

# The map asymptotics constant $t_g$

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## Abstract

The constant  $t_g$  appears in the asymptotic formulas for a variety of rooted maps on the orientable surface of genus  $g$ . Heretofore, studying this constant has been difficult. A new recursion derived by Goulden and Jackson for rooted cubic maps provides a much simpler recursion for  $t_g$  that leads to estimates for its asymptotics.

## 1 Introduction

Let  $\Sigma_g$  be the orientable surface of genus  $g$ . A *map* on  $\Sigma_g$  is a graph  $G$  embedded on  $\Sigma_g$  such that all components of  $\Sigma_g - G$  are simply connected regions. These components are called *faces* of the map. A map is rooted by distinguishing an edge, an end vertex of the edge and a side of the edge.

With  $M_{n,g}$  the number of rooted maps on  $\Sigma_g$  with  $n$  edges, Bender and Canfield [1] showed that

$$M_{n,g} \sim t_g n^{5(g-1)/2} 12^n \quad \text{as } n \rightarrow \infty, \quad (1)$$

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where the  $t_g$  are positive constants which can be calculated recursively using a complicated recursion involving, in addition to  $g$ , many other parameters. The first three values are

$$t_0 = \frac{2}{\sqrt{\pi}}, \quad t_1 = \frac{1}{24} \quad \text{and} \quad t_2 = \frac{7}{4320\sqrt{\pi}}.$$

Gao [3] showed that many other interesting families of maps also satisfy asymptotic formulas of the form

$$\alpha t_g (\beta n)^{5(g-1)/2} \gamma^n \tag{2}$$

and presented a table of  $\alpha$ ,  $\beta$  and  $\gamma$  for eleven families. Richmond and Wormald [5] showed that many families of unrooted maps have asymptotics that differ from the rooted asymptotics by a factor of four times the number of edges. See Goulden and Jackson [4] for a discussion of connections with mathematical physics.

Although  $\alpha$ ,  $\beta$  and  $\gamma$  in (2) seem relatively easy to compute, the common factor  $t_g$  has been difficult to study. A recursion for rooted ‘‘cubic’’ maps derived by Goulden and Jackson [4] leads to a much simpler recursion for  $t_g$  than that in [1]. We will use it to derive the following recursion and asymptotic estimate for  $t_g$ .

**Theorem 1** *Define  $u_g$  by  $u_1 = 1/10$  and*

$$u_g = u_{g-1} + \sum_{h=1}^{g-1} \frac{1}{R_1(g, h)R_2(g, h)} u_h u_{g-h} \quad \text{for } g \geq 2, \tag{3}$$

where

$$R_1(g, h) = \frac{[1/5]_g}{[1/5]_h [1/5]_{g-h}}, \quad R_2(g, h) = \frac{[4/5]_{g-1}}{[4/5]_{h-1} [4/5]_{g-h-1}}$$

and  $[x]_k$  is the rising factorial  $x(x+1)\cdots(x+k-1)$ . Then

$$\begin{aligned} t_g &= 8 \frac{[1/5]_g [4/5]_{g-1}}{\Gamma\left(\frac{5g-1}{2}\right)} \left(\frac{25}{96}\right)^g u_g \\ &\sim \frac{40 \sin(\pi/5) K}{\sqrt{2\pi}} \left(\frac{1440g}{e}\right)^{-g/2} \quad \text{as } g \rightarrow \infty, \end{aligned} \tag{4}$$

where  $u_g \sim K \doteq 0.1034$  is a constant.

## 2 Cubic Maps

A map is called cubic if all its vertices have degree 3. The dual of cubic maps are called triangular maps whose faces all have degree 3. Let  $T_{n,g}$  be the number of triangular maps on  $\Sigma_g$  with  $n$  vertices and let  $C_{n,g}$  be the number of cubic maps on  $\Sigma_g$  with  $2n$  vertices. It was shown in [2] that

$$T_{n,g} \sim 3 \left(3^7 \times 2^9\right)^{(g-1)/2} t_g n^{5(g-1)/2} (12\sqrt{3})^n \quad \text{as } n \rightarrow \infty. \tag{5}$$

Since a triangular map on  $\Sigma_g$  with  $v$  vertices has exactly  $2(v + 2g - 2)$  faces,

$$C_{n,g} = T_{n-2g+2,g} \sim 3 \times 6^{(g-1)/2} t_g n^{5(g-1)/2} (12\sqrt{3})^n \quad \text{as } n \rightarrow \infty. \quad (6)$$

Define

$$H_{n,g} = (3n + 2)C_{n,g} \quad \text{for } n \geq 1, \quad (7)$$

$$H_{-1,0} = 1/2, \quad H_{0,0} = 2 \quad \text{and} \quad H_{-1,g} = H_{0,g} = 0 \quad \text{for } g \neq 0.$$

Goulden and Jackson [4] derived the following recursion for  $(n, g) \neq (-1, 0)$ :

$$H_{n,g} = \frac{4(3n + 2)}{n + 1} \left( n(3n - 2)H_{n-2,g-1} + \sum_{i=-1}^{n-1} \sum_{h=0}^g H_{i,h} H_{n-2-i,g-h} \right). \quad (8)$$

This is significantly simpler than the recursion derived in [2]. We will use it to derive information about  $t_g$ .

### 3 Generating Functions

Define the generating functions

$$T_g(x) = \sum_{n \geq 0} T_{n,g} x^n, \quad C_g(x) = \sum_{n \geq 0} C_{n,g} x^n, \quad H_g(x) = \sum_{n \geq 0} H_{n,g} x^n \quad \text{and} \quad F_g(x) = x^2 H_g(x).$$

It was shown in [2] that  $T_g(x)$  is algebraic for each  $g \geq 0$ , and

$$T_0(x) = \frac{1}{2} t^3 (1 - t)(1 - 4t + 2t^2) \quad \text{with} \quad x = \frac{1}{2} t(1 - t)(1 - 2t), \quad (9)$$

where  $t = t(x)$  is a power series in  $x$  with non-negative coefficients.

It follows from (6) and (7) that

$$C_g(x) = x^{2g-2} T_g(x) \quad \text{for } g \geq 0, \quad (10)$$

$$F_g(x) = 3x^3 C'_g(x) + 2x^2 C_g(x) \quad \text{for } g \geq 1. \quad (11)$$

We also have

$$\begin{aligned} F_0(x) &= H_{0,0} x^2 + \sum_{n \geq 1} (3n + 2) C_{n,0} x^{n+2} \\ &= 2x^2 + 3x^3 C'_0(x) + 2x^2 C_0(x) \\ &= 2x^2 + 3x T'_0(x) - 4T_0(x) \\ &= \frac{1}{2} t^2 (1 - t), \end{aligned} \quad (12)$$

where we have used (9). Hence  $C_g(x)$  and  $F_g(x)$  are both algebraic for all  $g \geq 0$ .

In the following we assume  $g \geq 1$ . From the recursion (8), we have

$$\begin{aligned} \frac{1}{4} \sum_{n \geq 0} \frac{n+1}{3n+2} H_{n,g} x^n &= \sum_{n \geq 1} n(3n-2) H_{n-2,g-1} x^n \\ &+ 2 \sum_{n \geq 0} H_{-1,0} H_{n-1,g} x^n + x^2 \sum_{h=0}^g H_h(x) H_{g-h}(x). \end{aligned}$$

Using (7) with a bit manipulation, we can rewrite the above equation as

$$\begin{aligned} \frac{1}{4} \sum_{n \geq 0} (n+1) C_{n,g} x^n &= 3x^2 F''_{g-1}(x) + x F'_{g-1}(x) + x H_{-1,g-1} \\ &+ x^{-1} F_g(x) + x^{-2} \sum_{h=0}^g F_h(x) F_{g-h}(x). \end{aligned}$$

With  $\delta_{i,j}$  the Kronecker delta, this becomes

$$\begin{aligned} x^3 C'_g(x) + x^2 C_g(x) &= 12x^4 F''_{g-1}(x) + 4x^3 F'_{g-1}(x) + 2x^3 \delta_{g,1} \\ &+ 4x F_g(x) + 8F_0(x) F_g(x) + 4 \sum_{h=1}^{g-1} F_h(x) F_{g-h}(x). \end{aligned}$$

It follows from (11) that

$$\begin{aligned} (1 - 12x - 24F_0(x)) F_g(x) &= 36x^4 F''_{g-1}(x) + 12x^3 F'_{g-1}(x) + 6x^3 \delta_{g,1} \\ &+ 12 \sum_{h=1}^{g-1} F_h(x) F_{g-h}(x) - x^2 C_g(x). \end{aligned} \quad (13)$$

Substituting (12) and (9) into (13), we obtain

$$\begin{aligned} F_g(x) &= \frac{1}{1 - 6t + 6t^2} \left( 36x^4 F''_{g-1}(x) + 12x^3 F'_{g-1}(x) + 6x^3 \delta_{g,1} \right. \\ &\quad \left. + 12 \sum_{h=1}^{g-1} F_h(x) F_{g-h}(x) - x^2 C_g(x) \right). \end{aligned} \quad (14)$$

We now show that this equation can be used to calculate  $C_g(x)$  more easily than the method in [2]. For this purpose we set  $s = 1 - 6t + 6t^2$  and show inductively that  $C_g(x)$  is a polynomial in  $s$  divided by  $s^a$  for some integer  $a = a(g) > 0$ . (It can be shown that  $a = 5g - 3$  is the smallest such  $a$ , but we do not do so.) The method for calculating  $C_g(x)$  follows from the proof. Then we have

$$x^2 = \frac{1}{432} (s-1)^2 (2s+1) \quad \text{and} \quad \frac{ds}{dx} = \frac{144x}{s(s-1)}. \quad (15)$$

Thus

$$\begin{aligned} x \frac{d}{dx} &= x \frac{ds}{dx} \frac{d}{ds} = \frac{(s-1)(2s+1)}{3s} \frac{d}{ds}, \\ \frac{d^2}{dx^2} &= \left( \frac{ds}{dx} \right)^2 \frac{d^2}{ds^2} + \frac{d(ds/dx)}{dx} \frac{d}{ds} = \frac{48(2s+1)}{s^2} \frac{d^2}{ds^2} - \frac{48(s+1)}{s^3} \frac{d}{ds}. \end{aligned}$$

From the above and (11)

$$F_g(x) + \frac{x^2 C_g}{1 - 6t + 6t^2} = x^2 \left( 3x \frac{dC_g}{dx} + \frac{(2s+1)C_g}{s} \right) = \frac{x^2(2s+1)}{s} \frac{d((s-1)C_g)}{ds}.$$

With some algebra, (14) can be rewritten as

$$\begin{aligned} \frac{d((s-1)C_g)}{ds} &= \frac{4(s-1)^2(2s+1)}{s^2} \frac{d^2 F_{g-1}}{ds^2} + \frac{4(s-1)}{s^3} \frac{dF_{g-1}}{ds} \\ &+ \frac{5184}{(s-1)^2(2s+1)^2} \sum_{h=1}^{g-1} F_h F_{g-h} \quad \text{for } g \geq 2. \end{aligned} \quad (16)$$

In what follows  $P(s)$  stands for a polynomial in  $s$  and  $a$  a positive integer, both different at each occurrence. It was shown in [2] that

$$C_1(x) = T_1(x) = \frac{1-s}{12s^2}.$$

By (11), (15) and the induction hypothesis, the right hand side of (16) has the form  $P(s)/s^a$ . Integrating,  $(s-1)C_g = P(s)/s^a + K \log s$ . Since we know  $C_g(x)$  is algebraic, so is  $(s-1)C_g$  and hence  $K = 0$ . Since  $s = 1$  corresponds to  $x = 0$ ,  $C_g$  is defined there. It follows that  $P(s)$  in  $(s-1)C_g = P(s)/s^a$  is divisible by  $s-1$ , completing the proof.

Using Maple, we obtained

$$\begin{aligned} C_2 &= \frac{1}{2^6 3^4} \frac{(2s+1)(17s^2+60s+28)(1-s)^3}{s^7}, \\ C_3 &= \frac{1}{2^9 3^8} \frac{(5052s^4 - 747s^3 - 33960s^2 - 35620s - 9800)(2s+1)^2(s-1)^5}{s^{12}}, \\ C_4 &= \frac{1}{2^{14} 3^{11}} \frac{P_4(s)(2s+1)^3(s-1)^7}{s^{17}}, \\ C_5 &= \frac{1}{2^{17} 3^{14}} \frac{P_5(s)(2s+1)^4(1-s)^9}{s^{22}}, \end{aligned}$$

where

$$\begin{aligned} P_4(s) &= -12458544 - 63378560s - 103689240s^2 - 42864016s^3 \\ &\quad + 31477893s^4 + 20750256s^5 + 417636s^6, \\ P_5(s) &= 7703740800 + 50294009360s + 117178660480s^2 \\ &\quad + 100386081272s^3 - 16827627792s^4 - 67700509763s^5 \\ &\quad - 21455389524s^6 + 4711813020s^7 + 1394857272s^8. \end{aligned}$$

## 4 Generating Function Asymptotics

Suppose  $A(x)$  is an algebraic function and has the following asymptotic expansion around its dominant singularity  $1/r$ :

$$A(x) = \sum_{j=l}^k a_j (1-rx)^{j/2} + O\left((1-rx)^{(k+1)/2}\right),$$

where  $a_j$  are not all zero. Then we write

$$A(x) \approx \sum_{j=l}^k a_j (1 - rx)^{j/2}.$$

The following lemma is proved in [2].

**Lemma 1** For  $g \geq 0$ ,  $T_g(x)$  is algebraic,

$$T_0(x) \approx \frac{\sqrt{3}}{72} - \frac{5}{216} + \frac{1}{54\sqrt{6}}(1 - 12\sqrt{3}x)^{3/2},$$

$$T_g(x) \approx 3(3^7 \times 2^9)^{(g-1)/2} t_g \Gamma\left(\frac{5g-3}{2}\right) (1 - 12\sqrt{3}x)^{-(5g-3)/2} \quad \text{for } g \geq 1.$$

Let

$$f_g = 24^{-3/2} 6^{g/2} \Gamma\left(\frac{5g-1}{2}\right) t_g. \quad (17)$$

Using Lemma 1, (10) and (11), we obtain

$$C_g(x) \approx \frac{288}{(5g-3)} f_g (1 - 12\sqrt{3}x)^{-(5g-3)/2} \quad \text{for } g \geq 1,$$

$$F_g(x) \approx f_g (1 - 12\sqrt{3}x)^{-(5g-1)/2} \quad \text{for } g \geq 1.$$

As noted in [2], the function  $t(x)$  of (9) has the following asymptotic expansion around its dominant singularity  $x = \frac{1}{12\sqrt{3}}$ :

$$t \approx \frac{3 - \sqrt{3}}{6} - \frac{\sqrt{2}}{6} (1 - 12\sqrt{3}x)^{1/2}.$$

Using this and (12), we obtain

$$F_0(x) \approx \frac{3 - \sqrt{3}}{72} + f_0 (1 - 12\sqrt{3}x)^{1/2},$$

$$\frac{1}{1 - 6t + 6t^2} \approx \frac{\sqrt{6}}{2} (1 - 12\sqrt{3}x)^{-1/2}.$$

Comparing the coefficients of  $(1 - 12\sqrt{3}x)^{(5g-1)/2}$  on both sides of (14), we obtain

$$f_g = \frac{\sqrt{6}}{96} (5g-4)(5g-6) f_{g-1} + 6\sqrt{6} \sum_{h=1}^{g-1} f_h f_{g-h}. \quad (18)$$

Letting

$$u_g = f_g \left(\frac{25\sqrt{6}}{96}\right)^{-g} \frac{6\sqrt{6}}{[1/5]_g [4/5]_{g-1}}.$$

and using (17), the recursion (18) becomes (3).

## 5 Asymptotics of $t_g$

It follows immediately from (3) that  $u_g \geq u_{g-1}$  for all  $g \geq 2$ . To show that  $u_g$  approaches a limit  $K$  as  $g \rightarrow \infty$ , it suffices to show that  $u_g$  is bounded above. The value of  $K$  is then calculated using (3).

We use induction to prove  $u_g \leq 1$  for all  $g \geq 1$ . Since  $u_1 = \frac{1}{10}$  and  $u_2 = u_1 + \frac{1}{480}$ , we can assume  $g \geq 3$  for the induction step. From now on  $g \geq 3$ .

Note that

$$\begin{aligned} R_1(g, 1)R_2(g, 1) &= 5(g - \frac{4}{5})(g - \frac{6}{5}) > 5(g - \frac{4}{5})(g - \frac{9}{5}) \\ R_1(g, 2)R_2(g, 2) &= \frac{25}{24}(g - \frac{6}{5})(g - \frac{11}{5})\left(5(g - \frac{4}{5})(g - \frac{9}{5})\right) \\ &> \frac{25}{24}(g - \frac{6}{5} + \frac{4}{5})(g - \frac{11}{5} - \frac{4}{5})\left(5(g - \frac{4}{5})(g - \frac{9}{5})\right) \\ &\geq 2(g - 3)\left(5(g - \frac{4}{5})(g - \frac{9}{5})\right). \end{aligned}$$

Note that  $R_i(g, h) = R_i(g, g - h)$  and, for  $h < g/2$ ,  $\frac{R_i(g, h+1)}{R_i(g, h)} \geq 1$ . Combining all these observations and the induction hypothesis with (3) we have

$$\begin{aligned} u_g &= u_{g-1} + \sum_{h=1}^{g-1} \frac{u_h u_{g-h}}{R_1(g, h)R_2(g, h)} \\ &< u_{g-1} + \frac{2u_1 u_{g-1}}{5(g - \frac{4}{5})(g - \frac{9}{5})} + \sum_{h=2}^{g-2} \frac{1}{R_1(g, 2)R_2(g, 2)} \\ &< u_{g-1} + \frac{1/5}{5(g - \frac{4}{5})(g - \frac{9}{5})} + \frac{1/2}{5(g - \frac{4}{5})(g - \frac{9}{5})} \\ &< u_{g-1} + \frac{1}{5g - 9} - \frac{1}{5g - 4}. \end{aligned}$$

Hence

$$u_g < u_2 + \sum_{k=3}^g \left( \frac{1}{5k - 9} - \frac{1}{5k - 4} \right) < u_2 + \frac{1}{5 \times 3 - 9} < 1.$$

The asymptotic expression for  $t_g$  in (4) is obtained by using

$$[x]_k = \frac{\Gamma(x + k)}{\Gamma(x)}, \quad \Gamma(1/5)\Gamma(4/5) = \frac{\pi}{\sin(\pi/5)},$$

and Stirling's formula

$$\Gamma(ag + b) \sim \sqrt{2\pi}(ag)^{b-1/2} \left(\frac{ag}{e}\right)^{ag} \quad \text{as } g \rightarrow \infty,$$

for constants  $a > 0$  and  $b$ .

## 6 Open Questions

We list some open questions.

- From (18), we can show that  $f(z) = \sum_{g \geq 1} f_g z^g$  satisfies the following differential equation

$$f(z) = 72\sqrt{6}(f(z))^2 + \frac{\sqrt{6}}{96}z \left( 25z^2 f''(z) + 25zf'(z) - f(z) - \frac{\sqrt{6}}{72} \right).$$

The asymptotic expression of  $f_g$  implies that  $f(z)$  cannot be algebraic. Can one show that  $f(z)$  is not D-finite, that is,  $f(z)$  does not satisfy a linear differential equation?

- There is a constant  $p_g$  that plays a role for maps on non-orientable like  $t_g$  plays for maps on orientable surfaces [3]. Is there a recursion for maps on non-orientable surfaces that can be used to derive a theorem akin to Theorem 1 for  $p_g$ ?
- Find simple recursions akin to (8) for other classes of rooted maps that lead to simple recursive calculations of their generating functions as in (16).

## References

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## Corrigendum – submitted Jul 28, 2008

The last displayed equation in the paper contains two incorrect coefficients, namely the first and last. The correct equation is

$$f(z) = 6\sqrt{6}(f(z))^2 + \frac{\sqrt{6}}{96}z \left( 25z^2 f''(z) + 25z f'(z) - f(z) + \frac{\sqrt{6}}{72} \right).$$