Chromatic Graphs, Ramsey Numbers and the Flexible Atom Conjecture

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Abstract

Let K_N denote the complete graph on N vertices with vertex set $V = V(K_N)$ and edge set $E = E(K_N)$. For $x, y \in V$, let xy denote the edge between the two vertices x and y. Let L be any finite set and $\mathcal{M} \subseteq L^3$. Let $c : E \to L$. Let $[n]$ denote the integer set $\{1, 2, \ldots, n\}$.

For $x, y, z \in V$, let $c(xyz)$ denote the ordered triple $(c(xy), c(yz), c(xz))$. We say that c is *good with respect to* $\mathcal{M}$ if the following conditions obtain:

(i) $\forall x, y \in V$ and $\forall (c(xy), j, k) \in \mathcal{M}$, $\exists z \in V$ such that $c(xyz) = (c(xy), j, k)$;

(ii) $\forall x, y, z \in V$, $c(xyz) \in \mathcal{M}$; and

(iii) $\forall x \in V\ \forall \ell \in L\ \exists y \in V$ such that $c(xy) = \ell$.

We investigate particular subsets $\mathcal{M} \subseteq L^3$ and those edge colorings of K_N which are good with respect to these subsets $\mathcal{M}$. We also remark on the connections of these subsets and colorings to projective planes, Ramsey theory, and representations of relation algebras. In particular, we prove a special case of the flexible atom conjecture.

1 Motivation and background

Let K_N denote the complete graph on N vertices with vertex set $V = V(K_N)$ and edge set $E = E(K_N)$. For $x, y \in V$, let xy denote the edge between the two vertices x and

y. Let L be any finite set and $\mathcal{M} \subseteq L^3$. Let $c : E \to L$. Let $[n]$ denote the integer set $\{1, 2, \ldots, n\}$.

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(ii) $\forall x, y, z \in V$, $c(xyz) \in \mathcal{M}$; and

(iii) $\forall x \in V\ \forall \ell \in L\ \exists\, y \in V$ such that $c(xy) = \ell$.

If $K = K_N$ has a coloring c which is good with respect to $\mathcal{M}$, then we say that K *realizes* $\mathcal{M}$ (or that $\mathcal{M}$ is *realizable*).

If we take $R_\alpha = \{(x, y) : \ c(xy) = \alpha\}$, and let $|$ stand for ordinary composition of binary relations, ie. $R_\alpha | R_\beta := \{(x, z) : \exists y\ (x, y) \in R_\alpha, (y, z) \in R_\beta\}$, then conditions (i) and (ii) imply

$$(R_\alpha | R_\beta) \cap R_\gamma \neq \emptyset \Longrightarrow\ R_\gamma \subseteq R_\alpha | R_\beta.$$

Conditions (i) - (iii) are given in [1] where the author calls a coloring on K_N that realizes some $\mathcal{M}$ a *symmetric color scheme*. It is proved in [2] that if $\mathcal{M}$ is a set of triples that is closed under permutation such that there is at least one $\alpha \in L$ such that for all $\beta, \gamma \in L$, $(\alpha, \beta, \gamma) \in \mathcal{M}$, then $\mathcal{M}$ is realized by a coloring on K_ω, the complete graph on countably many vertices. Any such color α is called a *flexible color*, since it can participate in any triple.

Conditions (i) - (iii) may seem quite stringent, but in fact these conditions are satisfied in many natural situations. Recall the notation for the Ramsey numbers; that is, $R(k_1, k_2, \ldots, k_\ell)$ is the minimum integer n such that in any ℓ-coloring of the edges of K_n there is a monochromatic complete graph on k_j vertices in color j for some j. In particular, the coloring of K_5 which shows $R(3, 3) \geq 6$ satisfies (i) - (iii), as does the coloring of K_8 which shows $R(4, 3) \geq 9$, the colorings of K_{16} that show $R(3, 3, 3) \geq 17$, both "twisted" and "untwisted", and the coloring of K_{29} given in [7] and [4] that shows that $R(4, 3, 3) \geq 30$. In fact, the coloring of K_5 without monochromatic triangles is a realization of $\mathcal{M}_0 = \{(r, b, b), (b, r, b), (b, b, r), (r, r, b), (r, b, r), (b, r, r)\}$. The coloring of K_8 mentioned above is a realization of $\mathcal{M} = \mathcal{M}_0 \cup \{(r, r, r)\}$; the col orings of K_{16} are realizations of $\mathcal{M} = \{r, b, g\}^3 \backslash \{(r, r, r), (b, b, b), (g, g, g)\}$; the coloring of K_{29} is a realization of $\mathcal{M} = \{r, b, g\}^3 \backslash \{(b, b, b), (g, g, g)\}$.

In [1], Comer introduces the number $r(k)$ which is the largest N such that there is a coloring on K_N that realizes

$$\mathcal{M} = \{r_1, ..., r_k\}^3 \backslash \{(r_i, r_i, r_i) : i \in [k]\}.$$

Clearly, $r(k) \leq R(\overbrace{3, 3, \ldots, 3}^{k \text{ times}}) - 1$; equality holds for $k = 2$ and $k = 3$. An interesting open problem is whether equality holds for all values of k.

Realizations of color schemes arise in connection with projective planes as well. Let $L = \{r_1, \ldots, r_\ell\}$, and let

$$\mathcal{M}_\ell = \{(r_i, r_j, r_k) : |\{i, j, k\}| \in \{1, 3\}\}\,.$$

Lyndon proved in [5] that $\mathcal{M}_\ell$ is realizable in some complete graph if and only if there exists a projective plane of order $\ell - 1$, for $\ell > 2$. This result has been extremely important in the theory of relation algebras.

In [3], Maddux, Jipsen and Tuza show that for $\mathcal{M} = L^3$, K_N realizes $\mathcal{M}$ for arbitrarily large finite N. In the case when $\mathcal{M} = L^3$, every color in L is a flexible color.

2 The Main Result

The principal result of this paper is that $\mathcal{M}_n$ is realizable in K_N for some $N < \omega$, where $L = \{r, b_1, ..., b_n\}$ and

$$\mathcal{M}_n := \{(r,r,r), (r,r,b_i), (r,b_i,r), (b_i,r,r), (r,b_i,b_j), (b_i,r,b_j), (b_i,b_j,r) : i,j \in [n]\}.$$

(Observe that $\mathcal{M}_n = \{r, b_1, \ldots, b_n\}^3 \setminus \{b_1, \ldots, b_n\}^3$.) This is a special case of a problem that has come to be known as the flexible atom conjecture. This problem originates in relation algebra; an explanation of the conjecture in this context can be found in [6].

Theorem 1. *$\forall n \geq 1$ $\exists r = r(n)$ such that $\forall k > r$, K_N realizes $\mathcal{M}_n$ for $N = \binom{3k-4}{k}$.*

The proof will proceed as follows. First we will construct realizations of M_1 in K_N for arbitrarily large N. These colorings of K_N will exhibit quite a lot of redundancy; in particular, for any given edge $xy \in E$ and triple $(c(xy), j, k) \in \mathcal{M}_1$, there exist many vertices z such that $c(xyz) = (c(xy), j, k)$, while condition (i) only requires that there be *one* such vertex. The graph K_N, which is colored in colors r and b, can then be recolored by assigning edges colored b to a color from $\{b_1, ..., b_n\}$ uniformly at random. The probability that this recoloring realizes $\mathcal{M}_n$ is shown to be nonzero for sufficiently large N.

Note that r is a *flexible color* in $\mathcal{M}_n$. In the case that a flexible color is present, it is not hard to see that condition (iii) is automatically satisfied whenever (i) and (ii) are, and so we make no further mention of it.

3 Proof of theorem 1

Let $k \in \mathbb{N}$ and let $[3k-4]^k$ denote the collection of k-subsets of $[3k-4]$. Let G be the complete graph with vertex set $V = [3k-4]^k$.

Lemma 1. *G realizes $\mathcal{M}_1$ for any $k \geq 3$.*

Proof of lemma 1. Define an edge coloring $c : E(G) \to \{r, b\}$ by

$$c(xy) = \begin{cases} b, & \text{if } 0 \le |x \cap y| \le 1, \\ r, & \text{otherwise.} \end{cases}$$

Let $E_r = \{xy \in E(G) : c(xy) = r\}$ and $E_b = \{xy \in E(G) : c(xy) = b\}$. The following five claims establish that c satisfies condition (i) for $\mathcal{M}_1$.

Let $xy \in E_r$. Since $|x \cap y| \ge 2$, $|x \cup y| \le 2k - 2$.

Claim 1. $\exists z \in V$ *such that* $c(xyz) = (r, r, r)$.

Let $\overline{(x \cup y)}$ denote $[3k-4] \setminus (x \cup y)$ and let ℓ be any subset of $\overline{(x \cup y)}$ with $k - |x \cap y|$ elements. Set $z = \ell \cup (x \cap y)$. We have $|x \cap z| \ge 2$ and $|y \cap z| \ge 2$, so $c(xyz) = (r, r, r)$ and claim 1 is true.

Claim 2. $\exists z \in V$ *such that* $c(xyz) = (r, r, b)$.

Let $a_1 \in y \setminus x$, $a_2 \in x \cap y$, and ℓ be any $(k-2)$-subset of $\overline{(x \cup y)}$. Set $z = \ell \cup \{a_1, a_2\}$. We have $|x \cap z| = 1$ and $|y \cap z| = 2$, so $c(xyz) = (r, r, b)$ and claim 2 is true.

Claim 3. $\exists z \in V$ *such that* $c(xyz) = (r, b, b)$.

Let $a_1 \in x \setminus y$, $a_2 \in y \setminus x$. Let ℓ be as in the the proof of claim 2. Set $z = \ell \cup \{a_1, a_2\}$. We have $|x \cap z| = |y \cap z| = 1$, so $c(xyz) = (r, b, b)$ and claim 3 is true.

Now let $xy \in E_b$. Since $|x \cap y| \le 1$, $|x \cup y| \ge k - 3$.

Claim 4. $\exists z \in V$ *such that* $c(xyz) = (b, r, r)$.

If $k = 3$, then $|x \cap y| = 1$, so we can pick z to be the 3-subset consisting of $x \cap y$, one point from $x \setminus y$ and one point in $y \setminus x$. For $k \ge 4$, let ℓ_1 be any 2-subset of $x \setminus y$, ℓ_2 be any 2-subset of $y \setminus x$, and ℓ_3 be any $(k-4)$-subset of $\overline{(x \cup y)}$. Set $z = \ell_1 \cup \ell_2 \cup \ell_3$. We have $|x \cap z| = 2$ and $|y \cap z| = 2$, so $c(xyz) = (b, r, r)$ and claim 4 is true.

Claim 5. $\exists z \in V$ *such that* $c(xyz) = (b, b, r)$.

If $k = 3$, then $|x \cap y| = 1$, so we can pick z to be the 3-subset consisting of $y \setminus x$ together with one point from $x \setminus y$. For $k \ge 4$, let ℓ_1 be any 3-subset of $x \setminus y$ and $a \in y \setminus x$. Let ℓ_3 be as in the proof of claim 4. Set $z = \ell_1 \cup \{a\} \cup \ell_3$. We have $|x \cap z| \ge 2$ and $|y \cap z| = 1$, so $c(xyz) = (b, b, r)$ and claim 5 is true.

Observe that claims 1-5 imply that c satisfies condition (i) for $\mathcal{M}_1$. It remains to show that c satisfies condition (ii) for $\mathcal{M}_1$, which we show in claim 6 below.

Claim 6. $\forall x, y, z \in V$, $c(xyz) \in \mathcal{M}_1$.

By way of contradiction, suppose $\exists x, y, z \in V$ with $c(xyz) = (b, b, b)$. Since $|x \cup y \cup z| \le 3k - 4$, the pigeonhole principle implies that one of $|x \cap y|$, $|x \cap z|$, or $|y \cap z|$ is greater than or equal to 2, a contradiction.

Observe that claims 1-6 imply that c is good with respect to $\mathcal{M}_1$, and thus G realizes $\mathcal{M}_1$. □

Let $n \geq 2$ and let E_r and E_b be as in the proof of lemma 1. Let $\mathcal{S}$ be the set of all n-ary sequences of length $m = |E_b|$ taking digits from $[n]$. Choose a sequence s from $\mathcal{S}$ at random. Enumerate the edges of $E_b : e_1, \ldots, e_m$. Let $s(j) \in [n]$ denote the j^{th} position of the sequence s. Define a partition of E_b into n (possibly empty) parts $E_{b_1}, \ldots, E_{b_n}$ as follows:

$$E_{b_i} = \{e_j : s(j) = i\}, i \in [n]$$

Define a new edge coloring of G given by

$$c'(xy) = \begin{cases} b_i & \text{if } xy \in E_{b_i}, \\ r & \text{if } xy \in E_r. \end{cases}$$

It is not hard to see that the probability that a given edge has color i is $1/n$; and furthermore that, given two distinct edges, the assignment of their colors is independent.

We claim that for sufficiently large k, c' is good with respect to $\mathcal{M}_n$, and thus G realizes $\mathcal{M}_n$; for this reason, we assume that $k \geq 4$. Since c satisfies condition (ii) for $\mathcal{M}_1$, it is easy to see c' satisfies condition (ii) for $\mathcal{M}_n$. We show that the probability that c' does not satisfy condition (i) for $\mathcal{M}_n$ is less than 1.

Claim 7. *The probability P_1 that given $xy \in E_r$, $\exists i, j \in [n]$ such that $\forall z \in V$ $c'(xyz) \neq (r, b_i, b_j)$ is bounded from above by $n^2(1 - 1/n^2)^{(k-2)^2}$.*

Proof of claim 7. Let $Z := \{z \in V : c(xyz) = (r, b, b)\}$. For fixed $i, j \in [n]$ and $z \in Z$, the probability

$$(xz \in E_{b_i}) \wedge \left(yz \in E_{b_j}\right)$$

is $1/n^2$, so the probability

$$(xz \notin E_{b_i}) \vee \left(yz \notin E_{b_j}\right)$$

is $1 - 1/n^2$. Considering all $z \in Z$, we have that the probability

$$\bigwedge_{z \in Z} \left[(xz \notin E_{b_i}) \vee \left(yz \notin E_{b_j}\right)\right]$$

is $(1 - 1/n^2)^{|Z|}$. Summing over all n^2 combinations of i and j, we arrive at

$$P_1 = n^2 \left(1 - 1/n^2\right)^{|Z|}. \tag{1}$$

For an upper bound on P_1 we compute a lower bound on $|Z|$. Since we seek a lower bound, we may assume $|x \cap y| = 2$. Note that $|\overline{(x \cup y)}| = k - 2$. Let $a_x \in x \setminus y$ and $a_y \in y \setminus x$. If $z = \overline{(x \cup y)} \cup \{a_x, a_y\}$, then $z \in Z$. Since there are $(k-2)^2$ distinct z of this form, $(k-2)^2 \leq |Z|$. This fact together with (1) gives $P_1 \leq n^2 \left(1 - 1/n^2\right)^{(k-2)^2}$, as desired. □

Claim 8. *The probability P_2 that given $xy \in E_r$, $\exists j \in [n]$ such that $\forall z \in V$ $c'(xyz) \neq (r, r, b_j)$ is bounded from above by $n \left(1 - 1/n\right)^{\binom{k-2}{2}}$.*

Proof of claim 8. Let $Z := \{z \in V : c(xyz) = (r, r, b)\}$. For fixed $j \in [n]$ and $z \in Z$, the probability

$$(xz \in E_{b_j}) \wedge (yz \in E_r) = (xz \in E_{b_j})$$

is $1/n$, so the probability

$$(xz \notin E_{b_j})$$

is $1 - 1/n$. Considering all $z \in Z$, we have that the probability

$$\bigwedge_{z \in Z} (xz \notin E_{b_j})$$

is $(1 - 1/n)^{|Z|}$. Summing over all j, we arrive at

$$P_2 = n(1 - 1/n)^{|Z|}. \tag{2}$$

For an upper bound on P_2, we compute a lower bound on $|Z|$. As in claim 7, we may assume $|x \cap y| = 2$ so $|\overline{(x \cup y)}| = k - 2$. Let ℓ be any 2-subset of $\overline{(x \cup y)}$. If $z = (y \setminus x) \cup \ell$, then $z \in Z$. Since there are $\binom{k-2}{2}$ distinct z of this form, $\binom{k-2}{2} \leq |Z|$. This fact together with (2) gives $P_2 \leq n\,(1 - 1/n)^{\binom{k-2}{2}}$, as desired. □

Claim 9. *The probability P_3 that given $i \in [n]$ and $xy \in E_{b_i}$, $\exists j \in [n]$ such that $\forall z \in V$, $c'(xyz) \neq (b_i, r, b_j)$ is bounded from above by $n\,(1 - 1/n)^{\binom{k}{4}}$.*

Proof of claim 9. Fix $i \in [n]$ and $xy \in E_{b_i}$. Let

$$Z := \{z \in V : c(yz) = r \text{ and } c(xz) = b\}.$$

For $j \in [n]$, the probability that $xz \in E_{b_j}$ is $1/n$, so the probability that $xz \notin E_{b_j}$ is $1 - 1/n$. Continuing as in claim 8, we have

$$P_3 = n(1 - 1/n)^{|Z|}. \tag{3}$$

Again, we seek a lower bound for $|Z|$, so we may assume $|x \cap y| = 0$. Note that $|\overline{(x \cup y)}| = k - 4$. Let ℓ be any 4-subset of y. If $z = \overline{(x \cup y)} \cup \ell$, then $z \in Z$. Since there are $\binom{k}{4}$ distinct z of this form, $\binom{k}{4} \leq |Z|$. This fact together with (3) gives $P_3 \leq n(1 - 1/n)^{\binom{k}{4}}$. □

Observe that $\forall \ell$, $P_1 \geq P_\ell$. Hence, we can use the upper bound in claim 7 for P_1 as an upper bound for the probability that condition (i) does not obtain for given $xy \in E$. Since G has less than $\binom{3k-4}{k}^2$ edges, an upper bound for the probability P that c' fails to satisfy condition (i) for $\mathcal{M}_n$ is

$$P \leq \sum_{e \in E} P_1 \leq \binom{3k-4}{k}^2 P_1 \leq \binom{3k-4}{k}^2 n^2 \left(1 - \frac{1}{n^2}\right)^{(k-2)^2}. \tag{4}$$

Next, we show that the right hand side of the expression in (4) can be made less than 1 by choosing k large enough. Since $1 - x \leq e^{-x}$ for all x, we have

$$\begin{aligned} \binom{3k-4}{k}^2 n^2 \left(1 - \frac{1}{n^2}\right)^{(k-2)^2} &\leq \binom{3k-4}{k}^2 n^2 \left(e^{-(k-2)^2/n^2}\right) \\ &\leq \left(2^{3k-4}\right)^2 n^2 \left(e^{-(k-2)^2/n^2}\right) \\ &\leq 2^{6k} n^2 \left(e^{-(k-2)^2/n^2}\right). \end{aligned} \tag{5}$$

Note that the expression in (5) is less than 1 if and only if $\log\left[2^{6k} n^2 \left(e^{-(k-2)^2/n^2}\right)\right] < 0$, which holds just in case

$$6k \log 2 + 2 \log n - \frac{(k-2)^2}{n^2} < 0. \tag{6}$$

To ensure that the inequality in (6) will hold, we first assume that $k = cn^2$ for some $c \in \mathbb{R}$ and realize the above as a quadratic polynomial in c. Since the coefficient of c^2 is negative, the function is concave down. By finding the zeros of this polynomial in terms of n and then maximizing (over n) the greatest of them, we can find the c which will guarantee the inequality in (6). For $n \geq 2$, it is sufficient to take $c \geq 5.2$.

For such k, we have that $P < 1$, so there exists an edge coloring $c : E(G) \to \{r, b_1, \ldots, b_n\}$ which is good with respect to $\mathcal{M}_n$. Hence, G realizes $\mathcal{M}_n$ and the proof of theorem 1 is complete.

Corollary 1. *Any finite integral symmetric relation algebra with one flexible atom and with all (mandatory) diversity cycles involving the flexible atom is representable on arbitrarily large finite sets.*

It is possible to obtain Corollary 1 with "symmetric" deleted in the following way. We alter the proof of Theorem 1 to construct $n+1$ binary relations instead of an edge-colored graph in $n+1$ colors. Referring to the graph colored in two colors constructed lemma 1, let $R = \{(x, y) : c(xy) = r\}$ and $B = \{(x, y) : c(xy) = b\}$. Partition B into n disjoint subsets in the following way: Let 2ℓ be the number of asymmetric diversity atoms, so that $b_1, \ldots, b_{2\ell}$ are asymmetric and $b_{2\ell+1}, \ldots, b_n$ are all symmetric. Order the vertices $v_1, \ldots, v_N$. Let $V_< = \{(v_i, v_j) : i < j\}$. Assign pairs from $V_<$ to sets $B_1, \ldots, B_n$ uniformly at random. Now we assign the remaining pairs (v_j, v_i) as follows. For i, j with $i < j$,

(i) if $(v_i, v_j) \in B_m$ and $m > 2\ell$, then $(v_j, v_i) \in B_m$;

(ii) if $(v_i, v_j) \in B_m$ and $1 \leq m \leq \ell$, then $(v_j, v_i) \in B_{m+\ell}$;

(iii) if $(v_i, v_j) \in B_m$ and $\ell < m \leq 2\ell$, then $(v_j, v_i) \in B_{m-\ell}$.

Thus we have $B_m\breve{} = B_{m+\ell}$ for $m \leq \ell$. Then by making superficial changes to the remainder of the proof we establish the result for the nonsymmetric case. Thus we have the following:

Theorem 2. *Any finite integral relation algebra with one flexible atom and with all (mandatory) diversity cycles involving the flexible atom is representable on arbitrarily large finite sets.*

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References

[1] Stephen D. Comer. Color schemes forbidding monochrome triangles. In *Proceedings of the fourteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1983)*, volume 39, pages 231–236, 1983.

[2] Stephen D. Comer. Combinatorial aspects of relations. *Algebra Universalis*, 18(1):77–94, 1984.

[3] P. Jipsen, R. D. Maddux, and Z. Tuza. Small representations of the relation algebra $\mathcal{E}_{n+1}(1,2,3)$. *Algebra Universalis*, 33(1):136–139, 1995.

[4] J.G. Kalbfleisch. *Chromatic Graphs and Ramsey's Theorem.* Ph.D. dissertation, University of Waterloo, 1966.

[5] R. C. Lyndon. Relation algebras and projective geometries. *Michigan Math. J.*, 8(1):21–28, 1961.

[6] Roger D. Maddux. A perspective on the theory of relation algebras. *Algebra Universalis*, 31(3):456–465, 1994.

[7] Konrad Piwakowski and Stanisław P. Radziszowski. $30 \leq R(3,3,4) \leq 31$. *J. Combin. Math. Combin. Comput.*, 27:135–141, 1998.