Hurwitz Equivalence in Tuples of Generalized Quaternion Groups and Dihedral Groups

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Abstract

Let Q_{2^m} be the generalized quaternion group of order 2^m and D_N the dihedral group of order $2N$. We classify the orbits in $Q_{2^m}^n$ and $D_{p^m}^n$ (p prime) under the Hurwitz action.

1 The Hurwitz Action

Let G be a group. For $a, b \in G$, let $a^b = b^{-1}ab$ and ${}^b a = bab^{-1}$. The Hurwitz action on G^n ($n \geq 2$) is an action of the n-string braid group B_n on G^n. Recall that B_n is given by the presentation

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| > 2;\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \leq i \leq n-2\rangle.$$

The action of σ_i on G^n is defined by

$$\sigma_i(a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, a_{i+1}, a_i^{a_{i+1}}, a_{i+2}, \dots, a_n),$$

where $(a_1, \dots, a_n) \in G^n$. Note that

$$\sigma_i^{-1}(a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, {}^{a_i}a_{i+1}, a_i, a_{i+2}, \dots, a_n).$$

An action by σ_i or σ_i^{-1} on G^n is called a Hurwitz move. Two tuples $(a_1, \dots, a_n)$, $(b_1, \dots, b_n) \in G^n$ are called (Hurwitz) equivalent, denoted as $(a_1, \dots, a_n) \sim (b_1, \dots, b_n)$, if they are in the same B_n-orbit. The (Hurwitz) equivalence class of $(a_1, \dots, a_n) \in G^n$, i.e., the B_n-orbit of $(a_1, \dots, a_n)$, is denoted by $[a_1, \dots, a_n]$.

If G is a nonabelian group, in general, the B_n-orbits in G^n are not known. In [1], Ben-Itzhak and Teicher determined all B_n-orbits in S_m^n represented by $(t_1, \dots, t_n)$, where S_m is the symmetric group, each t_i is a transposition and $t_1 \cdots t_n = 1$. It is obvious that if

$a_1, \ldots, a_n \in G$ generate a finite subgroup, then the B_n-orbit of $(a_1, \ldots, a_n)$ in G^n is finite. It has been proved that if $s_1, \ldots, s_n \in \mathrm{GL}(\mathbb{R}^n)$ are reflections such that the B_n-orbit of $(s_1, \ldots, s_n)$ is finite, then the group generated by $s_1, \ldots, s_n$ is finite; see [2] and [3].

It is natural to ask which types of nonabelian group G allow complete determination of the B_n-orbits in G^n. In this paper, we show that when G is the generalized quaternion group Q_{2^m} or the dihedral group D_{p^m} of order $2p^m$, where p is a prime, the answer to the above question is affirmative.

2 The Generalized Quaternion Group

Let $m \geq 2$. The generalized quaternion group Q_{2^m} of order 2^m is given by the presentation

$$Q_{2^m} = \langle \alpha, \beta \mid \alpha^{2^{m-1}} = 1,\ \alpha^{2^{m-2}} = \beta^2,\ \beta\alpha\beta^{-1} = \alpha^{-1}\rangle.$$

Each element of Q_{2^m} can be uniquely written as $\alpha^i\beta^j$, where $0 \leq i < 2^{m-1}$ and $0 \leq j \leq 1$. We have

$$(\alpha^i\beta^j)^{\alpha^k\beta^l} = \alpha^{(-1)^l(i-2kj)}\beta^j, \tag{2.1}$$

$$^{\alpha^i\beta^j}(\alpha^k\beta^l) = \alpha^{(-1)^j k+2il}\beta^l. \tag{2.2}$$

Thus in $Q_{2^m}^n$, a Hurwitz move gives one of the following equivalences:

$$\begin{aligned}(\cdots, \alpha^i\beta^j, \alpha^k\beta^l, \cdots) &\sim (\cdots, \alpha^k\beta^l, \alpha^{(-1)^l(i-2kj)}\beta^j, \cdots),\\ (\cdots, \alpha^i\beta^j, \alpha^k\beta^l, \cdots) &\sim (\cdots, \alpha^{(-1)^j k+2il}\beta^l, \alpha^i\beta^j, \cdots).\end{aligned}$$

For easier reading, we rewrite the above equivalences, omitting the $\cdots$'s, with $(j, l) = (0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ respectively.

$$(\alpha^i, \alpha^k) \sim (\alpha^k, \alpha^i), \tag{2.3}$$

$$\begin{cases}(\alpha^i,\ \alpha^k\beta) \sim (\alpha^k\beta,\ \alpha^{-i}),\\ (\alpha^i,\ \alpha^k\beta) \sim (\alpha^{k+2i}\beta,\ \alpha^i),\end{cases} \tag{2.4}$$

$$\begin{cases}(\alpha^i\beta,\ \alpha^k) \sim (\alpha^k,\ \alpha^{i-2k}\beta),\\ (\alpha^i\beta,\ \alpha^k) \sim (\alpha^{-k},\ \alpha^i\beta),\end{cases} \tag{2.5}$$

$$\begin{cases}(\alpha^i\beta,\ \alpha^k\beta) \sim (\alpha^k\beta,\ \alpha^{-i+2k}\beta) = (\alpha^{i+(k-i)}\beta,\ \alpha^{k+(k-i)}\beta),\\ (\alpha^i\beta,\ \alpha^k\beta) \sim (\alpha^{-k+2i}\beta,\ \alpha^i\beta) = (\alpha^{i-(k-i)}\beta,\ \alpha^{k-(k-i)}\beta).\end{cases} \tag{2.6}$$

Lemma 2.1. (i) $(\alpha^i,\ \alpha^j\beta) \sim (\alpha^{-i},\ \alpha^{j+2i}\beta)$ *for all* $i, j \in \mathbb{Z}$.

(ii) $(\alpha^i\beta,\ \alpha^j\beta) \sim (\alpha^{i+k(j-i)}\beta,\ \alpha^{j+k(j-i)}\beta)$ *for all* $i, j, k \in \mathbb{Z}$.

(iii) *Let* $\tau, \nu, e, f \in \mathbb{Z}$ *such that* $0 \leq \nu \leq m-2$ *and* $e \not\equiv f \pmod 2$. *Then for every* $g \in \mathbb{Z}$,

$$(\alpha^{\tau+2^\nu e}\beta,\ \alpha^{\tau+2^\nu f}\beta) \sim (\alpha^{\tau+2^\nu(e+g)}\beta,\ \alpha^{\tau+2^\nu(f+g)}\beta).$$

Proof. (i) We have

$$\begin{aligned}(\alpha^i, \alpha^j\beta) &\sim (\alpha^j\beta, \alpha^{-i}) && \text{(the first eq. of (2.4))}\\ &\sim (\alpha^{-i}, \alpha^{j+2i}\beta) && \text{(the first eq. of (2.5)).}\end{aligned}$$

(ii) follows from (2.6).

(iii) In (ii) let $i = \tau + 2^\nu e$, $j = \tau + 2^\nu f$ and choose $k \in \mathbb{Z}$ such that $k2^\nu(f-e) \equiv g2^\nu \pmod{2^{m-1}}$. □

3 B_n-Orbits in $Q_{2^m}^n$

Let G be a group. For $\boldsymbol{a} = (a_1, \dots, a_n) \in G^n$, define $\pi(\boldsymbol{a}) = a_1 \cdots a_n \in G$. $\pi(\boldsymbol{a})$ is an invariant of the Hurwitz action on G^n.

For $\boldsymbol{a} = (\alpha^{i_1}\beta^{j_1}, \dots, \alpha^{i_n}\beta^{j_n}) \in Q_{2^m}^n$, where $0 \le i_k < 2^{m-1}$ and $0 \le j_k \le 1$, let

$$\Lambda(\boldsymbol{a}) = \text{the multi set } \big\{\min\{i_k, 2^{m-1} - i_k\} : j_k = 0\big\},$$

$$\Gamma(\boldsymbol{a}) = \{i_k : j_k = 1\}.$$

For example, if $\boldsymbol{a} = (\alpha^3\beta, \alpha^4\beta, \alpha^3\beta, \alpha\beta)$, $\boldsymbol{b} = (\alpha^6, \alpha\beta, 1, \alpha^2) \in Q_{2^4}^4$, then $\Lambda(\boldsymbol{a}) = \varnothing$, $\Lambda(\boldsymbol{b}) = \{0, 2, 2\}$, $\Gamma(\boldsymbol{a}) = \{1, 3, 4\}$, $\Gamma(\boldsymbol{b}) = \{1\}$. $\Lambda(\boldsymbol{a})$ is an invariant of the Hurwitz action on $Q_{2^m}^n$. In fact, it is easy to see that $\Lambda(\boldsymbol{a})$ is invariant under each of the Hurwitz moves in (2.3) – (2.6).

To determine the B_n-orbits in $Q_{2^m}^n$, we first partition $Q_{2^m}^n$ into suitable subsets. Let

$$\mathcal{A} = \{\boldsymbol{a} \in Q_{2^m}^n : \Gamma(\boldsymbol{a}) = \varnothing\}.$$

For each $1 \le \nu \le m-1$ and $0 \le \tau < 2^\nu$, let

$$\mathcal{B}_{\nu,\tau} = \big\{\boldsymbol{a} \in Q_{2^m}^n : \min(\{\nu_2(i) : i \in \Lambda(\boldsymbol{a})\} \cup \{m-2\}) = \nu - 1,\ \varnothing \ne \Gamma(\boldsymbol{a}) \subset \tau + 2^\nu\mathbb{Z}\big\},$$

where ν_2 is the 2-adic order. For each $0 \le \nu \le m-2$ and $0 \le \tau < 2^\nu$, let

$$\begin{aligned}\mathcal{C}_{\nu,\tau} = \big\{\boldsymbol{a} \in Q_{2^m}^n : \min(\{\nu_2(i) : i \in \Lambda(\boldsymbol{a})\} \cup \{m-2\}) \ge \nu,\ \Gamma(\boldsymbol{a}) \subset \tau + 2^\nu\mathbb{Z},\\ \exists j, j' \in \Gamma(\boldsymbol{a}) \text{ such that } \nu_2(j - j') = \nu\big\}.\end{aligned}$$

Then

$$Q_{2^m}^n = \mathcal{A} \,\dot\cup \Big(\dot{\bigcup_{\substack{1 \le \nu \le m-1\\ 0 \le \tau < 2^\nu}}} \mathcal{B}_{\nu,\tau}\Big) \,\dot\cup \Big(\dot{\bigcup_{\substack{0 \le \nu \le m-2\\ 0 \le \tau < 2^\nu}}} \mathcal{C}_{\nu,\tau}\Big).$$

It is routine to check that each of $\mathcal{A}$, $\mathcal{B}_{\nu,\tau}$, $\mathcal{C}_{\nu,\tau}$ is invariant under the Hurwitz moves in (2.3) – (2.6). Thus, $\mathcal{A}$, $\mathcal{B}_{\nu,\tau}$ and $\mathcal{C}_{\nu,\tau}$ are invariant under the Hurwitz equivalence. Therefore, to determine the B_n-orbits in $Q_{2^m}^n$, it suffices to find a set of representatives of the B_n-orbits in each of $\mathcal{A}$, $\mathcal{B}_{\nu,\tau}$ and $\mathcal{C}_{\nu,\tau}$.

For $\boldsymbol{a} = (\alpha^{i_1}\beta^{j_1}, \dots, \alpha^{i_n}\beta^{j_n}) \in \mathcal{C}_{\nu,\tau}$, where $0 \le i_k < 2^{m-1}$ and $0 \le j_k \le 1$, let

$$t(\boldsymbol{a}) = |\{k : j_k = 1 \text{ and } i_k \equiv \tau \pmod{2^{\nu+1}}\}|.$$

We claim that $t(\boldsymbol{a})$ is an invariant under the Hurwitz equivalence. Once again, it is easy to see that $t(\boldsymbol{a})$ is invariant under the Hurwitz moves in (2.3) – (2.6).

Theorem 3.1. (i) *The B_n-orbits in $\mathcal{A}$ are represented by*

$$(\alpha^{i_1}, \dots, \alpha^{i_n}),$$

where $0 \le i_1 \le \dots \le i_n < 2^{m-1}$.

(ii) *Let $1 \le \nu \le m-1$ and $0 \le \tau < 2^\nu$. The B_n-orbits in $\mathcal{B}_{\nu,\tau}$ are represented by*

$$(\alpha^{i_1}, \dots, \alpha^{i_s},\ \alpha^{\tau+2^\nu e}\beta,\ \alpha^\tau\beta, \dots, \alpha^\tau\beta), \tag{3.1}$$

where $0 \le i_1 \le \dots \le i_s \le 2^{m-2}$, $\min\{\nu_2(i_1), \dots, \nu_2(i_s), m-2\} = \nu - 1$, $0 \le e < 2^{m-1-\nu}$.

(iii) *Let $0 \le \nu \le m-2$ and $0 \le \tau < 2^\nu$. The B_n-orbits in $\mathcal{C}_{\nu,\tau}$ are represented by*

$$(\alpha^{i_1}, \dots, \alpha^{i_s},\ \alpha^{\tau+2^\nu e}\beta,\ \alpha^{\tau+2^\nu}\beta,\ \dots,\ \alpha^{\tau+2^\nu}\beta,\ \underbrace{\alpha^\tau\beta, \dots, \alpha^\tau\beta}_{t}), \tag{3.2}$$

where $0 \le i_1 \le \dots \le i_s \le 2^{m-2}$, $\min\{\nu_2(i_1), \dots, \nu_2(i_s), m-2\} \ge \nu$, $0 \le e < 2^{m-1-\nu}$, $e \equiv 1 \pmod 2$, $t > 0$.

Proof. (i) is obvious.

(ii) We first observe that different tuples in (3.1) have different combinations of invariants $\Lambda(\boldsymbol{a})$ and $\pi(\boldsymbol{a})$. Thus, different tuples in (3.1) are nonequivalent.

Next, we show that every $\boldsymbol{a} \in \mathcal{B}_{\nu,\tau}$ is equivalent to one of the tuples in (3.1). We may assume that

$$\boldsymbol{a} = (\alpha^{i'_1}, \dots, \alpha^{i'_{s-1}},\ \alpha^{\tau+2^\nu e_1}\beta,\ \dots,\ \alpha^{\tau+2^\nu e_t}\beta,\ \alpha^{j_0}), \tag{3.3}$$

where $\nu_2(j_0) = \nu - 1$. Using (2.5) repeatedly, we have

$$\begin{aligned} &(\alpha^{\tau+2^\nu e_1}\beta,\ \dots,\ \alpha^{\tau+2^\nu e_t}\beta,\ \alpha^{j_0}) \\ \sim\ &(\alpha^{\tau+2^\nu e_1}\beta,\ \dots,\ \alpha^{\tau+2^\nu e_{t-1}}\beta,\ \alpha^{j_1},\ \alpha^{\tau+2^\nu e'_t}\beta) \\ \sim\ &\cdots \\ \sim\ &(\alpha^{j_t},\ \alpha^{\tau+2^\nu e'_1}\beta,\ \dots,\ \alpha^{\tau+2^\nu e'_t}\beta), \end{aligned} \tag{3.4}$$

where $\nu_2(j_0) = \dots = \nu_2(j_t) = \nu - 1$, $e'_1, \dots, e'_{t-1}$ are even and e'_t is odd. Using Lemma 2.1 (iii) repeatedly, we have

$$\begin{aligned} &(\alpha^{\tau+2^\nu e'_1}\beta,\ \dots,\ \alpha^{\tau+2^\nu e'_{t-1}}\beta,\ \alpha^{\tau+2^\nu e'_t}\beta) \\ \sim\ &(\alpha^{\tau+2^\nu e'_1}\beta,\ \dots,\ \alpha^{\tau+2^\nu f_1}\beta,\ \alpha^\tau\beta) \qquad (f_1 \text{ odd}) \\ \sim\ &\cdots \\ \sim\ &(\alpha^{\tau+2^\nu f_{t-1}}\beta,\ \alpha^\tau\beta,\ \dots,\ \alpha^\tau\beta) \qquad (f_{t-1} \text{ odd}). \end{aligned} \tag{3.5}$$

Combining (3.3) – (3.5), we have

$$\begin{aligned}\boldsymbol{a} &\sim (\alpha^{i'_1}, \dots, \alpha^{i'_{s-1}}, \alpha^{j_t}, \alpha^{\tau+2^\nu f_{t-1}}\beta, \alpha^\tau\beta, \dots, \alpha^\tau\beta)\\ &\sim (\alpha^{i_1}, \dots, \alpha^{i_s}, \alpha^{\tau+2^\nu e}\beta, \alpha^\tau\beta, \dots, \alpha^\tau\beta) \qquad \text{(by Lemma 2.1 (i))},\end{aligned}$$

where $0 \le i_1 \le \dots \le i_s \le 2^{m-2}$ and $0 \le e < 2^{m-1-\nu}$.

(iii) Different tuples in (3.2) have different combinations of invariants $\Lambda(\boldsymbol{a})$, $t(\boldsymbol{a})$ and $\pi(\boldsymbol{a})$. Hence different tuples in (3.2) are nonequivalent.

It remains to show that every $\boldsymbol{a} \in \mathcal{C}_{\nu,\tau}$ is equivalent to one of the tuples in (3.2). We may assume that

$$\boldsymbol{a} = (\alpha^{i_1}, \dots, \alpha^{i_s}, \alpha^{\tau+2^\nu e_1}\beta, \dots, \alpha^{\tau+2^\nu e_u}\beta, \alpha^{\tau+2^\nu f_1}\beta, \dots, \alpha^{\tau+2^\nu f_t}\beta), \tag{3.6}$$

where $0 \le i_1 \le \dots \le i_s \le 2^{m-2}$, $u > 0$, $t > 0$, $e_1, \dots, e_u$ are odd and $f_1, \dots, f_t$ are even. We have

$$\begin{aligned}&(\alpha^{\tau+2^\nu e_1}\beta, \dots, \alpha^{\tau+2^\nu e_u}\beta, \alpha^{\tau+2^\nu f_1}\beta, \dots, \alpha^{\tau+2^\nu f_t}\beta)\\ \sim &(\alpha^{\tau+2^\nu f'_0}\beta, \alpha^{\tau+2^\nu e_1}\beta, \dots, \alpha^{\tau+2^\nu e_u}\beta, \alpha^{\tau+2^\nu f_2}\beta, \dots, \alpha^{\tau+2^\nu f_t}\beta) \qquad (f'_0 \text{ even}),\end{aligned}$$

where

$$\begin{aligned}&(\alpha^{\tau+2^\nu f'_0}\beta, \alpha^{\tau+2^\nu e_1}\beta, \dots, \alpha^{\tau+2^\nu e_u}\beta)\\ \sim &(\alpha^{\tau+2^\nu}\beta, \alpha^{\tau+2^\nu f'_1}\beta, \alpha^{\tau+2^\nu e_2}\beta, \dots, \alpha^{\tau+2^\nu e_u}\beta) \qquad (f'_1 \text{ even, Lemma 2.1 (iii)})\\ \sim &\cdots\\ \sim &(\alpha^{\tau+2^\nu}\beta, \dots, \alpha^{\tau+2^\nu}\beta, \alpha^{\tau+2^\nu f'_u}\beta) \qquad (f'_u \text{ even}).\end{aligned}$$

Hence

$$\begin{aligned}&(\alpha^{\tau+2^\nu e_1}\beta, \dots, \alpha^{\tau+2^\nu e_u}\beta, \alpha^{\tau+2^\nu f_1}\beta, \dots, \alpha^{\tau+2^\nu f_t}\beta)\\ \sim &(\alpha^{\tau+2^\nu}\beta, \dots, \alpha^{\tau+2^\nu}\beta, \alpha^{\tau+2^\nu f'_u}\beta, \alpha^{\tau+2^\nu f_2}\beta, \dots, \alpha^{\tau+2^\nu f_t}\beta).\end{aligned} \tag{3.7}$$

By a similar argument,

$$\begin{aligned}&(\alpha^{\tau+2^\nu}\beta, \alpha^{\tau+2^\nu f'_u}\beta, \alpha^{\tau+2^\nu f_2}\beta, \dots, \alpha^{\tau+2^\nu f_t}\beta)\\ \sim &(\alpha^{\tau+2^\nu h}\beta, \alpha^\tau\beta, \dots, \alpha^\tau\beta) \qquad (h \text{ odd}).\end{aligned} \tag{3.8}$$

By (3.7) and (3.8),

$$\begin{aligned}&(\alpha^{\tau+2^\nu e_1}\beta, \dots, \alpha^{\tau+2^\nu e_u}\beta, \alpha^{\tau+2^\nu f_1}\beta, \dots, \alpha^{\tau+2^\nu f_t}\beta)\\ \sim &(\alpha^{\tau+2^\nu}\beta, \dots, \alpha^{\tau+2^\nu}\beta, \alpha^{\tau+2^\nu h}\beta, \alpha^\tau\beta, \dots, \alpha^\tau\beta)\\ \sim &(\alpha^{\tau+2^\nu e}\beta, \alpha^{\tau+2^\nu}\beta, \dots, \alpha^{\tau+2^\nu}\beta, \alpha^\tau\beta, \dots, \alpha^\tau\beta) \qquad (e \text{ odd}).\end{aligned} \tag{3.9}$$

Combining (3.6) and (3.9), we see that α is equivalent to the tuple in (3.2). □

Theorem 3.1 has an immediate corollary.

Corollary 3.2. (i) *$\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}$ are equivalent $\Leftrightarrow$ $\boldsymbol{a}$ is a permutation of $\boldsymbol{b}$.*

(ii) *$\boldsymbol{a}, \boldsymbol{b} \in \mathcal{B}_{\nu,\tau}$ are equivalent $\Leftrightarrow \Lambda(\boldsymbol{a}) = \Lambda(\boldsymbol{b})$ and $\pi(\boldsymbol{a}) = \pi(\boldsymbol{b})$.*

(iii) *$\boldsymbol{a}, \boldsymbol{b} \in \mathcal{C}_{\nu,\tau}$ are equivalent $\Leftrightarrow \Lambda(\boldsymbol{a}) = \Lambda(\boldsymbol{b})$, $t(\boldsymbol{a}) = t(\boldsymbol{b})$ and $\pi(\boldsymbol{a}) = \pi(\boldsymbol{b})$.*

Theorem 3.1 and Corollary 3.2 allow us to compute the number of B_n-orbits in $\mathbb{Q}_{2^m}^n$ and the cardinality of each B_n-orbit.

Corollary 3.3. *The total number of equivalence classes in $Q_{2^m}^n$ is*

$$\left|Q_{2^m}^n/{\sim}\right| = \binom{n+2^{m-1}-1}{n} + 2^{m-1}\binom{n+2^{m-2}}{n-1} + 2^{m-2}\sum_{\nu=0}^{m-2}\binom{n+2^{m-2-\nu}}{n-2}.$$

Proof. By Theorem 3.1 (i),

$$|\mathcal{A}/{\sim}| = \binom{n+2^{m-1}-1}{n}.$$

In (3.1), the number $(i_1, \ldots, i_s)$, where $s \leq n-1$ is not fixed, with $0 \leq i_1 \leq \cdots \leq i_s \leq 2^{m-2}$ is $\binom{n+2^{m-2}}{n-1}$, which is the number of "$2^{m-2}+2$ choose $n-1$ with repetition". When $i_1, \ldots, i_s$ are chosen, the number of choices for (τ, e) in (3.1) is 2^{m-1}. So,

$$\left|\Big(\bigcup_{\substack{1\leq\nu\leq m-1\\ 0\leq\tau<2^\nu}} \mathcal{B}_{\nu,\tau}\Big)\Big/{\sim}\right| = 2^{m-1}\binom{n+2^{m-2}}{n-1}.$$

In (3.2), for each $0 \leq \nu \leq m-2$, the number of $(i_1, \ldots, i_s; t)$, where $s \leq n-2$ is not fixed, with $0 \leq i_1 \leq \cdots \leq i_s \leq 2^{m-2}$, $\min\{\nu_2(i_1), \ldots, \nu_2(i_s)\} \geq \nu$ and $1 \leq t \leq n-s-1$ is $\binom{n+2^{m-2-\nu}}{n-2}$, which is the number of "$2^{m-2-\nu}+3$ choose $n-2$ with repetition". When ν and $(i_1, \ldots, i_s; t)$ are chosen, the number of choices for (τ, e) in (3.2) is 2^{m-2}. So,

$$\left|\Big(\bigcup_{\substack{0\leq\nu\leq m-2\\ 0\leq\tau<2^\nu}} \mathcal{C}_{\nu,\tau}\Big)\Big/{\sim}\right| = 2^{m-2}\sum_{\nu=0}^{m-2}\binom{n+2^{m-2-\nu}}{n-2}.$$

Therefore,

$$\begin{aligned}\left|Q_{2^m}^n/{\sim}\right| &= |\mathcal{A}/{\sim}| + \left|\Big(\bigcup_{\substack{1\leq\nu\leq m-1\\ 0\leq\tau<2^\nu}} \mathcal{B}_{\nu,\tau}\Big)\Big/{\sim}\right| + \left|\Big(\bigcup_{\substack{0\leq\nu\leq m-2\\ 0\leq\tau<2^\nu}} \mathcal{C}_{\nu,\tau}\Big)\Big/{\sim}\right| \\ &= \binom{n+2^{m-1}-1}{n} + 2^{m-1}\binom{n+2^{m-2}}{n-1} + 2^{m-2}\sum_{\nu=0}^{m-2}\binom{n+2^{m-2-\nu}}{n-2}.\end{aligned}$$

$\square$

Corollary 3.4. (i) *Let* $n_0, \ldots, n_{2^{m-1}-1} \in \mathbb{N}$ *such that* $n_0 + \cdots + n_{2^{m-1}-1} = n$. *Then*

$$\big|[\underbrace{\alpha^0, \ldots, \alpha^0}_{n_0}, \ldots, \underbrace{\alpha^{2^{m-1}-1}, \ldots, \alpha^{2^{m-1}-1}}_{n_{2^{m-1}-1}}]\big| = \binom{n}{n_0, \ldots, n_{2^{m-1}-1}}.$$

(ii) *Let* $1 \le \nu \le m-1$, $0 \le \tau < 2^\nu$, *and* $0 \le e < 2^{m-1-\nu}$. *Let* $n_0, \ldots, n_{2^{m-2}} \in \mathbb{N}$ *such that* $n_0 + \cdots + n_{2^{m-2}} \le n-1$ *and* $\min(\{\nu_2(i) : n_i > 0\} \cup \{m-2\}) = \nu - 1$. *Then*

$$\big|[\underbrace{\alpha^0, \ldots, \alpha^0}_{n_0}, \ldots, \underbrace{\alpha^{2^{m-2}}, \ldots, \alpha^{2^{m-2}}}_{n_{2^{m-2}}}, \alpha^{\tau+2^\nu e}\beta, \alpha^\tau\beta, \ldots, \alpha^\tau\beta]\big|$$
$$= \binom{n}{n_0, \ldots, n_{2^{m-2}}, n - n_0 - \cdots - n_{2^{m-2}}} 2^{(m-1-\nu)(n-n_0-\cdots-n_{2^{m-2}}-1)+n_0+\cdots+n_{2^{m-2}-1}}.$$

(iii) *Let* $0 \le \nu \le m-2$, $0 \le \tau < 2^\nu$, *and* $0 \le e < 2^{m-1-\nu}$, $e \equiv 1 \pmod 2$. *Let* $n_0, \ldots, n_{2^{m-2}} \in \mathbb{N}$ *and* $t > 0$ *such that* $n_0 + \cdots + n_{2^{m-2}} + t \le n-1$ *and* $\min(\{\nu_2(i) : n_i > 0\} \cup \{m-2\}) \ge \nu$. *Then*

$$\big|[\underbrace{\alpha^0, \ldots, \alpha^0}_{n_0}, \ldots, \underbrace{\alpha^{2^{m-2}}, \ldots, \alpha^{2^{m-2}}}_{n_{2^{m-2}}}, \alpha^{\tau+2^\nu e}\beta, \alpha^{\tau+2^\nu}\beta, \ldots, \alpha^{\tau+2^\nu}\beta, \underbrace{\alpha^\tau\beta, \ldots, \alpha^\tau\beta}_{t}]\big|$$
$$= \binom{n}{n_0, \ldots, n_{2^{m-2}}, t, n - n_0 - \cdots - n_{2^{m-2}} - t} 2^{(m-2-\nu)(n-n_0-\cdots-n_{2^{m-2}}-1)+n_0+\cdots+n_{2^{m-2}-1}}.$$

Proof. The formulas follow from Corollary 3.2 and simple counting arguments. □

4 B_n-orbits in Tuples of Dihedral Groups

The dihedral group D_N of order $2N$ is given by the presentation

$$D_N = \langle \alpha\beta \mid \alpha^N = 1 = \beta^2,\ \beta\alpha\beta^{-1} = \alpha^{-1} \rangle.$$

Each element of D_N can be uniquely written as $\alpha^i\beta^j$ with $0 \le i < N$ and $0 \le j \le 1$. Clearly, equations (2.1) and (2.2), hence (2.3) – (2.6), also hold for D_N. In these equations, the only difference between D_N and Q_{2^m} that affects the Hurwitz action is that $o(\alpha) = N$ in D_N but $o(\alpha) = 2^{m-1}$ in Q_{2^m}. When $N = 2^{m-1}$, there is no difference. Therefore, under the bijection $D_{2^{m-1}} \to Q_{2^m}$, $\alpha^i\beta^j \mapsto \alpha^i\beta^j$, $0 \le i < 2^{m-1}$, $0 \le j \le 1$, the action of B_n on $D_{2^{m-1}}^n$ is identical to that on $Q_{2^m}^n$. Hence, all results in section 3 hold with Q_{2^m} replaced by $D_{2^{m-1}}$.

When $N = p^m$, where p is an odd prime, the B_n-orbits in $D_{p^m}^n$ can be determined using a method similar to that of section 3.

For $\boldsymbol{a} = (\alpha^{i_1}\beta^{j_1}, \ldots, \alpha^{i_n}\beta^{j_n}) \in D_{p^m}^n$, where $0 \le i_k < p^m$ and $0 \le j_k \le 1$, let

$$\lambda(\boldsymbol{a}) = \text{the multi set } \big\{\min\{i_k, p^m - i_k\} : j_k = 0\big\},$$
$$\gamma(\boldsymbol{a}) = \{i_k : j_k = 1\}.$$

$\lambda(\boldsymbol{a})$ is an invariant of the Hurwitz action on $D^n_{p^m}$. Let

$$\mathfrak{A}=\{\boldsymbol{a}\in D^n_{p^m}:\gamma(\boldsymbol{a})=\varnothing\}.$$

Moreover, for $0\le\nu\le m$ and $0\le\tau<p^\nu$, let

$$\mathfrak{B}_{\nu,\tau}=\big\{\boldsymbol{a}\in D^n_{p^m}:\min(\{\nu_p(i):i\in\lambda(\boldsymbol{a})\}\cup\{m\})=\nu,\ \varnothing\ne\gamma(\boldsymbol{a})\subset\tau+p^\nu\mathbb{Z}\big\};$$

for $0\le\nu\le m-1$ and $0\le\tau<p^\nu$, let

$$\mathfrak{C}_{\nu,\tau}=\big\{\boldsymbol{a}\in D^n_{p^m}:\min(\{\nu_p(i):i\in\lambda(\boldsymbol{a})\}\cup\{m\})\ge\nu+1,\ \varnothing\ne\gamma(\boldsymbol{a})\subset\tau+p^\nu\mathbb{Z},\\ \exists j,j'\in\gamma(\boldsymbol{a})\text{ such that }\nu_p(j-j')=\nu\big\}.$$

Then $\mathfrak{A}$, $\mathfrak{B}_{\nu,\tau}$ and $\mathfrak{C}_{\nu,\tau}$ are all invariant under the Hurwitz equivalence and

$$D_{p^m}=\mathfrak{A}\,\dot\cup\Big(\dot{\bigcup_{\substack{0\le\nu\le m\\0\le\tau<p^\nu}}}\mathfrak{B}_{\nu,\tau}\Big)\,\dot\cup\Big(\dot{\bigcup_{\substack{0\le\nu\le m-1\\0\le\tau<p^\nu}}}\mathfrak{C}_{\nu,\tau}\Big).$$

For $\boldsymbol{a}\in\mathfrak{C}_{\nu,\tau}$, collect the components of $\boldsymbol{a}$ of the form $\alpha^i\beta$ and let the result be $(\alpha^{i_1}\beta,\dots,\alpha^{i_t}\beta)$, where $0\le i_k<p^m$. Let $e_k\in\mathbb{Z}_p$, $1\le k\le t$, be defined by $i_k\equiv\tau+p^\nu e_k \pmod{p^{\nu+1}}$. Put

$$\sigma(\boldsymbol{a})=\sum_{k=1}^t(-1)^{k-1}e_k.$$

For example, let $p=5$, $m=4$, $n=5$, and let

$$\boldsymbol{a}=(\alpha^{9+5^2\cdot4}\beta,\ \alpha^{5^3\cdot3},\ \alpha^{9+5^2\cdot2}\beta,\ \alpha^{9+5^2\cdot8}\beta,\alpha^{9+5^2}\beta)\in\mathfrak{C}_{2,9}.$$

Then $\sigma(\boldsymbol{a})=4-2+8-1=4\in\mathbb{Z}_5$. From (2.3) – (2.6), it is easy to see that $\sigma(\boldsymbol{a})$ is an invariant under the Hurwitz equivalence. Further partition $\mathfrak{C}_{\nu,\tau}$ as

$$\mathfrak{C}^0_{\nu,\tau}=\{\boldsymbol{a}\in\mathfrak{C}_{\nu,\tau}:\sigma(\boldsymbol{a})=0\}$$

and

$$\mathfrak{C}^1_{\nu,\tau}=\{\boldsymbol{a}\in\mathfrak{C}_{\nu,\tau}:\sigma(\boldsymbol{a})\ne0\}.$$

Lemma 4.1. *Let $\tau,\nu,e,f\in\mathbb{Z}$ such that $0\le\nu\le m-1$ and $e\not\equiv f \pmod p$. Then for every $g\in\mathbb{Z}$,*

$$(\alpha^{\tau+p^\nu e}\beta,\ \alpha^{\tau+p^\nu f}\beta)\sim(\alpha^{\tau+p^\nu(e+g)}\beta,\ \alpha^{\tau+p^\nu(f+g)}\beta).$$

The proof of Lemma 4.1 is the same as that of Lemma 2.1 (iii).

Theorem 4.2. (i) *The B_n-orbits in $\mathfrak{A}$ are represented by*

$$(\alpha^{i_1},\dots,\alpha^{i_n}),$$

where $0\le i_1\le\dots\le i_n<p^m$.

(ii) *Let $0 \le \nu \le m$ and $0 \le \tau < p^\nu$. The B_n-orbits in $\mathfrak{B}_{\nu,\tau}$ are represented by*

$$(\alpha^{i_1},\dots,\alpha^{i_s},\ \alpha^{\tau+p^\nu e}\beta,\ \alpha^\tau\beta,\ \dots,\ \alpha^\tau\beta),$$

where $0 \le i_1 \le \dots \le i_s < \frac{1}{2}p^m$, $\min\{\nu_p(i_1),\dots,\nu_p(i_s),m\} = \nu$, $0 \le e < p^{m-\nu}$.

(iii) *Let $0 \le \nu \le m-1$ and $0 \le \tau < p^\nu$.*

(iii-1) *The B_n-orbits in $\mathfrak{C}^0_{\nu,\tau}$ are represented by*

$$(\alpha^{i_1},\dots,\alpha^{i_s},\ \alpha^{\tau+p^\nu e}\beta,\ \alpha^{\tau+p^\nu}\beta,\ \underbrace{\alpha^\tau\beta,\ \dots,\ \alpha^\tau\beta}_{n-2-s}), \tag{4.1}$$

where $0 \le i_1 \le \dots \le i_s < \frac{1}{2}p^m$, $n-2-s > 0$, $\min\{\nu_p(i_1),\dots,\nu_p(i_s),m\} \ge \nu+1$, $0 \le e < p^{m-\nu}$, $e \equiv 1 \pmod p$.

(iii-2) *The B_n-orbits in $\mathfrak{C}^1_{\nu,\tau}$ are represented by*

$$(\alpha^{i_1},\dots,\alpha^{i_s},\ \alpha^{\tau+p^\nu e}\beta,\ \underbrace{\alpha^\tau\beta,\dots,\alpha^\tau\beta}_{n-1-s}), \tag{4.2}$$

where $0 \le i_1 \le \dots \le i_s < \frac{1}{2}p^m$, $n-1-s > 0$, $\min\{\nu_p(i_1),\dots,\nu_p(i_s),m\} \ge \nu+1$, $0 \le e < p^{m-\nu}$, $e \not\equiv 0 \pmod p$.

Proof. The proofs of (i) and (ii) are identical to those of the corresponding cases in Theorem 3.1.

(iii) Different tuples in (4.1) are nonequivalent since they have different combinations of invariants $\lambda(\boldsymbol{a})$ and $\pi(\boldsymbol{a})$. The same is true for the tuples in (4.2). Therefore, it remains to show that every tuple $\boldsymbol{a} \in \mathfrak{C}_{\nu,\tau}$ is equivalent to one of the tuples in (4.1) or (4.2).

By (2.3) – (2.5), we may write

$$\boldsymbol{a} = (\alpha^{i_1},\ \dots,\ \alpha^{i_s},\ \alpha^{\tau+p^\nu e_1}\beta,\ \dots,\ \alpha^{\tau+p^\nu e_t}\beta),$$

where $0 \le i_1 \le \dots \le i_s < \frac{1}{2}p^m$ and there exist k, l such that $e_k \not\equiv e_l \pmod p$. It suffices to show that either

$$(\alpha^{\tau+p^\nu e_1}\beta,\ \dots,\ \alpha^{\tau+p^\nu e_t}\beta) \sim (\alpha^{\tau+p^\nu e}\beta,\ \alpha^{\tau+p^\nu}\beta,\ \alpha^\tau\beta,\ \dots,\ \alpha^\tau\beta) \tag{4.3}$$

for some $0 \le e < p^{m-\nu}$ with $e \equiv 1 \pmod p$ or

$$(\alpha^{\tau+p^\nu e_1}\beta,\ \dots,\ \alpha^{\tau+p^\nu e_t}\beta) \sim (\alpha^{\tau+p^\nu e}\beta,\ \alpha^\tau\beta,\ \dots,\ \alpha^\tau\beta) \tag{4.4}$$

for some $0 \le e < p^{m-\nu}$ with $e \not\equiv 0 \pmod p$. We prove this claim by induction on t.

If $t = 2$, by Lemma 4.1, we have

$$(\alpha^{\tau+p^\nu e_1}\beta,\ \alpha^{\tau+p^\nu e_2}\beta) \sim (\alpha^{\tau+p^\nu(e_1-e_2)}\beta,\ \alpha^\tau\beta);$$

hence (4.4) holds.

Now assume $t > 2$. Assume that $e_k \not\equiv e_{k+1} \equiv \cdots \equiv e_t \pmod p$. By Lemma 4.1,

$$\begin{aligned}
&(\alpha^{\tau+p^\nu e_k}\beta,\ \alpha^{\tau+p^\nu e_{k+1}}\beta,\ \ldots,\ \alpha^{\tau+p^\nu e_t}\beta)\\
\sim\,&(\alpha^{\tau+p^\nu e'_k}\beta,\ \alpha^{\tau+p^\nu e_k}\beta,\ \ldots,\ \alpha^{\tau+p^\nu e_t}\beta)\\
\sim\,&\cdots\\
\sim\,&(\alpha^{\tau+p^\nu e'_k}\beta,\ \cdots,\ \alpha^{\tau+p^\nu e'_{t-2}}\beta,\ \alpha^{\tau+p^\nu e_k}\beta,\ \alpha^{\tau+p^\nu e_t}\beta)\\
\sim\,&(\alpha^{\tau+p^\nu e'_k}\beta,\ \cdots,\ \alpha^{\tau+p^\nu e'_{t-1}}\beta,\ \alpha^{\tau}\beta).
\end{aligned}$$

So,

$$(\alpha^{\tau+p^\nu e_1}\beta,\ \cdots,\ \alpha^{\tau+p^\nu e_t}\beta) \sim (\alpha^{\tau+p^\nu f_1}\beta,\ \cdots,\ \alpha^{\tau+p^\nu f_{t-1}}\beta, \alpha^{\tau}\beta).$$

If $f_1, \ldots, f_{t-1}$ are not all the same modulo p, the induction hypothesis applies to $(\alpha^{\tau+p^\nu f_1}\beta,\ \cdots,\ \alpha^{\tau+p^\nu f_{t-1}}\beta)$. So, assume $f_1 \equiv \cdots \equiv f_{t-1} \not\equiv 0 \pmod p$. Let $x \in \mathbb{Z}$ such that $x \not\equiv -f_{t-1} \pmod p$. Then

$$\begin{aligned}
&(\alpha^{\tau+p^\nu f_{t-2}}\beta,\ \alpha^{\tau+p^\nu f_{t-1}}\beta,\ \alpha^{\tau}\beta)\\
\sim\,&(\alpha^{\tau+p^\nu f_{t-2}}\beta,\ \alpha^{\tau+p^\nu (f_{t-1}+1)}\beta,\ \alpha^{\tau+p^\nu}\beta)\\
\sim\,&(\alpha^{\tau+p^\nu (f_{t-2}+x)}\beta,\ \alpha^{\tau+p^\nu (f_{t-1}+x+1)}\beta,\ \alpha^{\tau+p^\nu}\beta)\\
\sim\,&(\alpha^{\tau+p^\nu (f_{t-2}+x)}\beta,\ \alpha^{\tau+p^\nu (f_{t-1}+x)}\beta,\ \alpha^{\tau}\beta).
\end{aligned}$$

If $t = 3$, choose $x = -f_{t-1} + 1$, then (4.4) holds. If $t > 3$, choose $x \in \mathbb{Z}$ such that $x \not\equiv -f_{t-1}, 0 \pmod p$. Then the induction hypothesis applies to

$$(\alpha^{\tau+p^\nu f_1}\beta,\ \cdots,\ \alpha^{\tau+p^\nu f_{t-3}}\beta,\ \alpha^{\tau+p^\nu (f_{t-2}+x)}\beta,\ \alpha^{\tau+p^\nu (f_{t-1}+x)}\beta). \qquad \square$$

Corollary 4.3. *(i)* $\boldsymbol{a}, \boldsymbol{b} \in \mathfrak{A}$ *are equivalent* $\Leftrightarrow$ $\boldsymbol{a}$ *is a permutation of* $\boldsymbol{b}$.

(ii) $\boldsymbol{a}, \boldsymbol{b} \in \mathfrak{B}_{\nu,\tau}$ *are equivalent* $\Leftrightarrow \lambda(\boldsymbol{a}) = \lambda(\boldsymbol{b})$ *and* $\pi(\boldsymbol{a}) = \pi(\boldsymbol{b})$.

(iii) $\boldsymbol{a}, \boldsymbol{b} \in \mathfrak{C}_{\nu,\tau}$ *are equivalent* $\Leftrightarrow \lambda(\boldsymbol{a}) = \lambda(\boldsymbol{b})$, $\sigma(\boldsymbol{a}) = \sigma(\boldsymbol{b})$ *and* $\pi(\boldsymbol{a}) = \pi(\boldsymbol{b})$.

We remark that the B_n-orbits of $D_{2p^m}^n$, where p is an odd prime, can also be determined due to the fact that $D_{2p^m} \cong \mathbb{Z}_2 \times D_{p^m}$. However, for an arbitrary positive integer N, determination of the B_n-orbits of D_N^n seems to be a difficult problem.

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