac{|E(G^{(l)})|}{l^2}$. This disproves a conjecture of J. Czipszer, P. Erdős and A. Hajnal. On the other hand, we will show that $\liminf_{l\to\infty}\frac{|E(G^{(l)})|}{l^2}\le\frac{1}{3}$ for any k and such $G^{(\infty)}$.

1 Introduction

1.1 Preliminaries

Let $G^{(\infty)}=\big(V(G^{(\infty)}),E(G^{(\infty)})\big)$ be an infinite graph with the vertex set corresponding to the set of natural numbers, *i.e.*, $V(G^{(\infty)})=\mathbb{N}$, and the set of edges $E(G^{(\infty)})$. Denote by $G^{(l)}$ the subgraph of $G^{(\infty)}$ induced on the set $\{1,\dots,l\}$. Let $G^{(\infty)}$ be a K_{k+1}–free graph. Then, by Turán's theorem for finite graphs we get that $\liminf_{l\to\infty}\frac{|E(G^{(l)})|}{l^2}\le\limsup_{l\to\infty}\frac{|E(G^{(l)})|}{l^2}\le\frac{1}{2}\big(1-\frac{1}{k}\big)$. On the other hand, a K_{k+1}–free graph $G^{(\infty)}$ with edges $\{i,j\}\in E(G^{(\infty)})$ if $j-i\neq 0 \mod k$, achieves this bound. Hence, the Turán density for finite and infinite K_{k+1}–free graphs is the same.

In this paper we study the edge density of graphs without an increasing path of length k. We say that $I_k=i_1i_2\dots i_{k+1}$ is an increasing path of $G^{(\infty)}$ if $i_1<i_2<\dots<i_{k+1}$ and $\{i_j,i_{j+1}\}\in E(G^{(\infty)})$. One can easily see that for any fixed l there exists a graph $G^{(l)}$ not containing I_k such that $|E(G^{(l)})|$ equals to the Turán number for K_{k+1}–free graphs. Hence, for finite graphs forbidding I_k leads to the same restriction on number of edges

as forbidding K_{k+1}. While the maximum value $\limsup_{l\to\infty} \frac{|E(G^{(l)})|}{l^2}$ can achieve over all I_k-free infinite graphs G is $\frac{1}{2}\left(1-\frac{1}{k}\right)$, the corresponding value for the limit inferior is harder to find. Set

$$p(G) = \liminf_{l\to\infty} \frac{|E(G^{(l)})|}{l^2}.$$

Furthermore, let the *path Turán number* be defined as

$$p(k) = \sup\{p(G) \mid G \text{ is } I_k\text{-free}\}.$$

J. Czipszer, P. Erdős and A. Hajnal were the first ones who examined these numbers. In [1], they showed that $p(2) = \frac{1}{8}$ and $p(3) = \frac{1}{6}$. The following was stated in [1] as a question and in [2, 3] as a conjecture.

Conjecture 1.1 ([1, 2, 3]) *For any $k \geq 2$ the following holds*

$$p(k) = \frac{1}{4}\left(1 - \frac{1}{k}\right). \tag{1}$$

In this paper we will show that in general this fails to be true. In fact, for sufficiently large k the value of $p(k)$ exceeds $\frac{1}{4}$.

Theorem 1.2 *For any $k \geq 162$ the path Turán number satisfies*

$$p(k) > \frac{1}{4} + \frac{1}{200}.$$

We were unable to decide if (1) holds for $k = 4$. Here we will show that (1) fails for $k = 16$.

Theorem 1.3 *The path Turán number $p(16)$ satisfies*

$$p(16) > \frac{1}{4}\left(1 - \frac{1}{16}\right).$$

Moreover, complementing Theorems 1.2 and 1.3 we will show the following upper bound, confirming that the Turán number for I_k–free infinite graphs differs significantly from those for finite graphs.

Theorem 1.4 *For any $k \geq 2$ the path Turán number satisfies*

$$p(k) \leq \frac{1}{3}.$$

1.2 Reformulation

In order to prove Theorems 1.2, 1.3 and 1.4 we will work with infinite sequences of k symbols rather than with infinite graphs. Let $\mathcal{C} = \{c_n\}_{n=1}^{\infty}$ be a sequence of integers with $c_n \in \{1, 2, \ldots, k\}$, and

$$S_{\mathcal{C}}(k,l) = \left|\{(i,j) \mid 1 \leq i < j \leq l \text{ and } c_i < c_j\}\right|.$$

Furthermore, let

$$s_{\mathcal{C}}(k) = \liminf_{l\to\infty} \frac{|S_{\mathcal{C}}(k,l)|}{l^2},$$

and

$$s(k) = \sup_{\mathcal{C}} s_{\mathcal{C}}(k).$$

The following statement shows the equivalence between path Turán numbers and the numbers $s(k)$ for a fixed k.

Lemma 1.5 *Let $k \geq 2$. Then, $p(k) = s(k)$.*

Proof. For a given sequence $\mathcal{C} = \{c_n\}_{n=1}^{\infty}$ of k symbols, let $G^{(\infty)}$ be the infinite graph which corresponds to this sequence, *i.e.*, $V(G^{(\infty)}) = \mathbb{N}$ and $E(G^{(\infty)}) = \big\{\{i,j\} \mid i < j \text{ and } c_i < c_j\big\}$. Note, $G^{(\infty)}$ is an I_k–free. Hence, $|E(G^{(l)})| = S_{\mathcal{C}}(k,l)$, and consequently $p(k) \geq s(k)$. Conversely, let $G^{(\infty)}$ be an I_k–free infinite graph. Then, $G^{(\infty)}$ defines a partition of $\mathbb{N}$

$$\mathbb{N} = \bigcup_{j=1}^{k} N_j(G^{(\infty)}),$$

where

$$N_1(G^{(\infty)}) = \Big\{\alpha \in \mathbb{N} \mid \forall\beta \in \mathbb{N} : \{\alpha,\beta\} \in E(G^{(\infty)}) \Rightarrow \alpha < \beta\Big\},$$

and

$$N_i(G^{(\infty)}) = \Big\{\alpha \in \mathbb{N} \setminus \bigcup_{j=1}^{i-1} N_j(G^{(\infty)}) \mid \forall\beta \in \mathbb{N} : \{\alpha,\beta\} \in E(G^{(\infty)}) \Rightarrow$$
$$\alpha < \beta \text{ or } \beta \in \bigcup_{j=1}^{i-1} N_j(G^{(\infty)})\Big\},$$

for $i \in \{2,\ldots,k\}$. Let $\mathcal{C} = \{c_n\}_{n=1}^{\infty}$ be the sequence which corresponds to the above partition, *i.e.*, $c_n = i$ if $n \in N_i(G^{(\infty)})$. Note, $|E(G^{(l)})| \leq S_{\mathcal{C}}(k,l)$, and consequently, $p(k) \leq s(k)$. □

2 Auxiliary sequences

2.1 Sequence $\mathcal{A} = \{a_n\}_{n=1}^{\infty}$

The sequence $\mathcal{A} = \{a_n\}_{n=1}^{\infty}$ on the symbols $\{1,\ldots,k\}$, which we define below, will consists of infinitely many blocks. For $j \in \mathbb{N}$, the j-th block is a subsequence of $k2^j$ consecutive symbols, which consists of 2^j one's followed by 2^j two's, etc. We abbreviate such block of length $k2^j$ by $2^j \otimes \{1,2,\ldots,k\}$. Below are the first three blocks of the sequence $\mathcal{A}$:

$$1|1|2|2|\ldots|k|k\Big|1|1|1|1|2|2|2|2|\ldots|k|k|k|k\Big|1|1|1|1|1|1|1|1|2|2|2|2|2|2|2|2|\ldots|k|k|k|k|k|k|k|k\Big|$$

Formally, for a given k, $\mathcal{A} = \{a_n\}_{n=1}^{\infty}$ is a sequence of integers with $a_n \in \{1, \ldots, k\}$ defined as follows:

(i) for any $n \leq 2k$, $a_n = i$ if and only if $2(i-1) < n \leq 2i$, otherwise

(ii) for any $n > 2k$, $a_n = i$ if and only if there exists an integer number $m \in \mathbb{N} \cup \{0\}$ such that

$$k(2 + 2^2 + \cdots + 2^m) + (i-1)2^{m+1} < n \leq k(2 + 2^2 + \cdots + 2^m) + i2^{m+1}.$$

For $i \in \{0, 1, \ldots, k\}$, we identify the indices $n_i(m)$ for which value of the sequence changes from i to $i+1$, *i.e.*, $n_i(m) = k(2 + 2^2 + \cdots + 2^m) + i2^{m+1}$. Note, $n_k(m) = n_0(m+1)$ and

$$n_i(m) = (2k + 2i)2^m + o(2^m). \tag{2}$$

Proposition 2.1 *For any $i \in \{0, 1, \ldots, k-1\}$ we have*

$$S_{\mathcal{A}}(k, n_i(m)) = \left(\frac{4}{3}k(k-1) + 4i(i-1)\right)4^m + o(4^m).$$

Proof. First, we will find a formula for $s_{\mathcal{A}}(k, n_0(m))$. Note that setting $S_{\mathcal{A}}(k, n_0(0)) = 0$ we obtain that for $m \geq 1$,

$$\begin{aligned} S_{\mathcal{A}}(k, n_0(m)) &= S_{\mathcal{A}}(k, n_0(m-1)) + \sum_{j=1}^{k-1} j(2 + 2^2 + \cdots + 2^m)2^m \\ &= S_{\mathcal{A}}(k, n_0(m-1)) + k(k-1)2^m(2^m - 1), \end{aligned}$$

and hence by induction,

$$S_{\mathcal{A}}(k, n_0(m)) = k(k-1)\sum_{j=1}^{m} 2^j(2^j - 1) = \frac{4}{3}k(k-1)4^m + o(4^m).$$

Similarly, for $i \geq 1$,

$$\begin{aligned} S_{\mathcal{A}}(k, n_i(m)) &= S_{\mathcal{A}}(k, n_0(m)) + \sum_{j=1}^{i-1} j(2 + 2^2 + \cdots + 2^{m+1})2^{m+1} \\ &= S_{\mathcal{A}}(k, n_0(m)) + i(i-1)2^{m+1}(2^{m+1} - 1) \\ &= \left(\frac{4}{3}k(k-1) + 4i(i-1)\right)4^m + o(4^m). \end{aligned}$$

$\square$

By Proposition 2.1 and equation (2) we obtain the following.

Corollary 2.2 *For any $i \in \{0, 1, \dots, k-1\}$ we have*

$$\lim_{m\to\infty} \frac{S_{\mathcal{A}}(k, n_i(m))}{n_i(m)^2} = \frac{\frac{1}{3}k(k-1) + i(i-1)}{(k+i)^2}.$$

Denote the above limit by $t_{\mathcal{A}}(i)$.

Remark 2.3 Note that the existence of the limit $t_{\mathcal{A}}(i)$ means that the behavior of $\frac{S_{\mathcal{A}}(k,x)}{x^2}$, as a function of x with domain equal to the sequence $n_1(1) < \cdots < n_k(1) < n_1(2) < \cdots < n_k(2) < \cdots < n_i(m) < \cdots$, becomes close to periodic (with period k) for m large. In particular, $t_{\mathcal{A}}(0) = t_{\mathcal{A}}(k)$.

2.2 Sequence $\mathcal{B} = \{b_n\}_{n=1}^{\infty}$

Now, we define the second auxiliary sequence. For an even number k, let $\mathcal{B} = \{b_n\}_{n=1}^{\infty}$ be a sequence of integers with $b_n \in \{1, \dots, k\}$ such that

$$b_n = \begin{cases} a_n + \frac{k}{2} \mod k, & \text{if } a_n + \frac{k}{2} \neq 0 \mod k, \\ k, & \text{otherwise.} \end{cases}$$

Proposition 2.4 *For any $i \in \{0, 1, \dots, k-1\}$ we have*

$$S_{\mathcal{B}}(k, n_i(m)) = \begin{cases} \left(k^2 - \frac{4}{3}k + 2i(k+2i-2)\right) 4^m + o(4^m), & \text{if } 0 \leq i \leq \frac{k}{2}, \\ \left(3k^2 - \frac{10}{3}k + (2i-k-2)(2i-k)\right) 4^m + o(4^m), & \text{otherwise.} \end{cases}$$

Proof. First, we will find a formula for $S_{\mathcal{B}}(k, n_0(m))$. Recall that the m-th block is now of the form $2^m \otimes \{\frac{k}{2}+1, \dots, k, 1, \dots, \frac{k}{2}\}$. The number of pairs $b_\alpha < b_\beta$, where b_α belongs to the first $m-1$ blocks and $b_\beta = j+1$, for $j = 1, \dots, k-1$, and belongs to the last block, is equal to

$$j(2 + 2^2 + \cdots + 2^{m-1})2^m.$$

Setting $S_{\mathcal{B}}(k, n_0(0)) = 0$ yields for $m \geq 1$,

$$S_{\mathcal{B}}(k, n_0(m)) = S_{\mathcal{B}}(k, n_0(m-1)) + \sum_{j=1}^{k-1} j(2 + 2^2 + \cdots + 2^{m-1})2^m + 2\binom{\frac{k}{2}}{2}(2^m)^2,$$

where the last quantity counts the pairs $b_\alpha < b_\beta$ of the last block. Hence,

$$S_{\mathcal{B}}(k, n_0(m)) = S_{\mathcal{B}}(k, n_0(m-1)) + k(k-1)2^m(2^{m-1}-1) + \frac{k}{2}\left(\frac{k}{2}-1\right)4^m,$$

and by induction

$$\begin{aligned} S_{\mathcal{B}}(k, n_0(m)) &= k(k-1)\sum_{j=1}^{m} 2^j(2^{j-1}-1) + \frac{k}{2}\left(\frac{k}{2}-1\right)\sum_{j=1}^{m} 4^j \\ &= \frac{2}{3}k(k-1)4^m + \frac{2}{3}k\left(\frac{k}{2}-1\right)4^m + o(4^m) \\ &= \left(k^2 - \frac{4}{3}k\right)4^m + o(4^m). \end{aligned}$$

Now suppose that $i \in \{0, \ldots, \frac{k}{2}\}$. Then,

$$\begin{aligned} S_{\mathcal{B}}(k, n_i(m)) &= S_{\mathcal{B}}(k, n_0(m)) + \sum_{j=\frac{k}{2}}^{i+\frac{k}{2}-1} j(2 + 2^2 + \cdots + 2^m)2^{m+1} + \binom{i}{2}(2^{m+1})^2 \\ &= S_{\mathcal{B}}(k, n_0(m)) + i(k+i-1)2^{m+1}(2^m - 1) + 2i(i-1)4^m \\ &= S_{\mathcal{B}}(k, n_0(m)) + 2i(k+2i-2)4^m + o(4^m) \\ &= \left(k^2 - \frac{4}{3}k + 2i(k+2i-2)\right)4^m + o(4^m). \end{aligned}$$

Similarly, for $i \in \{\frac{k}{2}+1, \ldots, k-1\}$, we get

$$\begin{aligned} S_{\mathcal{B}}(k, n_i(m)) &= S_{\mathcal{B}}(k, n_{\frac{k}{2}}(m)) + \sum_{j=1}^{i-\frac{k}{2}-1} j(2 + 2^2 + \cdots + 2^{m+1})2^{m+1} \\ &= S_{\mathcal{B}}(k, n_{\frac{k}{2}}(m)) + (2i-k-2)(2i-k)4^m + o(4^m) \\ &= \left(3k^2 - \frac{10}{3}k + (2i-k-2)(2i-k)\right)4^m + o(4^m). \end{aligned}$$

$\square$

By Proposition 2.4 and equation (2) we get the following.

Corollary 2.5 *For any $i \in \{0, 1, \ldots, k-1\}$ we have*

$$t_{\mathcal{B}}(i) = \lim_{m\to\infty} \frac{S_{\mathcal{B}}(k, n_i(m))}{n_i(m)^2} = \begin{cases} \frac{\frac{1}{4}k^2 - \frac{1}{3}k + \frac{i}{2}(k+2i-2)}{(k+i)^2}, & \text{if } 0 \le i \le \frac{k}{2}, \\ \frac{\frac{3}{4}k^2 - \frac{5}{6}k + \frac{1}{4}(2i-k-2)(2i-k)}{(k+i)^2}, & \text{otherwise.} \end{cases}$$

The meaning of the existence of $t_{\mathcal{B}}(i)$ is similar as in Remark 2.3.

3 Proof of Theorem 1.2

We start with an outline of the proof. First, we redefine the sequence $\mathcal{B}$ by adding k to all its terms, *i.e.*, $b_n := b_n + k$. For an even integer k we construct a new sequence $\mathcal{C} = \{c_n\}_{n=1}^{\infty}$ setting $c_{2l-1} = a_l$ and $c_{2l} = b_l$, for $l \in \mathbb{N}$. Note, $\mathcal{C}$ is a sequence defined on symbols $\{1, 2, \ldots, 2k\}$. It is easy to see that, if $c_i < c_j$, then $c_i, c_j \in \mathcal{A}$, or $c_i, c_j \in \mathcal{B}$, or $c_i \in \mathcal{A}$ and $c_j \in \mathcal{B}$. Hence,

$$S_{\mathcal{C}}(2k, 2l) = S_{\mathcal{A}}(k, l) + S_{\mathcal{B}}(k, l) + \left|\{(a_i, b_j) \mid a_i \in \{a_n\}_{n=1}^{l}, b_j \in \{b_n\}_{n=1}^{l}, 1 \le i \le j \le l\}\right|.$$

Consequently,

$$\frac{S_{\mathcal{C}}(2k, 2l)}{(2l)^2} \ge \frac{1}{4}\left(\frac{S_{\mathcal{A}}(k, l)}{l^2} + \frac{S_{\mathcal{B}}(k, l)}{l^2}\right) + \frac{1}{8}. \tag{3}$$

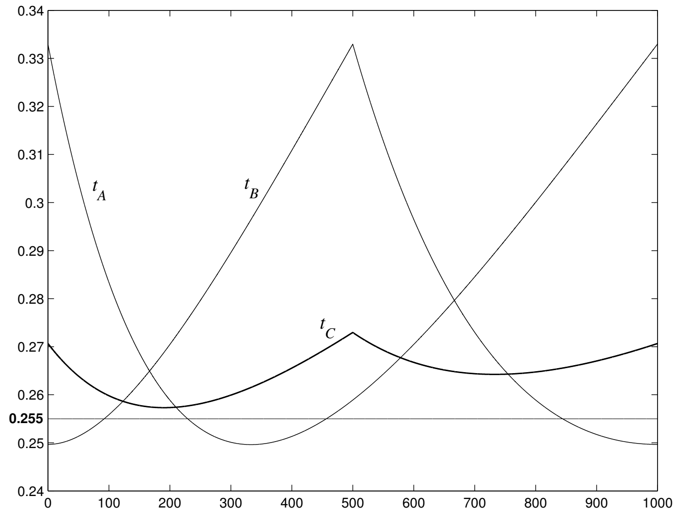

Figure 1: Density functions of sequences $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

Figure 1 describes the behavior of the density function for the sequences $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ on one block for $k = 1000$ and a large value of m. More precisely, it gives the graphs of $t_{\mathcal{A}}(1000)$, $t_{\mathcal{B}}(1000)$ and $t_{\mathcal{C}}(2000)$ as a function of $i = 1, \ldots, 1000$. Based on this, one can anticipate that $s_{\mathcal{C}}(2000) > 0.255$. Indeed, we will show that for k large enough ($k \geq 162$) the sequence $\{c_n\}$ implies the statement of Theorem 1.2.

Proof of Theorem 1.2. In view of Lemma 1.5 we need to show that $s_{\mathcal{C}}(2k) > 0.255$. To this end, we will verify that the limit inferior of the right side of (3) is larger than 0.255, as l goes to infinity. For any l, there are integers m and i such that $n_i(m) \leq l < n_{i+1}(m)$. Since $S_{\mathcal{A}}(k, l)$ is increasing in l, we thus have

$$\frac{S_{\mathcal{A}}(k,l)}{l^2} \geq \frac{S_{\mathcal{A}}(k, n_i(m))}{(n_{i+1}(m))^2}, \tag{4}$$

and similarly,

$$\frac{S_{\mathcal{B}}(k,l)}{l^2} \geq \frac{S_{\mathcal{B}}(k, n_i(m))}{(n_{i+1}(m))^2}. \tag{5}$$

In view of Propositions 2.1 and 2.4 and equation (2), for any $i \in \{0, \ldots, k-1\}$, we have,

$$\lim_{m\to\infty} \frac{S_{\mathcal{A}}(k, n_i(m))}{(n_{i+1}(m))^2} = \frac{\frac{k(k-1)}{3} + i(i-1)}{(k+i+1)^2},$$

and

$$\lim_{m\to\infty} \frac{S_{\mathcal{B}}(k, n_i(m))}{(n_{i+1}(m))^2} = \begin{cases} \frac{k^2 - \frac{4}{3}k + 2i(k+2i-2)}{4(k+i+1)^2}, & \text{if } i \in \{0, \ldots, \frac{k}{2} - 1\}, \\ \frac{3k^2 - \frac{10}{3}k + (2i-k-2)(2i-k)}{4(k+i+1)^2}, & \text{if } i \in \{\frac{k}{2}, \ldots, k-1\}. \end{cases}$$

Consequently,

$$\liminf_{l\to\infty}\left(\frac{S_{\mathcal{A}}(k,l)}{l^2}+\frac{S_{\mathcal{B}}(k,l)}{l^2}\right)\geq \min_i\left\{\lim_{m\to\infty}\frac{S_{\mathcal{A}}(k,n_i(m))}{(n_{i+1}(m))^2}+\lim_{m\to\infty}\frac{S_{\mathcal{B}}(k,n_i(m))}{(n_{i+1}(m))^2}\right\}.$$

Hence, it suffices to verify that for any $i\in\{0,\dots,\frac{k}{2}-1\}$

$$\frac{\frac{k(k-1)}{3}+i(i-1)}{(k+i+1)^2}+\frac{k^2-\frac{4}{3}k+2i(k+2i-2)}{4(k+i+1)^2}>0.52, \tag{6}$$

and for any $i\in\{\frac{k}{2},\dots,k-1\}$

$$\frac{\frac{k(k-1)}{3}+i(i-1)}{(k+i+1)^2}+\frac{3k^2-\frac{10}{3}k+(2i-k-2)(2i-k)}{4(k+i+1)^2}>0.52, \tag{7}$$

for k large enough. Multiplying both sides of (6) and (7) by $(k+i+1)^2$ one can verify that inequality (6) holds for $k\geq 162$ and inequality (7) for $k\geq 35$. □

4 Proof of Theorem 1.3

To prove Theorem 1.3 we need to refine some of the estimates made above. Observe that our main "tool" was the fact that for any l there are integers $m\in\mathbb{N}$ and $i\in\{0,\dots,k-1\}$ such that $n_i(m)\leq l<n_{i+1}(m)$, and consequently that (4) and (5) hold. In order to strengthen inequalities (4) and (5) we choose an integer r, which is a power of 2, and further subdivide the interval $\{n_i(m),\dots,n_{i+1}(m)-1\}$ into r disjoint intervals of the same length, *i.e.*,

$$\{n_i(m),\dots,n_{i+1}(m)-1\}=\bigcup_{j=0}^{r-1}\left\{n_i(m)+\frac{j}{r}2^{m+1},\dots,n_i(m)+\frac{j+1}{r}2^{m+1}-1\right\}.$$

Let $n_i^j(m)=n_i(m)+\frac{j}{r}2^{m+1}$ for $j\in\{0,1,\dots,r\}$. Note, $n_i^r(m)=n_{i+1}^0(m)$ and

$$n_i^j(m)=\left(2k+2i+2\frac{j}{r}\right)2^m+o(2^m). \tag{8}$$

Then, the following two statements hold.

Proposition 4.1 *Let r be a power of 2. Then, for any $i\in\{0,1,\dots,k-1\}$ and $j\in\{0,1,\dots,r-1\}$ we have*

$$S_{\mathcal{A}}(k,n_i^j(m))=\left(\frac{4}{3}k(k-1)+4i(i-1)+8i\frac{j}{r}\right)4^m+o(4^m),$$

for m sufficiently large.

Proof. Note that

$$\begin{aligned} S_{\mathcal{A}}(k, n_i^j(m)) &= S_{\mathcal{A}}(k, n_i(m)) + i(2 + 2^2 + \cdots + 2^{m+1})\frac{j}{r}2^{m+1} \\ &= S_{\mathcal{A}}(k, n_i(m)) + 8i\frac{j}{r}4^m + o(4^m), \end{aligned}$$

which in view of Proposition 2.1 yields the required statement. □

Proposition 4.2 *Let r be a power of 2. Then, for any $i \in \{0, 1, \ldots, k-1\}$ and $j \in \{0, 1, \ldots, r-1\}$ we have*

$$S_{\mathcal{B}}(k, n_i^j(m)) = \begin{cases} \left(k^2 - \frac{4}{3}k + 2i(k + 2i - 2) + \frac{j}{r}(2k + 8i)\right) 4^m + o(4^m), & \text{if } 0 \le i \le \frac{k}{2} - 1, \\ \left(3k^2 - \frac{10}{3}k + (2i - k - 2)(2i - k) + \frac{j}{r}(8i - 4k)\right) 4^m + o(4^m), & \text{otherwise.} \end{cases}$$

for m sufficiently large.

Proof. For $i \in \{0, \ldots, \frac{k}{2} - 1\}$ we get

$$\begin{aligned} S_{\mathcal{B}}(k, n_i^j(m)) &= S_{\mathcal{B}}(k, n_i(m)) + \left(\frac{k}{2} + i\right)(2 + 2^2 + \cdots + 2^m)\frac{j}{r}2^{m+1} + i2^{m+1}\frac{j}{r}2^{m+1} \\ &= S_{\mathcal{B}}(k, n_i(m)) + \frac{j}{r}(2k + 8i)4^m + o(4^m). \end{aligned}$$

Similarly, for $i \in \{\frac{k}{2}, \ldots, k-1\}$, we have

$$\begin{aligned} S_{\mathcal{B}}(k, n_i^j(m)) &= S_{\mathcal{B}}(k, n_i(m)) + \left(i - \frac{k}{2}\right)(2 + 2^2 + \cdots + 2^{m+1})\frac{j}{r}2^{m+1} \\ &= S_{\mathcal{B}}(k, n_i(m)) + \frac{j}{r}(8i - 4k)4^m + o(4^m). \end{aligned}$$

The required statement follows now from Proposition 2.4. □

Based on the above propositions we will prove Theorem 1.3.

Proof of Theorem 1.3. In view of Lemma 1.5 we need to show that $s_{\mathcal{C}}(16) > \frac{1}{4}\left(1 - \frac{1}{16}\right)$. In order to prove it, we will show that the limit inferior of the right side of (3) is strictly greater than $\frac{1}{4}\left(1 - \frac{1}{16}\right)$, as l goes to infinity. For any l there are integers $m \in \mathbb{N}$, $i \in \{0, \ldots, k-1\}$ and $j \in \{0, \ldots, r-1\}$ such that $n_i^j(m) \le l < n_i^{j+1}(m)$, and consequently

$$\frac{S_{\mathcal{A}}(k, l)}{l^2} \ge \frac{S_{\mathcal{A}}(k, n_i^j(m))}{(n_i^{j+1}(m))^2}, \tag{9}$$

and similarly,

$$\frac{S_{\mathcal{B}}(k, l)}{l^2} \ge \frac{S_{\mathcal{B}}(k, n_i^j(m))}{(n_i^{j+1}(m))^2}. \tag{10}$$

Let $k = 8$ and $r = 64$. Then, one can check[1] that for any $i \in \{0, \ldots, \frac{k}{2} - 1\}$ and $j \in \{0, \ldots, r-1\}$

$$\frac{\frac{k(k-1)}{3} + i(i-1) + 2i\frac{j}{r}}{\left(k + i + \frac{j+1}{r}\right)^2} + \frac{k^2 - \frac{4}{3}k + 2i(k + 2i - 2) + \frac{j}{r}(2k + 8i)}{4\left(k + i + \frac{j+1}{r}\right)^2} > \frac{28}{64}, \tag{11}$$

and for any $i \in \{\frac{k}{2}, \ldots, k-1\}$ and $j \in \{0, \ldots, r-1\}$

$$\frac{\frac{k(k-1)}{3} + i(i-1) + 2i\frac{j}{r}}{\left(k + i + \frac{j+1}{r}\right)^2} + \frac{3k^2 - \frac{10}{3}k + (2i - k - 2)(2i - k) + \frac{j}{r}(8i - 4k)}{4\left(k + i + \frac{j+1}{r}\right)^2} > \frac{28}{64}. \tag{12}$$

Hence, by Propositions 4.1 and 4.2, equation (8) and inequalities (9) and (10) we obtain

$$\liminf_{l\to\infty}\left(\frac{S_{\mathcal{A}}(k,l)}{l^2} + \frac{S_{\mathcal{B}}(k,l)}{l^2}\right) \geq \min_{i,j}\left\{\lim_{m\to\infty}\frac{S_{\mathcal{A}}(k, n_i^j(m))}{(n_i^{j+1}(m))^2} + \lim_{m\to\infty}\frac{S_{\mathcal{B}}(k, n_i^j(m))}{(n_i^{j+1}(m))^2}\right\} > \frac{28}{64},$$

which in view of (3) yields the statement of Theorem 1.3, *i.e.*,

$$s_{\mathcal{C}}(16) > \frac{1}{4}\cdot\frac{28}{64} + \frac{1}{8} = \frac{1}{4}\left(1 - \frac{1}{16}\right).$$

□

Remark 4.3 Analogously, one can show that Conjecture 1.1 fails for any $k \geq 24$. In order to do it, take a sequence $\mathcal{C}$ of $2k$ symbols from the proof of Theorem 1.3, for $k \geq 12$ and even. Then an approach similar to the one used in Theorem 1.3 yields

$$s_{\mathcal{C}}(2k) > \frac{1}{4}\left(1 - \frac{1}{2k+3}\right).$$

5 Proof of Theorem 1.4

For a given k, let $\mathcal{C} = \{c_n\}_{n=1}^{\infty}$ be a sequence of integers with $c_n \in \{1, \ldots, k\}$. Let $t > 1$ be an integer and $\varepsilon = \frac{1}{t}$. Let

$$\mathbb{N} = \bigcup_{i=1}^{k} N^{(i)} \tag{13}$$

be a partition such that $c_n = i$ for every $n \in N^{(i)}$. We further subdivide each $N^{(i)} = \{n_1^i < n_2^i < \cdots\}$, $i = 1, \ldots, k$, into

$$N^{(i)} = \bigcup_{j=1}^{t} N^{(i,j)},$$

[1]The authors used MATLAB [4] to verify (11) and (12).

where $N^{(i,j)} = \{n_j^i < n_{t+j}^i < n_{2t+j}^i < \cdots\}$. Now, we construct a new sequence $\mathcal{D} = \{d_n\}_{n=1}^{\infty}$ with values in $\{1, \ldots, \tilde{k}\}$, for $\tilde{k} = kt$, such that

$$d_n = (i-1)t + j,$$

for $n \in N^{(i,j)}$. Let

$$\mathcal{S}_{\mathcal{D}}(l) = \{(n_1, n_2) \mid 1 \leq n_1 < n_2 \leq l, i_1 < i_2 \text{ for } n_1 \in N^{(i_1)}, n_2 \in N^{(i_2)}\}.$$

Note,

$$S_{\mathcal{C}}(k, l) = |\mathcal{S}_{\mathcal{D}}(l)|. \tag{14}$$

For any $i \in \{1, \ldots, \tilde{k}\}$, let r_i be a function defined on $\mathbb{N}$ such that

$$r_i(l) = \frac{|\{n \in \mathbb{N} \mid d_n = i \text{ and } n \leq l\}|}{l}.$$

Consequently, for any $l \in \mathbb{N}$, we have $\big(r_1(l), \ldots, r_{\tilde{k}}(l)\big) \in [0, \varepsilon]^{\tilde{k}}$ and $\sum_{i=1}^{\tilde{k}} r_i(l) = 1$. Since the set $[0, \varepsilon]^{\tilde{k}}$ is compact, the sequence $\big\{\big(r_1(l), \ldots, r_{\tilde{k}}(l)\big)\big\}_{l=1}^{\infty}$ has a limit point in $[0, \varepsilon]^{\tilde{k}}$, say $\big(r_1, \ldots, r_{\tilde{k}}\big)$. Therefore, there exists a sequence $\{l_j\}_{j=1}^{\infty}$ such that

$$\lim_{j \to \infty} \big(r_1(l_j), \ldots, r_{\tilde{k}}(l_j)\big) = \big(r_1, \ldots, r_{\tilde{k}}\big).$$

Obviously, $\big(r_1, \ldots, r_{\tilde{k}}\big) \in [0, \varepsilon]^{\tilde{k}}$ and $\sum_{i=1}^{\tilde{k}} r_i = 1$. Thus, the set $\mathcal{R} = \{1, \ldots, \tilde{k}\}$ can be divided into 3 disjoint sets for some $i_1, i_2 \in \{1, \ldots, \tilde{k}\}$ as follows:

$$\begin{aligned} \mathcal{R}_1 &= \{1, \ldots, i_1\}, \\ \mathcal{R}_2 &= \{i_1 + 1, \ldots, i_2\}, \\ \mathcal{R}_3 &= \{i_2 + 1, \ldots, \tilde{k}\}, \end{aligned}$$

and

$$\Big|\sum_{i \in \mathcal{R}_q} r_i - \frac{1}{3}\Big| < \varepsilon,$$

for $q = 1, 2, 3$. Consequently, there exists j_0 such that for all $j \geq j_0$ the similar inequality holds, *i.e.*,

$$\Big|\sum_{i \in \mathcal{R}_q} r_i(l_j) - \frac{1}{3}\Big| < \varepsilon, \tag{15}$$

for $q = 1, 2, 3$. Let $\mathcal{S}_{\mathcal{D}}^1(l)$ denote the set of pairs $(n_1, n_2) \in \mathcal{S}_{\mathcal{D}}(l)$ for which $d_{n_1}, d_{n_2} \in \mathcal{R}_q$, for some $q \in \{1, 2, 3\}$. Similarly, let $\mathcal{S}_{\mathcal{D}}^2(l)$ denote the set of pairs $(n_1, n_2) \in \mathcal{S}_{\mathcal{D}}(l)$ for which $d_{n_1} \in \mathcal{R}_{q_1}$ and $d_{n_2} \in \mathcal{R}_{q_2}$ for some $q_1 \neq q_2 \in \{1, 2, 3\}$. Hence,

$$|\mathcal{S}_{\mathcal{D}}(l)| = |\mathcal{S}_{\mathcal{D}}^1(l)| + |\mathcal{S}_{\mathcal{D}}^2(l)|. \tag{16}$$

Furthermore, since

$$|\mathcal{S}^1_{\mathcal{D}}(l_j)| \leq \sum_{q=1}^{3} \binom{l_j \sum_{i\in\mathcal{R}_q} r_i(l_j)}{2},$$

and (15) we infer that

$$\limsup_{j\to\infty} \frac{|\mathcal{S}^1_{\mathcal{D}}(l_j)|}{{l_j}^2} = \frac{1}{3^2} \limsup_{j\to\infty} \frac{|\mathcal{S}^1_{\mathcal{D}}(l_j)|}{\left(\frac{l_j}{3}\right)^2} \leq \frac{1}{3^2}\cdot\frac{1}{2}\cdot 3 + O(\varepsilon) = \frac{1}{6} + O(\varepsilon). \tag{17}$$

On the other hand, due to the result of J. Czipszer, P. Erdős and A. Hajnal (see Theorem 2 in [1]), which states that Conjecture 1.1 is true for $k=3$, we obtain that

$$\liminf_{j\to\infty} \frac{|\mathcal{S}^2_{\mathcal{D}}(l_j)|}{{l_j}^2} \leq \frac{1}{4}\left(1-\frac{1}{3}\right) = \frac{1}{6}. \tag{18}$$

Consequently, by (14), (16), (17) and (18) we get

$$\begin{aligned}\liminf_{l\to\infty} \frac{S_{\mathcal{C}}(k,l)}{l^2} = \liminf_{l\to\infty} \frac{|\mathcal{S}_{\mathcal{D}}(l)|}{l^2} &\leq \liminf_{j\to\infty} \frac{|\mathcal{S}_{\mathcal{D}}(l_j)|}{{l_j}^2} \\ &\leq \limsup_{j\to\infty} \frac{|\mathcal{S}^1_{\mathcal{D}}(l_j)|}{{l_j}^2} + \liminf_{j\to\infty} \frac{|\mathcal{S}^2_{\mathcal{D}}(l_j)|}{{l_j}^2} \leq \frac{1}{3} + O(\varepsilon).\end{aligned}$$

Since this is true with ε arbitrarily small, we infer that

$$s_{\mathcal{C}}(k) \leq \frac{1}{3}.$$

This completes the proof of Theorem 1.4. □

Remark 5.1 Slightly modifying the above proof, one can show that for given k and m we have

$$p(k) \leq p(m) + \frac{1}{2m}. \tag{19}$$

Indeed, note that the set $\{1,\ldots,\tilde{k}\}$ can be partitioned into m pairwise disjoint sets

$$\{1,\ldots,\tilde{k}\} = \bigcup_{q=1}^{m} \mathcal{R}_q,$$

which satisfy

$$\Big|\sum_{i\in\mathcal{R}_q} r_i(l_j) - \frac{1}{m}\Big| < \varepsilon,$$

for every $q \in \{1,\ldots,m\}$ and every $j \geq j_0$, for some fixed j_0. Consequently, the left sides of (17) and (18) can be bounded by $\frac{1}{2m} + O(\varepsilon)$ and $p(m)$, respectively. Hence,

$$s_{\mathcal{C}}(k) \leq p(m) + \frac{1}{2m},$$

i.e., (19) holds.

Note that if Conjecture 1.1 would be true, say for $p(4)$, one could combine it with (19) to improve Theorem 1.4. More precisely, we would have

$$p(k) \leq \frac{1}{4}\left(1 - \frac{1}{16}\right) + \frac{1}{8} = \frac{5}{16},$$

for every $k \geq 2$.

6 Concluding remarks

One can extend the definition of $p(k)$ by allowing k to be also infinity, with the corresponding parameter

$$p(\infty) = \sup_{G^{(\infty)}} \left(\liminf_{l \to \infty} \frac{|E(G^{(l)})|}{l^2} \right),$$

where the supremum is taken over all graphs $G^{(\infty)}$ without infinite increasing paths. J. Czipszer, P. Erdős and A. Hajnal showed in [1] that

$$\frac{1}{4} + \frac{1}{36} \leq p(\infty) \leq \frac{1}{4} + \frac{3}{16}.$$

In this paper we showed that for k large enough the path Turán number satisfies

$$\frac{1}{4} + \frac{1}{200} \leq p(k) \leq \frac{1}{4} + \frac{1}{12}.$$

Determining the precise values of $p(k)$ and $p(\infty)$ does not seem to be an easy problem. However, it is easy to see that for any fixed k

$$p(k) < p(\infty). \tag{20}$$

Indeed, let $\mathcal{C} = \{c_n\}_{n=1}^{\infty}$ be a sequence of k symbols. Furthermore, let $\mathcal{D} = \{d_n\}_{n=1}^{\infty}$ be a sequence from the proof of Theorem 1.4 for some integer t. Then, $\mathcal{D}$ is a sequence of kt symbols. Recalling the definition of $N^{(i)}$ (*cf.* (13)), for $i = 1, \ldots, k$, set

$$l_i(l) = \left|\{\alpha \in N^{(i)} \mid \alpha \leq l\}\right|,$$

and

$$E_i(l) = \left|\{(\alpha, \beta) \in N^{(i)} \times N^{(i)} \mid \ d_\alpha < d_\beta \text{ and } 1 \leq \alpha < \beta \leq l\}\right|.$$

Since $N^{(i)}$ induces a subsequence of the form $1|2|\ldots|\mathrm{t}|1|2|\ldots|\mathrm{t}|1|2|\ldots|\mathrm{t}|\ldots$ we infer that $E_i(l) = \frac{1}{4}\big(1 - \frac{1}{t}\big) l_i(l)^2 + o\big(l_i(l)^2\big)$ and $\sum_{i=1}^{k} l_i(l) = l$. Consequently,

$$\sum_{i=1}^{k} E_i(l) = \frac{1}{4}\left(1 - \frac{1}{t}\right) \sum_{i=1}^{k} \left(l_i(l)^2 + o\big(l_i(l)^2\big)\right) \geq \frac{1}{4}\left(1 - \frac{1}{t}\right) k \left(\frac{l}{k}\right)^2 + o(l^2),$$

and

$$\frac{\sum_{i=1}^{k} E_i(l)}{l^2} \geq \frac{1}{4}\left(1 - \frac{1}{t}\right)\frac{1}{k} + o(1).$$

Hence,

$$s_{\mathcal{D}}(kt) \geq s_{\mathcal{C}}(k) + \frac{1}{4}\left(1 - \frac{1}{t}\right)\frac{1}{k},$$

and finally by Lemma 1.5 we obtain (20), *i.e.*,

$$p(k) = s(k) < s(kt) = p(kt) \leq p(\infty).$$

Letting $t \to \infty$, we have just showed that

$$p(k) + \frac{1}{4k} \leq p(\infty).$$

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