Determining Lower Bounds for Packing Densities of Non-layered Patterns Using Weighted Templates

Cathleen Battiste Presutti

Department of Mathematics
Bryn Mawr College, Bryn Mawr, Pennsylvania, USA
Ohio University - Lancaster, Lancaster, Ohio, USA
battiste@ohio.edu

Submitted: Sep 24, 2007; Accepted: Mar 17, 2008; Published: Mar 27, 2008
Mathematics Subject Classifications: 05A16

Abstract

The packing density of a permutation pattern π is the limiting value, $n \to \infty$, of the maximum proportion of subsequences of $\sigma \in S_n$ that are order-isomorphic to π. We generalize methods for obtaining lower bounds for the packing density of any pattern and demonstrate the methods' usefulness when patterns are non-layered.

1 Introduction

The permutation 52134 contains five subsequences 523, 524, 513, 514, 534 that are *order-isomorphic* (i.e. have the same relative order) to the permutation 312. In this situation, we can call the permutation 312 a *pattern*. In 1992, Herb Wilf first introduced the study of pattern containment during his address to the SIAM meeting on Discrete Mathematics. Since then, there has been a great deal of published results on pattern containment that deal with *pattern avoidance*, or the enumeration of permutations that do not contain any occurrences of a particular pattern. However, there is significantly less research on pattern containment involving permutations that contain the greatest number of subsequences which are order-isomorphic to a given pattern, commonly known as the *packing density*. Virtually all of this research has focused on a specific type of pattern known as *layered patterns*. Then in 2002, Albert, Atkinson, Handley, Holton & Stromquist (hereafter referred to as AAHHS) [1] determined a lower bound for the packing density of the non-layered pattern 2413 by using the permutation $\sigma = 35827146$, which contains a relatively large number of 2413-occurrences. In this paper, we will improve this lower bound and introduce a generalized method involving *weighted templates* as a way for computing lower bounds for $\delta(\pi)$, the packing density of a non-layered pattern π. Furthermore, we will show, via Theorem 3.6, that in order to achieve a better estimate for $\delta(\pi)$, it is necessary for the weights of the template to be non-uniform rather than uniform.

Let $\pi \in S_m$ be a permutation pattern and $\sigma \in S_n$ be a permutation. An *occurrence* of π in σ (or a π-*occurrence* in σ), is a subsequence of σ that is order-isomorphic to π. It is clear that $m \leq n$ in order for a π-occurrence in σ to exist and we will assume this to be the case unless stated otherwise. We denote $\nu(\pi, \sigma)$ to be the number of π-occurrences in σ. The *packing density of* π in σ, $\delta(\pi, \sigma)$, is the probability that a subsequence of σ is order-isomorphic to π with

$$\delta(\pi, \sigma) = \frac{\nu(\pi, \sigma)}{\binom{n}{m}}.$$

Since we are interested in finding the greatest number of π-occurrences among all permutations in S_n, let

$$\delta_n(\pi) = \max_{\sigma \in S_n} \delta(\pi, \sigma).$$

Clearly, $\{\delta_n(\pi)\}$ is a bounded sequence. In an unpublished work, Fred Galvin proved that this sequence is non-increasing and therefore approaches a limit. (Varying forms of the proof appear in a variety of published works, e.g.[1, 4].) Thus we define the *packing density of* π as

$$\delta(\pi) = \lim_{n \to \infty} \delta_n(\pi).$$

In [1], the lower bound for the packing density of the pattern 2413, $\delta(2413)$, was determined by starting with the permutation 35827146, which contains 17 occurrences of 2413-pattern and restricting consideration to permutations of the form $\sigma = \sigma_3\sigma_5\sigma_8\sigma_2\sigma_7\sigma_1\sigma_4\sigma_6$ where $n = |\sigma|$, $^n\!/_8 = |\sigma_i|$ for each i, $\sigma_i < \sigma_j$ whenever $i < j$, $|\sigma_k| = |\sigma_{k+1}|$ for each k, and each subpermutation σ_k is recursively composed in this same fashion. That is, the elements of each σ_k can be expressed in the form $\sigma_k = (\sigma_k)_3(\sigma_k)_5(\sigma_k)_8(\sigma_k)_2(\sigma_k)_7(\sigma_k)_1(\sigma_k)_4(\sigma_k)_6$ such that $(\sigma_k)_i < (\sigma_k)_j$ whenever $i < j$, and $|(\sigma_k)_i| = |(\sigma_{k+1})_j|$ for each k. This recursive composition continues until we are left with blocks of order 1.

Let p_n be the probability that an occurrence of 2413 is obtained when a four-term subsequence is chosen at random from σ. These occurrences arise in two ways. The first way is when all four points are picked from the same σ_i. The probability of this occurring is the product of the number of σ_i's, the probability of picking all four points in the same σ_i, and the probability $p_{n/8}$ that the four points formed a 2413-pattern in σ_i, which equals

$$8\left(\frac{1}{8}\right)^4 p_{n/8} = \frac{1}{512} p_{n/8}.$$

The second way is when each point is selected from a distinct σ_i. The probability of this occurring is the product of the number of occurrences of 2413 in 35827146, the number of different orderings of four elements, and the probability of picking one point from each σ_i, which equals

$$17 \cdot 4!\left(\frac{1}{8}\right)^4 = \frac{52}{512}.$$

Therefore $p_n = \frac{1}{512} p_{\frac{n}{8}} + {}^{51}/_{512}$. By taking the limit as n approaches infinity, we know that p_n approaches a limit p which in turn satisfies

$$p = \frac{1}{512} p + \frac{51}{512}, \text{ resulting with } p = \frac{51}{511}.$$

It is worth noting that in both cases, we assume when four points are selected at random, the points are distinct. While it is possible for a point to be repeated when we are randomly picking four points, the result is a subsequence that does not form a 2413-pattern in σ. Furthermore, as $n \to \infty$, the probability of repetition approaches 0.

Since $p_n \leq \delta_n(2413)$, it is clear that $p \leq \delta(2413)$ and finally $\frac{51}{511} \leq \delta(2413)$.

2 Template method

The technique used by AAHHS can be formalized by calling the permutation 35827146 a *template* for 2413. In general, any permutation $T = t_1 t_2 \ldots t_n \in S_n$ (or T_n , for short), can be a template for a pattern $\pi \in S_m$ where the structure of T and the number of occurrences of π in T (or π-occurrences in T) are used to determine a lower bound for $\delta(\pi)$. Although it will be preferable for $\nu(\pi, \sigma) = \max_{\sigma \in S_n} \nu(\pi, \sigma)$, it is not necessary. Also, without loss of generality, we will assume that $m \leq n$ and $\nu(\pi, T) > 0$.

Example 2.1. In [1], the template for $\pi = 2413$ was $T_8 = 35827146$ with $\nu(\pi, T_8) = 17$.

Using the template, we can always find a lower bound for $\delta(\pi)$. This will be particularly useful for cases when our pattern π is non-layered.

Theorem 2.2. *Given any pattern $\pi \in S_m$ and a template $T \in S_n$,*

$$\frac{\nu(\pi, T) \cdot m!}{n^m - n} \leq \delta(\pi).$$

Proof. Consider the permutations of the form $\sigma_T = \sigma_{t_1} \sigma_{t_2} \ldots \sigma_{t_n} \in S_k$ where $\sigma_i \leq \sigma_j$ whenever $i < j$, $|\sigma_i| = |\sigma_{i+1}|$ for each i, and each σ_i is recursively constructed in this same manner. Now let us determine the probability p_k ($k = |\sigma_T|$) of choosing m points from σ_T that form a π-pattern. One way for this to occur is when all of the selected m points of the π-pattern lie in a single σ_{t_i}. The probability of this happening is $n \left(\frac{1}{n} \right)^m p_{k/n}$.

Another way is when each of the m points of the π-pattern lies in a distinct σ_{t_i}. The probability of this is $\nu(\pi, T) \cdot m! \cdot \left(\frac{1}{n} \right)^m$. Hence

$$p_k = n \left(\frac{1}{n} \right)^m p_{k/n} + \nu(\pi, T) \cdot m! \cdot \left(\frac{1}{n} \right)^m. \tag{1}$$

It is worth noting that we are ignoring the possible case when an occurrence may involve less than m but more than 1 subpermutation. However, the probability of this case (or cases depending on the structure of π) can be computed similarly and added to equation (1) as necessary.

As $k \to \infty$, p_k approaches a limit p. In the limit,

$$p = n\left(\frac{1}{n}\right)^m p + \nu(\pi, T) \cdot m! \cdot \left(\frac{1}{n}\right)^m$$

which has a solution

$$p = \frac{\nu(\pi, T) \cdot m! \cdot \left(\frac{1}{n}\right)^m}{1 - n\left(\frac{1}{n}\right)^m} = \frac{\nu(\pi, T) \cdot m!}{n^m - n}$$

It is true that $p_k \leq \delta_k(\pi)$. Therefore $p = \dfrac{\nu(\pi, T) \cdot m!}{n^m - n} \leq \delta(\pi)$ as desired. □

Corollary 2.3. *For any $\pi \in S_m$,*

$$\frac{m!}{m^m - m} \leq \delta(\pi).$$

Proof. Use π as its own template, i.e. $\pi = T$. □

Example 2.4. Let $\pi = T_4 = 2413$. Then a lower bound for $\delta(2413)$ is

$$\frac{4!}{4^4 - 4} = \frac{2}{21}.$$

Example 2.5. Let $\pi = 2413$. Recall that for $T = T_8 = 35827146$ with $\nu(\pi, T) = 17$, the lower bound found for $\delta(2413)$ was $\dfrac{51}{511}$. Although this choice of T provided us with a substantial number of 2413-occurrences, a lower bound can be determined using any choice of T. Suppose we let $T_8 = 13862745$. Then $\nu(\pi, T_8) = 5$ and we find a lower bound for $\delta(2413)$ to be

$$p = \frac{\nu(\pi, T) \cdot m!}{n^m - n} = \frac{5 \cdot 4!}{8^4 - 8} = \frac{15}{511}.$$

This certainly makes a case for using a template that has as large of a number of π-occurrences as possible.

Example 2.6. Let $\pi = 2413$ and $T = T_{12} = 468(12)3(11)2(10)1579$. Then $\nu(\pi, T) = 86$. Now using the template method, we can find another lower bound for the packing density of 2413.

$$p = \frac{\nu(\pi, T) \cdot m!}{n^m - n} = \frac{86 \cdot 4!}{12^4 - 12} = \frac{172}{1727}.$$

While this lower bound is better than the one resulting from $T_4 = 2413$, the best lower bound for $\delta(2413)$ thus far comes from $T_8 = 35827146$.

3 Weighted templates

In Theorem 2.2, the permutations of the form $\sigma_T = \sigma_{t_1}\sigma_{t_2}\ldots\sigma_{t_n} \in S_k$ are structured such that $|\sigma_i| = |\sigma_{i+1}|$ for each i. In a probabilistic sense, each σ_i has the same likelihood of being chosen to be part of a π-occurrence. However in using a template to determine a lower bound for $\delta(\pi)$, we find that we can achieve better lower bounds by allowing the lengths of the σ_i's to vary. In doing so, we permit the probabilities that the σ_i's contain a point in a π-occurrence to vary.

Definition 3.1. A *weighted template*, $T = t_1t_2\ldots t_n \in S_n$, is a template together with a sequence of rational weights $\{w_i\}$, one for each t_i, such that $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$.

From here, we can construct a weighted lower bound for $\delta(\pi)$, again restricting our consideration to those permutations of the form $\sigma_T = \sigma_{t_1}\sigma_{t_2}\ldots\sigma_{t_n} \in S_k$. However the probability that σ_{t_i} will occur in a π-pattern will be equal to w_i, the weight of t_i. Now let us determine the weighted probability p_k of choosing m points from σ_T that form a π-pattern.

1. All of the selected m points of the π-pattern lie in a single σ_{t_i}:

   The points found in this type of π-occurrence all have the same weight, namely w_i. This yields
   $$\sum_{i=1}^{n}[(w_i)^m p_{kw_i}].$$

2. Each of the m points of the π-pattern lies in a distinct σ_{t_i}:

   In this case, each point in the π-occurrence has the weight of the σ_{t_i} in which it is located. Thus for $j = 1, 2, \ldots, \nu(\pi, T)$, let W_j be the product of the weights of the m distinct σ_{t_i}'s that make up the j^{th} π-occurrence. This yields
   $$m! \cdot \sum_{j=1}^{\nu(\pi,T)} W_j$$
   where
   $$W_j = \prod_{r=1}^{m} w_{(j,r)} = w_{(j,r)}w_{(j,r)}\cdots w_{(j,r)}$$
   is determined by the weights of the m points of the j^{th} π-occurrence. Hence our probability equation is
   $$p_k = \sum_{i=1}^{n}[(w_i)^m p_{kw_i}] + m! \cdot \sum_{j=1}^{\nu(\pi,T)} W_j$$
   As $k \to \infty$, p_k approaches a limit p. Thus
   $$p = \sum_{i=1}^{n}[(w_i)^m p] + m! \cdot \sum_{j=1}^{\nu(\pi,T)} W_j$$

with solution

$$p = \frac{m! \cdot \sum_{j=1}^{\nu(\pi,T)} W_j}{1 - \sum_{i=1}^{n} (w_i)^m}.$$

Theorem 3.2. *Given any pattern $\pi \in S_m$ and a weighted template $T \in S_n$,*

$$\frac{m! \cdot \sum_{j=1}^{\nu(\pi,T)} W_j}{1 - \sum_{i=1}^{n} (w_i)^m} \leq \delta(\pi).$$

Proof. This is proved in the same way as in Theorem 2.2. □

Naturally, we are interested in our choices for w_i. In [1], the weights were *uniform*; i.e. $w_i = \frac{1}{8}$ for $i = 1, 2, \ldots, 8$. More generally, it was shown from Theorem 2.2 that the uniform weighting is determined by the order of the template T. Thus given any $T \in S_n$, the uniform weighting would be $w_i = \frac{1}{n}$ for $i = 1, 2, \ldots, n$. Examples 2.4, 2.5, and 2.6 illustrated this.

Another option for assigning weights is based on the *multiplicity* of each t_i in the set of π-occurrences. We have found multiplicity weighting to provide us with improved lower bounds over those bounds from uniform weighting. We will prove this in Theorem 3.6, but we will set the stage with a few crucial lemmas first.

Lemma 3.3. *If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a sequence of non-negative real numbers, then*

$$n \cdot \left(\frac{\alpha_1 + \alpha_2 + \ldots + \alpha_n}{n} \right)^m = \frac{1}{n^{m-1}} \left(\sum_{i=1}^{n} \alpha_i \right)^m \leq \sum_{i=1}^{n} (\alpha_i)^m$$

Proof. Hölders inequality for sums [3] states that for a fixed real number $p > 1$ and $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots)$ in the real space l^p,

$$\sum_{i=1}^{\infty} |a_i b_i| \leq \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{\infty} |b_i|^q \right)^{1/q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore our inequality is achieved by letting $p = m$, $q = \frac{m}{m-1}$, $a = (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, \ldots)$, and $b = (1, 1, \ldots, 1, 0, \ldots)$ □

Lemma 3.4. *Let n be a positive integer. If $y_1, y_2, \ldots, y_n$ are positive real numbers and y is the arithmetic mean of the y_i's, then*

$$y^{(y_1+\ldots+y_n)} \leq (y_1)^{y_1} \cdots (y_n)^{y_n}.$$

Proof. For non-negative numbers, $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, the Log-sum inequality [2] states

$$\left(\sum_{i=1}^{n} a_i\right) \cdot \log\left(\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}\right) \leq \sum_{i=1}^{n} a_i \log\Big(\frac{a_i}{b_i}\Big)$$

Let $a_i = y_i$ and $b_i = 1$ for $i = 1, 2, \ldots, n$ to obtain

$$\left(\sum_{i=1}^{n} y_i\right) \cdot \log\left(\frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} 1}\right) \leq \sum_{i=1}^{n} y_i \log(y_i)$$

Hence,

$$\log\left[\left(\frac{y_1+\ldots+y_n}{n}\right)^{(y_1+\ldots+y_n)}\right] \leq \log\Big[(y_1)^{y_1} \cdots (y_n)^{y_n}\Big]$$

and therefore

$$\left(\frac{y_1+\ldots+y_n}{n}\right)^{(y_1+\ldots+y_n)} \leq (y_1)^{y_1} \cdots (y_n)^{y_n}$$

□

We also need the following version of the arithmetic mean – geometric mean inequality.

Lemma 3.5. *Let n be a positive integer. Suppose $x_1, \ldots, x_n$ and a are positive real numbers such that $a^n \leq x_1 \cdots x_n$. Then $na \leq x_1 + \ldots + x_n$.*

Now, we are ready to prove the main theorem of this paper.

Theorem 3.6. *Given any pattern $\pi \in S_m$ and a weighted template $T = t_1 t_2 \ldots t_n \in S_n$ with weights $\{w_i\}$ proportional to the multiplicities of the t_i's, then the uniformly weighted lower bound*

$$\frac{\nu(\pi, T) \cdot m! \cdot \Big(\frac{1}{n}\Big)^m}{1 - n\Big(\frac{1}{n}\Big)^m} \leq \frac{m! \cdot \sum_{j=1}^{\nu(\pi,T)} W_j}{1 - \sum_{i=1}^{n} (w_i)^m}.$$

Proof. For each t_i, let α_i be the number of times t_i appears among the $\nu(\pi,T)$ π-occurrences. For $i = 1, 2, \ldots, n$, let $w_i = \dfrac{\alpha_i}{\alpha_1 + \ldots + \alpha_n}$. Using Lemma 3.3, we have

$$\frac{\nu(\pi,\, T)\cdot m!\cdot\left(\frac{1}{n}\right)^m}{1-n\left(\frac{1}{n}\right)^m} = \frac{m!\cdot\sum_{j=1}^{\nu(\pi,\,T)}\left(\frac{1}{n}\right)^m}{1-\sum_{i=1}^{n}\left(\frac{\frac{\alpha_1+\ldots+\alpha_n}{n}}{\alpha_1+\ldots+\alpha_n}\right)^m} \tag{2}$$

$$\leq \frac{m!\cdot\sum_{j=1}^{\nu(\pi,\,T)}\left(\frac{1}{n}\right)^m}{1-\sum_{i=1}^{n}\left(\frac{\alpha_i}{\alpha_1+\ldots+\alpha_n}\right)^m}. \tag{3}$$

Now we focus on the numerator. Recall that the probability for the j^{th} π-occurrence is the product of the weights of the m distinct σ_{t_i}'s that make up the j^{th} π-occurrence, $W_j = \prod_{r=1}^{m} w_{(j,r)} = w_{(j,r)}w_{(j,r)}\cdots w_{(j,r)}$. Thus when we use multiplicity based weights, the sum of the probabilities for the $\nu(\pi,T)$ π-occurrences is

$$\sum_{j=1}^{\nu(\pi,T)} W_j = \sum_{j=1}^{\nu(\pi,T)}\prod_{r=1}^{m} w_{(r,j)}.$$

Let us consider the possible values this sum may take. When $\nu(\pi,T) = 0$, every α_i must equal 0 and therefore the sum is zero. For any $\nu(\pi,T) \geq 0$, at least m of the α_i's are greater than 0.

Suppose $\nu(\pi,T) \geq 1$. We proceed by assuming that all of the w_i's are non-zero knowing that the proof can be completed by omitting the weights that equal zero and adjusting the subscripts accordingly.

Using Lemma 3.4 for the inequality in (3) and recalling that $\sum_{i=1}^{n} w_i = 1$, we find that our weights satisfy

$$\frac{1}{n} = \left(\frac{w_1+\ldots+w_n}{n}\right)^{w_1+\ldots+w_n} \leq (w_1)^{w_1}\cdots(w_n)^{w_n}. \tag{4}$$

Since the w_i's are proportional to the multiplicities of the elements of our template T, we may write $\alpha_i = w_i\cdot m\cdot\nu(\pi,T)$ for each i. Thus raising both sides of (4) to the power $m\cdot\nu(\pi,T)$ gives us

$$\left(\frac{1}{n}\right)^{m\cdot\nu(\pi,T)} \leq (w_1)^{\alpha_1}\cdots(w_n)^{\alpha_n}.$$

Now we are able to regroup the weights in the product $(w_1)^{\alpha_1} \cdots (w_n)^{\alpha_n}$ with respect to the π-occurrences to get

$$(w_1)^{\alpha_1} \cdots (w_n)^{\alpha_n} = \prod_{r=1}^{m} w_{(r,1)} \cdots \prod_{r=1}^{m} w_{(r,\nu(\pi,T))} = W_1 \cdots W_{\nu(\pi,T)}.$$

Thus we have

$$\left(\frac{1}{n}\right)^{m\cdot\nu(\pi,T)} \leq W_1 \cdots W_{\nu(\pi,T)}.$$

This satisfies the hypothesis of Lemma 3.5 and yields

$$\nu(\pi,T)\left(\frac{1}{n}\right)^{m} \leq W_1 + \ldots + W_{\nu(\pi,T)}.$$

Therefore, we have proven the inequality

$$\frac{\nu(\pi,T)\cdot m!\cdot\left(\frac{1}{n}\right)^{m}}{1-n\left(\frac{1}{n}\right)^{m}} \leq \frac{m!\cdot\sum_{j=1}^{\nu(\pi,T)} W_j}{1-\sum_{i=1}^{n}(w_i)^m}.$$

□

Example 3.7. Let $\pi = 2413$ and $T_8 = 35827146$. Recall that there are 17 occurrences of π in T_8. By examination, it is easy to see that 1, 3, 6, and 8 occur in nine of the π-occurrences and 2, 4, 5, and 7 occur in eight of them. Thus $w_i = \frac{9}{68}$ for $i = 1, 3, 6, 8$ and $w_i = \frac{8}{68}$ for $i = 2, 4, 5, 7$. Substituting these into $p = \left(m!\cdot\sum_{j=1}^{\nu(\pi,T)} W_j\right) \Big/ \left(1-\sum_{i=1}^{n}(w_i)^m\right)$ yields

$$p = \frac{19954}{197581} > \frac{51}{511}.$$

However, we can do better than weighting by multiplicity with *optimized weights*. Using the general constraints of the weighted template, i.e. $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$, we can find w_i's that will maximize our weighted lower bound probability equation, $p = \left(m!\cdot\sum_{j=1}^{\nu(\pi,T)} W_j\right) \Big/ \left(1-\sum_{i=1}^{n}(w_i)^m\right)$. This can be done numerically by using the above constraints and *Maximize* in the *Optimization* package of the software *Maple*.

Example 3.8. Let $\pi = 2413$ and $T_8 = 35827146$. We discover that the *optimized lower bound* for packing density of 2413 calculated with *Maple* is 0.102473281354887008 and the weights are as follows:

$w_1 = 0.155447485901727828$	$w_5 = 0.0945525140982564488$
$w_2 = 0.0945525140982700074$	$w_6 = 0.155447485901717336$
$w_3 = 0.155447485901717366$	$w_7 = 0.0945525140982719504$
$w_4 = 0.0945525140982845098$	$w_8 = 0.155447485901732546$

Using this same technique for determining *optimal weights*, we have found better results for the lower bound of $\delta(2413)$ using $T_{12} = 468(12)3(11)2(10)1579$ and $T_{16} = 579(11)(16)4(15)3(14)2(13)168(10)(12)$, 0.103816093087368305 and 0.104250980068974874 respectively.

Example 3.9. Recall in Example 2.5, we demonstrated that the choice of template has a large impact on how good the lower bound is when using the uniformly weighted method. When we let $T_8 = 13862745$, we found the uniformly weighted lower bound to be $\frac{15}{511}$. However if we use this same template to find optimal weights, we get an optimized lower bound of 0.0949154510266377454 which is on par with the lower bounds calculated from uniform weights of other templates.

References

[1] M. H. Albert, M. D. Atkinson, C. C. Handley, D. A. Holton, W. Stromquist, On packing densities of permutations, *Electronic Journal of Combinatorics* **9** (2002), #R5.

[2] Cover, T. and Thomas, J., Elements of Information Theory, John Wiley & Sons, New York, 1991.

[3] Kreyszig, E., Introductory Functional Analysis with Applications, John Wiley & Sons, New York, 1989.

[4] Alkes Price: Packing densities of layered patterns, Ph.D. thesis, University of Pennsylvania, Philadelphia, PA, 1997.