Augmented Rook Boards and General Product Formulas

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Abstract

There are a number of so-called factorization theorems for rook polynomials that have appeared in the literature. For example, Goldman, Joichi and White [6] showed that for any Ferrers board $B = F(b_1, b_2, \ldots, b_n)$,

$$\prod_{i=1}^{n}(x + b_i - (i-1)) = \sum_{k=0}^{n} r_k(B)(x)\downarrow_{n-k}$$

where $r_k(B)$ is the k-th rook number of B and $(x)\downarrow_k = x(x-1)\cdots(x-(k-1))$ is the usual falling factorial polynomial. Similar formulas where $r_k(B)$ is replaced by some appropriate generalization of the k-th rook number and $(x)\downarrow_k$ is replaced by polynomials like $(x)\uparrow_{k,j} = x(x+j)\cdots(x+j(k-1))$ or $(x)\downarrow_{k,j} = x(x-j)\cdots(x-j(k-1))$ can be found in the work of Goldman and Haglund [5], Remmel and Wachs [9], Haglund and Remmel [7], and Briggs and Remmel [3]. We shall refer to such formulas as product formulas. The main goal of this paper is to develop a new rook theory setting in which we can give a uniform combinatorial proof of a general product formula that includes, as special cases, essentially all the product formulas referred to above. We shall also prove q-analogues and (p, q)-analogues of our general product formula.

Keywords: rook theory, rook placements, generating functions

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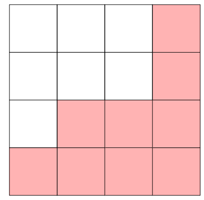

Figure 1: A Ferrers board $B = F(1, 2, 2, 4) \subseteq \mathcal{B}_n$, with $n = 4$.

1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers. For any positive integer a, we will set $[a] := \{1, 2, \ldots, a\}$. We let $\mathcal{B}_n = [n] \times [n]$ be the n by n array of squares. We number the rows of $\mathcal{B}_n$ from bottom to top and the columns of $\mathcal{B}_n$ from left to right with the numbers $1, \ldots, n$ and refer to the square or cell in the i-th row and j-th column of $\mathcal{B}_n$ as the (i, j)-th cell of $\mathcal{B}_n$. A *rook board* B is any subset of $\mathcal{B}_n$. If $B \subseteq \mathcal{B}_n$ is the rook board consisting of the first b_i cells of column i for $i = 1, \ldots, n$, then we will write $B = F(b_1, \ldots, b_n)$ and refer to B as a skyline board. In the special case where $0 \leq b_1 \leq b_2 \leq \cdots \leq b_n \leq n$, we will say that $B = F(b_1, b_2, \ldots, b_n)$ is a *Ferrers board*. For example, $F(1, 2, 2, 4)$ is pictured in Figure 1.

Given a board $B \subseteq \mathcal{B}_n$, we let $\mathcal{N}_k(B)$ denote the set of all placements $\mathbb{P}$ of k rooks in B such that no two rooks in $\mathbb{P}$ lie in the same row or column. We will refer to such a $\mathbb{P}$ as a *nonattacking placement* of k rooks in B. Similarly, we let $\mathcal{F}_k(B)$ denote the set of all placements Q of k rooks in B such that no two rooks in Q lie in the same column. We will refer to such a Q as a *file placement* of k rooks in B. Thus in a file placement Q, we do allow the possibility that two rooks lie in the same row. We then define the k-th *rook number of* B, $r_k(B)$, by setting $r_k(B) := |\mathcal{N}_k(B)|$. Similarly, we define the the k-th *file number of* B, $f_k(B)$, by setting $f_k(B) := |\mathcal{F}_k(B)|$. If $B = F(b_1, \ldots, b_n)$, then we shall sometimes write $r_k(b_1, b_2, \ldots, b_n)$ for $r_k(B)$ and $f_k(b_1, b_2, \ldots, b_n)$ for $f_k(B)$.

Given $x \in \mathbb{N}$, define $(x)\downarrow_n = (x) \uparrow_n = 1$ if $n = 0$ and $(x)\downarrow_n = x(x-1)\cdots(x-(n-1))$ and $(x) \uparrow_n = x(x+1)\cdots(x+(n-1))$ if $n > 0$. More generally, for any integer $m \geq 0$, let $(x) \downarrow_{0,m} = (x) \uparrow_{0,m} = 1$ and for $k \geq 1$, let $(x) \downarrow_{k,m} = x(x-m)\cdots(x-m(k-1))$ and $(x) \uparrow_{k,m} = x(x+m)\cdots(x+m(k-1))$. For each $B \subseteq \mathcal{B}_n$ and each $x \in \mathbb{N}$, we define $R_{n,B}(x)$, the *n-th rook polynomial of* B, and $F_{n,B}(x)$, the *n-th file polynomial of* B, by setting

$$R_{n,B}(x) = \sum_{k=0}^{n} r_{n-k}(B)(x)\downarrow_k \text{ and} \tag{1.1}$$

$$F_{n,B}(x) = \sum_{k=0}^{n} f_{n-k}(B)x^k. \tag{1.2}$$

Given a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ in the symmetric group S_n, we shall identify σ with the placement $\mathbb{P}_\sigma = \{(1, \sigma_1), (2, \sigma_2), \ldots, (n, \sigma_n)\}$. Then the *$k$-th hit number of* B, $h_k(B)$, is the number of $\sigma \in S_n$ such that the placement $\mathbb{P}_\sigma$ intersects the board in exactly k cells.

Rook numbers, file numbers, and hit numbers have been extensively studied. Here are three basic identities that hold for these numbers.

$$\sum_{k=0}^{n} h_k(B)x^k = \sum_{k=0}^{n} r_k(B)(n-k)!(x-1)^k, \tag{1.3}$$

$$\prod_{i=1}^{n}(x+b_i-(i-1)) = \sum_{k=0}^{n} r_{n-k}(B)(x)\downarrow_k, \text{ and} \tag{1.4}$$

$$\prod_{i=1}^{n}(x+b_i) = \sum_{k=0}^{n} f_{n-k}(B)x^k. \tag{1.5}$$

Identity (1.3) is due to Kaplansky and Riordan [8] and holds for any board $B \subseteq \mathcal{B}_n$. Identity (1.4) holds for all Ferrers boards $B = F(b_1, \ldots, b_n)$ and is due to Goldman, Joichi and White [6]. Identity (1.5) is due to Garsia and Remmel [4] and holds for all skyline boards $B = F(b_1, \ldots, b_n)$. Formulas (1.4) and (1.5) are examples of what we shall call *product formulas* in rook theory. That is, they say that for a Ferrers board $B = F(b_1, \ldots, b_n)$, the polynomials $R_{n,B}(x)$ and $F_{n,B}(x)$ factor as a product of linear factors.

We note that in the special case where $B = \mathbf{B}_n := F(0, 1, 2, \ldots, n-1)$, equations (1.4) and (1.5) become

$$x^n = \sum_{k=0}^{n} r_{n-k}(\mathbf{B}_n)(x)\downarrow_k \text{ and} \tag{1.6}$$

$$(x)\uparrow_n = \sum_{k=0}^{n} f_{n-k}(\mathbf{B}_n)x^k. \tag{1.7}$$

This shows that $r_{n-k}(\mathbf{B}_n) = S_{n,k}$ where $S_{n,k}$ is the *Stirling number of the second kind* and $(-1)^{n-k}f_{n-k}(\mathbf{B}_n) = s_{n,k}$ where $s_{n,k}$ is the *Stirling number of the first kind*.

There are natural q-analogues of formulas (1.3), (1.4) and (1.5). Let

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

The q-analogues of the factorials and falling factorials are defined by $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ and $[x]_q\downarrow_m = [x]_q[x-1]_q \cdots [x-(m-1)]_q$. Garsia and Remmel [4] defined q-analogues of the hit numbers, $h_k(B,q)$, the rook numbers, $r_k(B,q)$, and the file numbers, $f_k(B,q)$, for Ferrers boards B so that the following hold:

$$\sum_{k=0}^{n} h_k(B,q)x^{n-k} = \sum_{k=0}^{n} r_{n-k}(B,q)[k]_q!(1-xq^{k+1})\cdots(1-xq^n), \tag{1.8}$$

$$\prod_{i=1}^{n}[x+b_i-(i-1)]_q = \sum_{k=0}^{n} r_{n-k}(B,q)[x]_q\downarrow_k, \quad \text{and} \tag{1.9}$$

$$\prod_{i=1}^{n}[x+b_i]_q = \sum_{k=0}^{n} f_{n-k}(B,q)[x]_q^k. \tag{1.10}$$

Let

$$[n]_{p,q} = p^{n-1}+p^{n-2}q+\cdots+pq^{n-2}+q^{n-1} = \frac{p^n-q^n}{p-q}.$$

The (p,q)-analogues of the factorials and falling factorials are defined by $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q}$ and $[x]_{p,q}\downarrow_m = [x]_{p,q}[x-1]_{p,q}\cdots[x-(m-1)]_{p,q}$. There are also (p,q)-analogues of formulas (1.3)-(1.5) using such (p,q)-analogues; see the work of Wachs and White [10], Remmel and Wachs [9], Briggs and Remmel [2], and Briggs [1].

In recent years, a number of researchers have developed new rook theory models which give rise to new classes of product formulas. For example, Haglund and Remmel [7] developed a rook theory model where the analogue of the k-rook number is $m_k(B)$ which counts the number of k-element partial matchings in the complete graph $\mathcal{K}_n$. They defined an analogue of a Ferrers board $\tilde{B} = \tilde{F}(a_1,\ldots a_{2n-1})$ where $2n-1 \geq a_1 \geq \cdots \geq a_{2n-1} \geq 0$ and where the nonzero entries in $(a_1,\ldots,a_{2n-1})$ are strictly decreasing. In their setting, they proved the following identity,

$$\prod_{i=1}^{2n-1}(x+a_{2n-i}-2i+2) = \sum_{k=0}^{2n-1} m_{n-k}(\tilde{B})(x)\downarrow_{k,2}. \tag{1.11}$$

Remmel and Wachs [9] defined a more restricted class of rook numbers, $\tilde{r}_k^j(B)$, in their j-*attacking* rook model and proved that for Ferrers boards $B = F(b_1,\ldots,b_n)$, where $b_{i+1}-b_i \geq j-1$ if $b_i \neq 0$,

$$\prod_{i=1}^{n}(x+b_i-j(i-1)) = \sum_{k=0}^{n} \tilde{r}_{n-k}^j(B)(x)\downarrow_{k,j}. \tag{1.12}$$

Goldman and Haglund [5] developed an i-*creation rook theory model* and an appropriate notion of rook numbers $r_{n-k}^{(i)}$ for which the following identity holds for Ferrers boards:

$$\prod_{j=1}^{n}(x+b_i+(j-1)(i-1)) = \sum_{k=0}^{n} r_{n-k}^{(i)}(B)(x)\uparrow_{k,i-1}. \tag{1.13}$$

In all of these new models, the authors proved q-analogues and/or (p,q)-analogues of their product formulas.

In this paper, we define a new rook theory model in which we can prove a general product formula that can be specialized to give all the product formulas described above. That is, it is easy to see that for any $m \geq 0$, the polynomials $\{(x) \downarrow_{k,m}: k \geq 0\}$ and $\{(x) \uparrow_{k,m}: k \geq 0\}$ are basis for the polynomial ring $Q[x]$. Thus if we have product formulas of the form

$$\begin{aligned}\prod_{i=1}^{n}(x+a_i) &= \sum_{k=0}^{n} b_{n,k}(x)\downarrow_{k,m} \text{ and}\\ \prod_{i=1}^{n}(x+c_i) &= \sum_{k=0}^{n} d_{n,k}(x)\downarrow_{k,m},\end{aligned}$$

then the coefficients $c_{n,k}$ and $d_{n,k}$ are uniquely determined by the sequences $(a_1,\ldots,a_n)$ and $(c_1,\ldots,c_n)$. For example, in the special cases of (1.11) and (1.12) where $j=2$ and $(a_{2n-1},\ldots,a_1)=(b_1,\ldots,b_{2n-1})$, we can conclude that $m_t(\tilde{B})=\tilde{r}_t(B)$ for all t. In such a case, we shall say that (1.11) and (1.12) yield the same product formula even though the combinatorial interpretations of $m_t(\tilde{B})$ and $\tilde{r}_t(B)$ are not the same. It should be noted that in this case, these coefficients satisfy simple recursions that do allow us to construct bijections which show that the combinatorial interpretations of $m_t(\tilde{B})$ and $\tilde{r}_t(B)$ are equivalent in these cases. An example of this type of argument will be presented in section 3.1.2. Now suppose we are given any two sequences of natural numbers, $\mathcal{B}=\{b_i\}_{i=1}^n, \mathcal{A}=\{a_i\}_{i=1}^n \in \mathbb{N}^n$, and two functions, $sgn, \overline{sgn}:[n]\to\{-1,+1\}$. Let $B=F(b_1,b_2,\ldots,b_n)$ be a skyline board. The main goal of this paper is to define a rook theory model with an appropriate notion of rook numbers $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$ such that the following product formula holds:

$$\prod_{i=1}^{n}(x+sgn(i)b_i)=\sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})\prod_{j=1}^{k}(x+\sum_{s\leq j}\overline{sgn}(s)a_s). \tag{1.14}$$

We will refer to equation (1.14) as the *general product formula* and $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$ as the k-th *augmented rook number of* $\mathcal{B}$ *with respect to* $\mathcal{A}$, sgn, *and* $\overline{sgn}$.

This general product formula is new and vastly extends the range of any of the product formulas that have appeared in the literature. Our general product formula specializes to all the product formulas described above so that our new rook theory model provides a common framework in which we can give a uniform proof of all these product formulas. We shall also prove q-analogues and (p,q)-analogues of our general product formula and describe the connection between other q-analogues and (p,q)-analogues of product formulas that have appeared in the literature.

The outline of this paper is as follows. In section 2, we shall review the rook theory models of Garsia-Remmel [4], Remmel-Wachs [9], Briggs-Remmel [3], Haglund-Remmel [7], and Goldman-Haglund [5]. In particular, we shall give explicit definitions of the rook numbers, the q-rook numbers, and the product formulas in these models. In section 3, we shall define our new rook theory model and prove (1.14). We shall

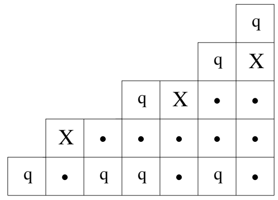

Figure 2: The q-weight of a rook placement in $B = F(1,2,2,3,3,4,5)$.

also compare our rook theory model with the rook theory models in section 2. In section 4, we shall prove several q-analogues of our general product formula and describe the connection between other q-analogues of product formulas that have appeared in the literature. Finally, in section 5, we shall describe how we can prove several (p,q)-analogues of our general product formula.

2 Previous product formulas

In this section, we shall define the appropriate analogues of rook and file numbers so that we can state the product formulas proved by Garsia-Remmel [4], Remmel-Wachs [9], Briggs-Remmel[3], Haglund-Remmel [7], and Goldman-Haglund [5].

2.1 The Garsia-Remmel Model

In [4], Garsia and Remmel defined q-analogues of rook numbers and file numbers. Given a Ferrers board $B = F(b_1, b_2, \ldots, b_n)$ and a placement $\mathbb{P} \in \mathcal{N}_k(B)$, we say that each rook in $\mathbb{P}$ cancels all of the cells in its row that lie to its right and all of the cells in its column that lie below it. We then set $u_B(\mathbb{P})$ to be the number of cells in B which do not contain a rook and which are not canceled by a rook in $\mathbb{P}$ and define the q-weight of $\mathbb{P}$ to be $W_{q,B}(\mathbb{P}) = q^{u_B(\mathbb{P})}$. Then Garsia and Remmel defined the k-th *q-rook number of B* for a Ferrers board $B = F(b_1, b_2, \ldots, b_n)$ by setting

$$r_k(B,q) = \sum_{\mathbb{P} \in \mathcal{N}_k(B)} W_{q,B}(\mathbb{P}). \tag{2.1}$$

For example, if $B = F(1,2,2,3,3,4,5)$ and $\mathbb{P} \in \mathcal{N}_3(B)$ is the placement pictured in Figure 2, then $W_{q,B}(\mathbb{P}) = q^7$. In Figure 2, we indicate the canceled cells by placing a $\bullet$ in those cells and we place a q in all those cells counted by $u_B(\mathbb{P})$.

For any Ferrers board $B \subseteq \mathcal{B}_n$, let B_x be the board B with x rows of length n appended below it as illustrated in Figure 3. We will call the part of the board B_x which we attached below B, the *x-part* of B_x. We shall refer to the line that separates the x-part of B_x from B as the *bar*. Let $\mathcal{N}_k(B_x)$ denote the set of all placements $\mathbb{P}$ of k rooks in B_x such that no two rooks in $\mathbb{P}$ lie in the same row or column and $\mathcal{F}_k(B_x)$ denote the set of all placements Q of k rooks in B_x such that no two rooks in Q lie in the same column.

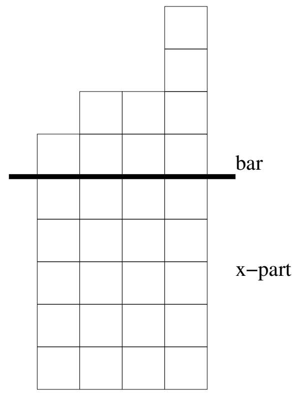

Figure 3: The board B_x, with $B = F(1,2,2,4)$ and x=5.

For any $\mathbb{P} \in \mathcal{N}_n(B_x)$, each rook in $\mathbb{P}$ cancels all of the cells in its row that lie to its right and all of the cells in its column that lie below it. We then define the q-weight of $\mathbb{P}$ to be $W_{q,B_x}(\mathbb{P}) = q^{u_{B_x}(\mathbb{P})}$ where $u_{B_x}(\mathbb{P})$ equals the number of cells in B_x which do not contain a rook and which are not canceled by a rook in $\mathbb{P}$. This given, the following q-analogue of (1.4) was proved by Garsia and Remmel [4] by summing

$$S(q) = \sum_{\mathbb{P} \in \mathcal{N}_n(B_x)} W_{q,B_x}(\mathbb{P}) \tag{2.2}$$

in two different ways.

Theorem 2.1. *For any $x, n \in \mathbb{N}$ with $x \geq n$ and any Ferrers board, $B = F(b_1, b_2, \ldots, b_n)$,*

$$\prod_{i=1}^{n} [x + b_i - (i-1)]_q = \sum_{k=0}^{n} r_{n-k}(B,q)[x]_q\downarrow_k. \tag{2.3}$$

Given a placement $\mathbb{P} \in \mathcal{F}_k(B)$, we let each rook in $\mathbb{P}$ cancel all of the cells of B in its column which lie below it. We then define the q-weight $\mathbb{P}$ by setting $w_{q,B}(\mathbb{P}) = q^{z_B(\mathbb{P})}$ where $z_B(\mathbb{P})$ equals the number of cells in B which do not contain a rook and are not canceled by a rook in $\mathbb{P}$. We define *q-file numbers* by setting

$$f_k(B,q) = \sum_{\mathbb{P} \in \mathcal{F}_k(B)} w_{q,B}(\mathbb{P}). \tag{2.4}$$

For example, if $B = F(2,2,3,4,4,5)$ and $\mathbb{P} \in \mathcal{F}_3(B)$ is the placement pictured in Figure 4, then we have that $w_{q,B}(\mathbb{P}) = q^{13}$. Again, in Figure 4, we indicate the canceled cells by placing a $\bullet$ in those cells and we place a q in all those cells which are counted by $z_B(\mathbb{P})$. We can extend this statistic to the board B_x by saying that each rook in $\mathbb{P}$ cancels all of the cells of B_x which lie below it in B_x. We then set $w_{q,B_x}(\mathbb{P}) = q^{z_{B_x}(\mathbb{P})}$

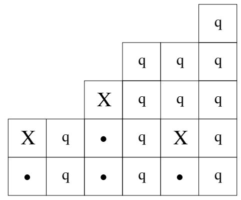

Figure 4: The q-weight of a file placement in $B = F(2, 2, 3, 4, 4, 5)$.

where $z_{B_x}(\mathbb{P})$ equals the number of cells in B_x which do not contain a rook and are not canceled by a rook in $\mathbb{P}$. Then one can prove a q-analogue of (1.5) by summing

$$F(q) = \sum_{\mathbb{P} \in \mathcal{F}_n(B_x)} w_{q,B_x}(\mathbb{P}) \tag{2.5}$$

in two different ways.

Theorem 2.2. *For any $x \in \mathbb{N}$ and and skyline board $B = F(b_1, b_2, \ldots, b_n)$,*

$$\prod_{i=1}^{n} [x + b_i]_q = \sum_{k=0}^{n} f_{n-k}(B, q)([x]_q)^k. \tag{2.6}$$

2.2 The Remmel-Wachs Model

Next, we will discuss the *j-attacking rook model* of Remmel and Wachs [9]. For a fixed integer $j \geq 1$, we say that a Ferrers board $F(a_1, \ldots, a_n)$ is a *j-attacking* board if for all $1 \leq i < n$, $a_i \neq 0$ implies $a_{i+1} \geq a_i + j - 1$. Suppose that $F(a_1, \ldots, a_n)$ is a j-attacking board and $\mathbb{P}$ is a placement of rooks in $F(a_1, \ldots, a_n)$ which has at most one rook in each column of $F(a_1, \ldots, a_n)$. Then for any individual rook $r \in \mathbb{P}$, we say that r *j-attacks* a cell $c \in F(a_1, \ldots, a_n)$ if c lies in a column which is strictly to the right of the column of r and c lies in the first j rows which are weakly above the row of r and which are not j-attacked by any rook which lies in a column that is strictly to the left of r.

For example, suppose $j = 2$ and $\mathbb{P}$ is the placement in $F(1, 2, 3, 5, 7, 8, 10)$ pictured in Figure 5. Here the rooks are indicated by placing an **X** in each cell that contains a rook. We place a 2 in each cell 2-attacked by the rook r_2 in column 2. In this case, since there are no rooks to the left of r_2, the cells c which are 2-attacked by r_2 lie in the first two rows which are weakly above the row of r_2, i.e., all the cells in rows 2 and 3 that are in columns 3, 4, 5, 6 and 7. Next consider the rook r_4 which lies in column 4. Again we place a 4 in each of the cells that are 2-attacked by r_4. In this case, the first two rows which lie weakly above r_4 that are not 2-attacked by any rook to the left of r_4 are rows 1 and 4. Thus r_4 2-attacks all the cells in rows 1 and 4 that lie in columns 5, 6 and 7. Finally the rook r_6, which lies in column 6, 2-attacks the cells (6,7) and (7,7) and we

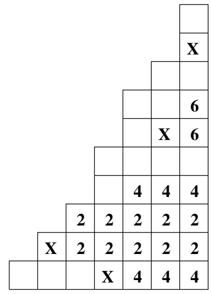

Figure 5: An example of cells that are 2-attacked in the board $B = F(1,2,3,5,7,8,10)$.

place a 6 in these cells. We say that a placement $\mathbb{P}$ is *j-nonattacking* if no rook in $\mathbb{P}$ is j-attacked by a rook to its left and there is at most one rook in each row and column.

Note that the condition that $F(a_1,\ldots,a_n)$ is j-attacking ensures that any placement $\mathbb{P}$ of j-nonattacking rooks in $F(a_1,\ldots,a_n)$, with at most one rook in each column, has the property that, for any rook $r \in \mathbb{P}$ which lies in a column $k < n$, there are j rows which lie weakly above r and which have no cells which are j-attacked by a rook to the left of r, namely, the row of r plus the top $j-1$ rows in column $k+1$ since $a_{k+1} \geq a_k + j - 1$.

Given a j-attacking board $B = F(a_1,\ldots,a_n)$, we let $\mathcal{N}_k^j(B)$ be the set of all placements $\mathbb{P}$ of k j-nonattacking rooks in B. For example, if $j=2$ and $B = F(0,2,3,4)$, then $|\mathcal{N}_1^2(B)| = 9$ since there are 9 cells in B. Also $|\mathcal{N}_2^2(B)| = 12$ and these 12 placements are pictured in Figure 6. Finally $|\mathcal{N}_3^2(B)| = 0$ since any placement $\mathbb{P}$ which has one rook in each nonempty column of B and at most one rook in each row has the property that the rooks in columns 2 and 3 would 2-attack 4 cells in column 4 and hence there would be no place to put a rook in column 4 that is not 2-attacked by a rook to its left. We then define the *k-th j-rook number* of B, $r_k^j(B)$, by setting $r_k^j(B) = |\mathcal{N}_k^j(B)|$.

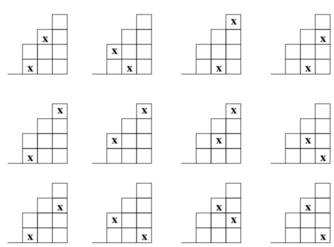

Figure 6: The 12 placements in $\mathcal{N}_2^2(F(0,2,3,4))$.

Let $B = F(a_1,\ldots,a_n)$ be a j-attacking board. Then for any placement $\mathbb{P} \in \mathcal{N}_k^j(B)$, we define

$$\tilde{W}^j_{p,q,B}(\mathbb{P}) = q^{a_B(\mathbb{P})} p^{b_B(\mathbb{P})} q^{e_B(\mathbb{P})} p^{-(c_1+\cdots+c_k)j} \tag{2.7}$$

where

1. $a_B(\mathbb{P})$ equals the number of cells of B which lie above a rook in $\mathbb{P}$ and which are not j-attacked by any rook in $\mathbb{P}$,

2. $b_B(\mathbb{P})$ equals the number of cells of B which lie below a rook in $\mathbb{P}$ and which are not j-attacked by any rook in $\mathbb{P}$,

3. $e_B(\mathbb{P})$ equals the number of cells of B which lie in a column with no rook in $\mathbb{P}$ and which are not j-attacked by any rook in $\mathbb{P}$, and

4. $c_1 < \cdots < c_k$ are the columns which contain rooks in $\mathbb{P}$.

For example, in Figure 7, we have pictured a placement $\mathbb{P} \in \mathcal{N}_3^3(B)$ where B is the 3-attacking board $F(2, 5, 8, 10, 12)$ such that $\mathbb{P}$ has rooks in columns 1, 3 and 4 and $a_B(\mathbb{P}) = 3$, $b_B(\mathbb{P}) = 5$, $e_B(\mathbb{P}) = 5$. Thus $\tilde{W}^3_{p,q,B}(\mathbb{P}) = q^3p^5q^5p^{-(1+3+4)3} = q^8p^{-19}$. Moreover, we have placed a p in each cell of B which contributes to the $b_B(\mathbb{P})$, a q in each cell that contributes to either $a_B(\mathbb{P})$ or $e_B(\mathbb{P})$, and a $\bullet$ in each cell that is j-attacked by some rook in $\mathbb{P}$.

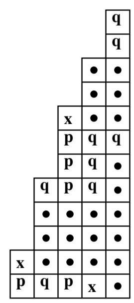

Figure 7: An example of $\tilde{W}_{p,q,B}(\mathbb{P})$

We then define the (p, q)-rook number of B by

$$\tilde{r}^j_{k,B}(p, q) = \sum_{\mathbb{P} \in \mathcal{N}^j_k(B)} \tilde{W}^j_{p,q,B}(\mathbb{P}). \tag{2.8}$$

Remmel and Wachs [9] proved the following (p, q)-extension of Theorem 2.1.

Theorem 2.3. *Let $B = F(a_1, \ldots, a_n)$ be a j-attacking board. Then*

$$\prod_{i=1}^{n}[x + a_i - j(i-1)]_{p,q} = \sum_{k=0}^{n} \tilde{r}^j_{k,B}(p, q)p^{kx+\binom{k+1}{2}j}[x]_{p,q}\downarrow_{n-k,j} \tag{2.9}$$

where $[x]_{p,q}\downarrow_{0,j} = 1$ and for $k > 0$, $[x]_{p,q}\downarrow_{k,j} = [x]_{p,q}[x-j]_{p,q}\cdots[x-(k-1)j]_{p,q}$.

When we talk of the q-analogue of the Remmel-Wachs model, we mean to take the q-statistic on placement of j-nonattacking rooks which results by setting $p = 1$ in the (p, q)-statistic $\tilde{W}^j_{p,q,B}(\mathbb{P})$.

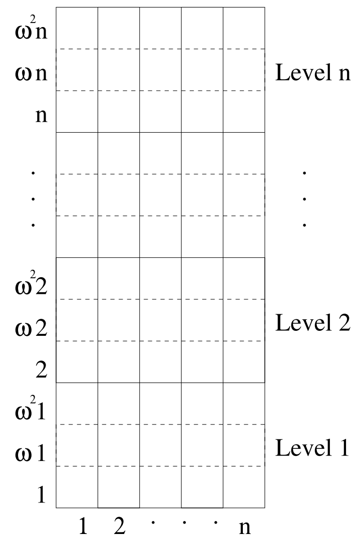

Figure 8: $B_{n\times 3n}$.

2.3 The Briggs-Remmel Model

In this section, we describe a variation of the Remmel-Wachs model that is appropriate for rook placements that correspond to partial permutations of the wreath product of the cyclic group of order m, C_m, with the symmetric group $\mathscr{S}_n$, denoted by $C_m \wr \mathscr{S}_n$.

If $\omega = e^{\frac{2\pi i}{m}}$, then we say that $C_m \wr \mathscr{S}_n$ is the group of $m^n n!$ signed permutations where there are m signs, $1 = \omega^0, \omega, \omega^2, \ldots, \omega^{m-1}$. We will usually write the signed permutations in either one-line notation or in disjoint cycle form. For example, if $\sigma \in C_3 \wr \mathscr{S}_8$ is the map with $1 \to \omega 5$, $2 \to 8$, $3 \to \omega^2 3$, $4 \to \omega^2 1$, $5 \to 4$, $6 \to \omega^2 7$, $7 \to \omega 2$, and $8 \to \omega 6$, then in one-line notation,

$$\sigma = \omega 5 \;\; 8 \;\; \omega^2 3 \;\; \omega^2 1 \;\; 4 \;\; \omega^2 7 \;\; \omega 2 \;\; \omega 6,$$

whereas in disjoint cycle form,

$$\sigma = (\omega^2 1 \;\; \omega 5 \;\; 4)(\omega 2 \;\; 8 \;\; \omega 6 \;\; \omega^2 7)(\omega^2 3).$$

That is, in disjoint cycle form, to determine where i is being mapped, we ignore the sign on i and only consider the sign on the element to which it is mapped.

Let $B_{n\times mn}$ be the $n \times mn$ array of squares where the n columns are labeled from left to right by $1, 2, \cdots, n$, and the mn rows are labeled from bottom to top by $1, \omega 1, \cdots, \omega^{m-1}1, 2, \omega 2, \cdots, \omega^{m-1}2, \cdots, n, \omega n, \cdots, \omega^{m-1}n$. For instance, the board $B_{n\times 3n}$ is illustrated in Figure 8. We let $(i, \omega^r j)$ identify the square in the column labeled with i and the row labeled with $\omega^r j$. Each such square will be called a *cell* and the rows labeled by $j, \omega j, \cdots, \omega^{m-1}j$ will be called *level* j.

A *board* will be a subset of cells in $B_{n\times mn}$. In particular, a *skyline board* in $B_{n\times mn}$ is a board whose column heights from left to right are $h_1, \ldots, h_n$, and is denoted by

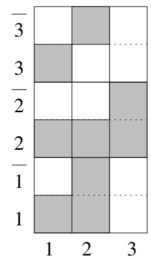

Figure 9: $B \subseteq B_{3\times 6}$.

$B_m(h_1,\ldots,h_n)$. That is, for each $1 \le i \le n$, if $h_i \ne 0$ and $h_i = a_i m + b_i$ with $0 \le a_i \le n-1$ and $1 \le b_i \le m$, then the i-th column contains all of the cells in the set

$$\big\{(i,\omega^s j) \mid 0 \le s < m, 1 \le j \le a_i\big\} \cup \big\{(i,\omega^s(a_i+1)) \mid 0 \le s < b_i\big\}.$$

Further, if $0 \le h_1 \le \cdots \le h_n \le mn$ and $h_{i+1} \ge (a_i+1)m$ whenever $h_i = a_i m + b_i$ where $1 \le b_i < m$, then $B_m(h_1,\ldots,h_n)$ is called an *m-Ferrers board* in $B_{n\times mn}$. We will denote the m-Ferrers board with column heights h_1, …, h_n by $F_m(h_1,\ldots,h_n)$.

Given a board $B \subseteq B_{n\times mn}$, we let $R^m_{k,n}(B)$ denote the set of all k element subsets $\mathbb{P}$ of B such that no two elements lie in the same level or column for nonnegative integers k. Such a subset $\mathbb{P}$ will be called a placement of nonattacking m-rooks in B. The cells in $\mathbb{P}$ are considered to contain m-rooks so that we call $r^m_{k,n}(B) = |R^m_{k,n}(B)|$ the k-th *m-rook number* of B. We note that for any board $B \subseteq B_{n\times mn}$, $r^m_{0,n}(B) = 1$, $r^m_{1,n}(B) = |B|$, and if $k > n$, then $r^m_{k,n}(B) = 0$. For example, consider the board of shaded cells in Figure 9. One can easily check that $r^2_{0,3}(B) = 1$, $r^2_{1,3}(B) = 9$, $r^2_{2,3}(B) = 18$, and $r^2_{3,3}(B) = 6$.

Suppose that $B = F_m(b_1,\ldots,b_n) \subseteq B_{n\times mn}$ is an m-Ferrers board and let $\mathbb{P} \in R^m_{k,n}(B)$. A rook in the cell $(i,\omega^r j) \in \mathbb{P}$ is said to *m-rook-cancel* those cells in the set

$$\big\{(a,\omega^s j) : i < a \le n, 0 \le s < m\big\}.$$

Then, Briggs and Remmel [3] proved the following product formula.

Theorem 2.4. *Let $B = F_m(b_1,\ldots,b_n) \subseteq B_{n\times mn}$ be an m-Ferrers board. Then*

$$\prod_{i=1}^{n}(mx + b_i - m(i-1)) = \sum_{k=0}^{n} r^m_{k,n}(B)(mx)\downarrow_{n-k,m}, \tag{2.10}$$

where $(x)\downarrow_{k,m} = x(x-m)\cdots(x-(k-1)m)$.

We note that for a given m, the Briggs-Remmel model is very similar to the m-attacking rook model of Remmel-Wachs. Each rook still cancels m cells in each column to its right so that the product formulas in these two models are essentially equivalent. However, it turns out that the Briggs-Remmel model has certain advantages, especially for formulas involving hit numbers. That is, given a permutation $\sigma \in C_m \wr \mathscr{S}_n$, we can identify σ with a placement $\mathbb{P}_\sigma$ of n m-rooks in $B_{n\times mn}$ by letting $\mathbb{P}_\sigma = \{(i,\omega^r j) : \sigma(i) =$

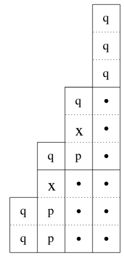

Figure 10: $\mathbb{P} \in R^3_{2,4}(B)$.

$\omega^r j\}$ for $1 \leq i \leq n$. We can then define a natural analogue of hit numbers in the Briggs-Remmel model by setting $H^m_{k,n}(B) = \{\mathbb{P}_\sigma : \sigma \in C_m \wr \mathscr{S}_n$ and $|\mathbb{P}_\sigma \cap B| = k\}$ and letting $h^m_{k,n} = |H^m_{k,n}(B)|$ denote the k-th m-*hit number* of B relative to $B_{n\times mn}$. We shall not pursue analogues of hit numbers in this paper so we refer the interested reader to [1] and [3] for details.

We can also define a (p,q)-analogue of the m-rook numbers and prove a (p,q)-analogue of Theorem 2.4. That is, we define the k-th (p,q)-m-rook number of B, denoted $r^m_{k,n}(B,p,q)$, as

$$r^m_{k,n}(B,p,q) = \sum_{\mathbb{P}\in R^m_{k,n}(B)} q^{\alpha_B(\mathbb{P})+\varepsilon_B(\mathbb{P})} p^{\beta_B(\mathbb{P})-m(c_1+\cdots+c_k)},$$

where the rooks of $\mathbb{P}$ lie in columns $c_1 < \ldots < c_k$ and where

1. $\alpha_B(\mathbb{P})$ is the number of cells of B which lie above a rook in $\mathbb{P}$ but are not m-rook-canceled by any other rook in $\mathbb{P}$,
2. $\beta_B(\mathbb{P})$ is the number of cells of B which lie below a rook in $\mathbb{P}$ but are not m-rook-canceled by any other rook in $\mathbb{P}$, and
3. $\varepsilon_B(\mathbb{P})$ is the number of cells of B which lie in a column with no rook in $\mathbb{P}$ and are not m-rook-canceled by any rook in $\mathbb{P}$.

For example, if $B = F_3(2,4,6,9) \subseteq B_{3\times 12}$ and $\mathbb{P} \in R^3_{2,4}(B)$ is the placement given in Figure 10, then $\alpha_B(\mathbb{P}) = 2$, $\beta_B(\mathbb{P}) = 3$, $\varepsilon_B(\mathbb{P}) = 5$, $c_1 = 2$, and $c_2 = 3$. So, the (p,q)-contribution of $\mathbb{P}$ to $R^3_{2,4}(B,p,q)$ is q^7p^{-12}.

With $[x]_{p,q}\downarrow_{k,m}$ denoting $[x]_{p,q}[x-m]_{p,q}\cdots[x-m(k-1)]_{p,q}$, Briggs and Remmel [3] proved the following (p,q)-analogue of the factorization theorem.

Theorem 2.5. *Let $B = F_m(b_1,\ldots,b_n) \subseteq B_{n\times mn}$ be an m-Ferrers board. Then*

$$\prod_{i=1}^n [mx+b_i-m(i-1)]_{p,q} = \sum_{k=0}^n r^m_{k,n}(B,p,q)p^{m\left(xk+\binom{k+1}{2}\right)}[mx]_{p,q}\downarrow_{n-k,m}. \tag{2.11}$$

2.4 The Haglund-Remmel Perfect Matching Model

The next model we will discuss is the *perfect matching model* of Haglund and Remmel [7]. In this model, we consider the board $\mathbf{B}_{2n}$ pictured in Figure 11 where the columns are labeled from 2 to $2n$ and the rows are labeled from 1 to $2n-1$.

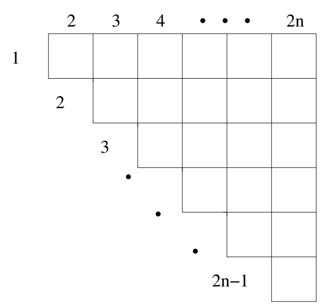

Figure 11: A perfect matching board $\mathbf{B}_{2n}$.

Let p_m denote the set of perfect matchings in the complete graph, K_{2m}, where we call $m = \{\{i_k, j_k\} : k = 1, \ldots n\}$ a perfect matching if $1 \leq i_k < j_k \leq 2n$ for every $1 \leq k \leq n$ and $\{i_u, j_u\} \bigcap \{i_v, j_v\} = \emptyset$ for every $u \neq v$. An example of a perfect matching of K_8 with $m = \{\{1,5\}, \{2,3\}, \{4,7\}, \{6,8\}\}$ can be seen in Figure 12. We define a rook placement in a board $B \subseteq \mathbf{B}_{2n}$ to be a subset of some perfect matching $m \in p_m$ which lies completely in B. Let $M_k(B) := \{S \subseteq B : (\exists m \in p_m)(m \cap B \supseteq S \text{ and } |S| = k)\}$. Then we define the k-th *rook number of* B to be $m_k(B) := |M_k(B)|$.

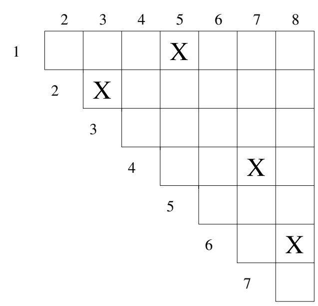

Figure 12: A example of a perfect matching of K_8.

The board $B = B(b_1, b_2, \ldots, b_{2n-1}) \subseteq \mathbf{B}_{2n}$ is defined as $B = \{(i, i+j) | 1 \leq j \leq b_j\}$, that is, B has row lengths, $b_1, b_2, \ldots, b_{2n-1}$ reading from top to bottom. If $2n-1 \geq b_1 \geq b_2 \geq \cdots \geq b_{2n-1} \geq 0$ and if $b_i > 0$ implies that $b_i > b_{i+1}$ for all $i = 1, 2, \ldots, 2n-2$, then $B = B(b_1, b_2, \ldots, b_{2n-1})$ is called a *shifted Ferrers board*. An example of the shifted Ferrers board $B = B(6,5,3,1,0,0,0) \subset B_8$ can be seen in Figure 13.

Haglund and Remmel also defined the notion of a *nearly Ferrers board*. That is, they defined a board B to be a nearly Ferrers board if for every cell $(i,j) \in B$, the cells $\{(s,j) : s < i\} \cup \{(t,i) : t < i\} \subseteq B$. By this description, every shifted Ferrers board is a nearly Ferrers board. Also, one can construct a nearly Ferrers board in the following

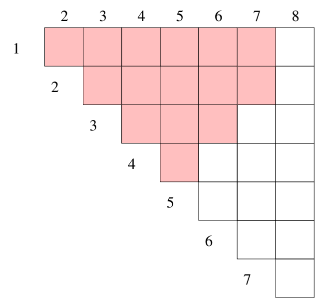

Figure 13: An example of the shifted Ferrers board $B = F(6, 5, 3, 1, 0, 0, 0) \subset B_8$.

manner. First fix an $a \in \mathbb{N}$, and then make a triangular arrangement of cells by letting $B' = \{(s,t)|s < t \leq a\}$. One may then add any columns to the right of B', and that board will be nearly Ferrers, as in Figure 14. Haglund and Remmel [7] proved the following theorem.

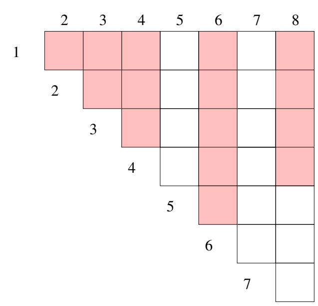

Figure 14: An example of the nearly Ferrers board $B \subset \mathbf{B}_8$.

Theorem 2.6. *Let $B \subseteq \mathbf{B}_{2n}$ be a nearly Ferrers board and let b_i denote the number of cells of B that lie in row i for each $1 \leq i \leq 2n-1$. Then*

$$\prod_{i=1}^{2n-1} (x + b_{2n-i} - 2i + 2) = \sum_{k=0}^{2n-1} m_{n-k}(B)(x) \downarrow_{2n-1-k,2} . \tag{2.12}$$

Haglund and Remmel also proved a q-analogue of Theorem 2.6. That is, suppose that we are given a nearly Ferrers board $B = F(b_1, b_2, \ldots, b_{2n-1})$. For any rook $\mathbf{r}$ in square (i,j), we say that $\mathbf{r}$ *rook cancels* the squares $\{(r,i) : r < i\} \cup \{(i,s) : i+1 \leq s < j\} \cup \{(t,j) : t < i\}$. For example, the squares other than $(4,7)$ that are rook canceled by a rook in $(4,7)$ in $\mathbf{B}_8$ are pictured in Figure 15 with a $\bullet$ in them. Then for any rook placement $\mathbb{P} \in M_k(B)$, we let $v_B(\mathbb{P})$ denote the number of squares in $B - \mathbb{P}$ that do not contain a rook in $\mathbb{P}$ and are not rook canceled by any rook in $\mathbb{P}$. If $k > 0$, we define the k-th q-rook number of B to be

$$m_k(B,q) = \sum_{\mathbb{P} \in M_k(B)} q^{v_B(\mathbb{P})}, \tag{2.13}$$

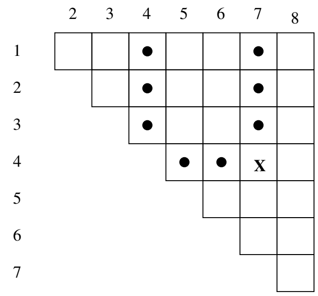

Figure 15: The cells rook canceled by (4,7) in $\mathbf{B}_8$.

and, if $k=0$, we set $m_0(B,q)=q^{|B|}$.

Then Haglund and Remmel [7] proved the following q-analogue of Theorem 2.6.

Theorem 2.7. *Let $B \subseteq \boldsymbol{B}_{2n}$ be a nearly Ferrers board and let b_i denote the number of cells of B that lie in row i for each $1 \le i \le 2n-1$. Then*

$$\prod_{i=1}^{2n-1} [x+b_{2n-i}-2i+2]_q = \sum_{k=0}^{2n-1} m_{n-k}(B,q)[x]_q \downarrow_{2n-1-k,2} \tag{2.14}$$

2.5 The Goldman-Haglund Generalized Rook Model

2.5.1 The i-Creation Model

A model which produces a product formula that has rising factorials on the right-hand side is the *i-creation model* due to Goldman and Haglund [5]. In this model, rooks are placed from left to right and new cells are created after an *i-creation rook* is placed in the board, rather than cells being canceled. For $i \in \mathbb{N}$, we call $B^{(i)} = F(b_1, b_2, \ldots, b_n)$ an *i-creation board* if $B = F(b_1, b_2, \ldots, b_n)$ is a Ferrers board, and, when an i-creation rook is placed in $B^{(i)}$, it replaces all the cells in its row to its right with i cells, the lowest of which get canceled - a process called *i-creation*. The next i-creation rook, reading from left to right, may then be placed in any available cell, both those that were part of the original board and those that have been *i-created*. An example of a 3-creation board and a placement of three 3-creation rooks can be seen in Figure 16.

Let $\mathcal{N}_k^{(i)}(B)$ denote the set of placements of k rooks in an i-creation board $B^{(i)}$ so that there is at most one rook in each column and no rook lies in a cell which is canceled by a rook to its left. Let $r_k^{(i)}(B) = |\mathcal{N}_k^{(i)}(B)|$. We call $r_k^{(i)}(B)$ the *k-th i-creation rook number of B*. In the special case where $B = F(b_1, b_2, \ldots, b_n)$, we shall write $r_k^{(i)}(b_1, b_2, \ldots, b_n)$ for $r_k^{(i)}(B)$. By classifying rook placements according to whether or not they have a rook in the last column, it is then easy to see that i-creation rook numbers satisfy the following recursion:

$$r_{n+1-k}^{(i)}(b_1, b_2, \ldots, b_{n+1}) = r_{n+1-k}^{(i)}(b_1, b_2, \ldots, b_n) + (b_{n+1} + k(i-1)) r_{n-k}^{(i)}(b_1, b_2, \ldots, b_n).$$

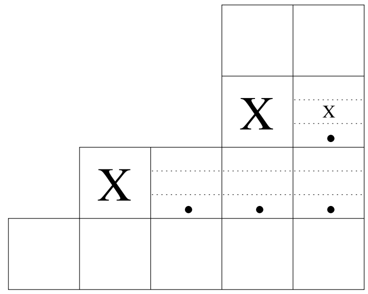

Figure 16: A placement of 3 i-creation rooks in $B^{(i)}$ where $B = F(1, 2, 2, 4, 4)$ and $i = 3$.

Note that for any Ferrers board B, $r_k^{(0)}(B) = r_k(B)$ and $r_k^{(1)}(B) = f_k(B)$.

The board $B_x^{(i)}$ is defined to be the board $B^{(i)}$ with an x-part appended below, and rooks placed in the x-part of $B_x^{(i)}$ will i-create and cancel cells exactly as would an i-creation rook placed in $B^{(i)}$. Using this construction, Haglund and Goldman [5] proved the following product formula.

Theorem 2.8. *Let $B^{(i)} = F(b_1, b_2, \ldots, b_n)$ be an i-creation board for some $i \in \mathbb{N}$. For all $x \in \mathbb{N}$,*

$$\prod_{j=1}^{n}(x + b_j + (j-1)(i-1)) = \sum_{k=0}^{n} r_{n-k}^{(i)}(B)x \uparrow_{k,i-1}, \tag{2.15}$$

where $x \uparrow_{n,m}= x(x+m)\cdots(x+(n-1)m)$ and $x \uparrow_{0,m}= 1$.

2.5.2 The α-Parameter

A more general rook placement setting was also defined by Goldman and Haglund in [5]. Here, given a Ferrers board $B = F(b_1, b_2, \ldots, b_n)$, we consider placements $\mathbb{P} \in \mathcal{F}_k(B)$. Given a placement $\mathbb{P} \in \mathcal{F}_k(B)$, we define the weight of $\mathbb{P}$, $w_{\alpha,B}(\mathbb{P})$, to be the product of the weights of all of the rows of the placement, where if a row r contains u rooks, then it has weight

$$w_{\alpha,B}(r) = \begin{cases} 1 & \text{if } 0 \leq u \leq 1, \text{ and} \\ \alpha(2\alpha-1)(3\alpha-2)\cdots((u-1)\alpha-(u-2)) & \text{if } u \geq 2. \end{cases}$$

We then set

$$r_k^{(\alpha)}(B) = \sum_{\mathbb{P}\in\mathcal{F}_k(B)} w_{\alpha,B}(\mathbb{P}),$$

and we call $r_k^{(\alpha)}(B)$ the k-th *α-rook number of* B. Goldman and Haglund [5] proved the following theorem.

Theorem 2.9. *If $B = F(b_1, b_2, \ldots, b_n)$ is a Ferrers board and α is an integer, then*

$$\prod_{j-1}^{n}(x + b_i + (j-1)(\alpha-1)) = \sum_{k=0}^{n} r_{n-k}^{(\alpha)}(B)x \uparrow_{k,\alpha-1}. \tag{2.16}$$

We note that if α is a nonnegative integer, then $r_k^{(\alpha)}(B)$ is the α-creation rook number described above and if α is a negative integer, then for a suitable board, $r_k^{(\alpha)}(B)$ is a α-attacking rook number as defined by Remmel and Wachs [9].

Finally Goldman and Haglund also proved a q-analogue of Theorem 2.9. Suppose that $B = F(b_1, b_2, \ldots, b_n)$ is a Ferrers board and consider $\mathbb{P} \in \mathcal{F}_k(B)$. Let c be any cell of B and define $\nu(c)$ to be the number of rooks which lie in the same row as c and are strictly to the left of c. We define the q-weight of c, denoted by $w_{q,\alpha,B}(c)$, to be

$$w_{q,\alpha,B}(c) = \begin{cases} 1 & \text{if there is a rook directly above } c \\ [(\alpha-1)\nu(c)+1]_q & \text{if there is a rook in } c\text{, and} \\ q^{(\alpha-1)\nu(c)+1} & \text{otherwise,} \end{cases}$$

and the weight of file placement $\mathbb{P}$, $w_{q,\alpha,B}(\mathbb{P})$, to be

$$w_{q,\alpha,B}(\mathbb{P}) = \prod_{c \in B} w_{q,\alpha,B}(c). \tag{2.17}$$

Then we define the k-th *q-α-rook number*, $r_k^{(\alpha)}(B,q)$ by

$$r_k^{(\alpha)}(B,q) = \sum_{\mathbb{P} \in \mathcal{F}_k(B)} w_B(\mathbb{P}). \tag{2.18}$$

This given, Goldman and Haglund [5] proved the following theorem.

Theorem 2.10. *If $B = F(b_1, b_2, \ldots, b_n)$ is a Ferrers board, then*

$$\prod_{i=1}^{n} [x + b_i - (j-1)(\alpha-1)]_q = \sum_{k=0}^{n} r_{n-k}^{(\alpha)}(B,q)[x]_q \uparrow_{k,\alpha-1}, \tag{2.19}$$

where $[x]_q \uparrow_{n,m} = [x]_q[x+m-1]_q \cdots [x+(n-1)(m-1)]_q$.

3 Augmented Rook Boards

The main goal of this section is to prove the generalized product formula (1.14). To do this, we must first present an appropriate rook model. Fix two sequences from $\mathbb{N}^n$, $\mathcal{B} = \{b_i\}_{i=1}^n$ and $\mathcal{A} = \{a_i\}_{i=1}^n$, and two functions $sgn, \overline{sgn} : [n] \to \{1, -1\}$. Let $A_i = a_1 + a_2 + \cdots + a_i$ be the i-th partial sum of the a_i's and let $B = F(b_1, b_2, \ldots, b_n)$. We will consider the *augmented rook board*, $\mathcal{B}^{\mathcal{A}}$, which is constructed by starting with the board B and then adding A_i cells on top of the i-th column for $i = 1, \ldots, n$. Thus $\mathcal{B}^{\mathcal{A}}$ can be thought of as the board $F(b_1 + A_1, b_2 + A_2, \ldots, b_n + A_n)$. For example, if $\mathcal{B} = (1, 2, 2, 3)$ and $\mathcal{A} = (1, 2, 1, 2)$, then Figure 17 pictures the board B and the board $\mathcal{B}^{\mathcal{A}}$. We will refer to the part of the board which corresponds to the b_i's as the *base part of $\mathcal{B}^{\mathcal{A}}$* and the part which corresponds to the a_i's as the *augmented part of $\mathcal{B}^{\mathcal{A}}$*. Moreover, for each column i, we will refer to the cells in rows $b_1 + 1, \ldots, b_1 + a_1$ as the a_1-st part

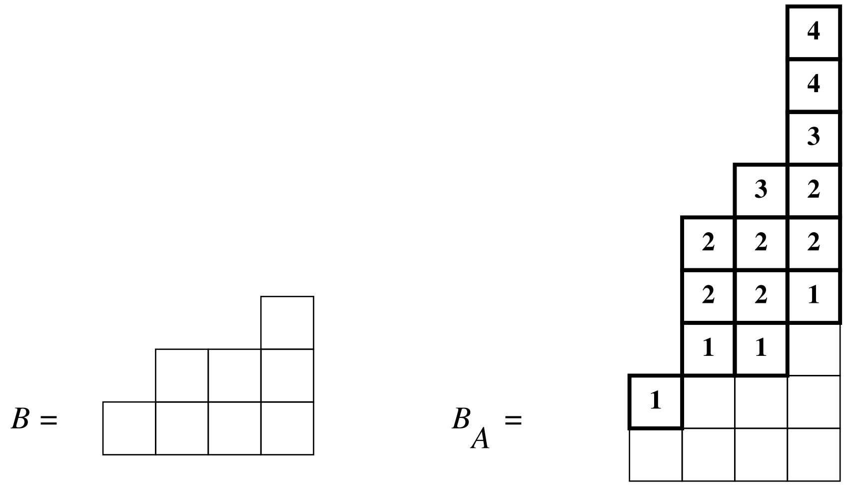

Figure 17: An Augmented Rook Board, $\mathcal{B}^{\mathcal{A}}$, with $n = 4$.

of i-th column, the cells in rows $b_1 + a_1 + 1, \ldots, b_1 + a_1 + a_2$ as the a_2-nd part of i-th column, etc. In Figure 17, we have indicated the a_s-th part of each column by putting an s in those cells.

Next we must define the appropriate notion of nonattacking rook placements in $\mathcal{B}^{\mathcal{A}}$. We first consider placements $\mathbb{P}$ of rooks in $\mathcal{B}^{\mathcal{A}}$ where there is at most one rook in each column. Now the leftmost rook of $\mathbb{P}$ will cancel all the cells in the columns to its right which correspond to the a_s-th part of that column of highest index. Thus, if the leftmost rook is in column i, then it will cancel the a_j-th part of column j for $j = i+1, \ldots, n$. In general, each rook will cancel all the cells in the columns to its right which correspond to the a_s-th part of that column where s is the highest index such that the cells of a_s-th part of the column have not been canceled by any rook to its left. We then let $\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ denote the set of placements of k rooks in the board $\mathcal{B}^{\mathcal{A}}$ such that (i) there is at most one rook per column and (ii) no rook lies in a cell which has been canceled by a rook to its left. For example, if $\mathcal{B} = (1, 2, 2, 3)$ and $\mathcal{A} = (1, 2, 1, 2)$, then we have illustrated in Figure 18 a placement $\mathbb{P} \in \mathcal{N}_2^{\mathcal{A}}(B)$ where we have placed a $\bullet$ in all cells canceled by rook in column 1 and a $*$ in all cells canceled by the rook in column 2. We shall refer to a placement $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(B)$ as a placement of k nonattacking rook in $\mathcal{B}^{\mathcal{A}}$.

We define

$$r_k(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}}) = \sum_{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(B)} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \tag{3.1}$$

where

$$w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = \prod_{r \in \mathbb{P}} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},\mathbb{P}}(r) \tag{3.2}$$

and, for any rook r, if r is the rook in the i-th column, then

1. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r) = sgn(i)$ if r is in base part of $\mathcal{B}^{\mathcal{A}}$ and

2. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r) = -\overline{sgn}(s)$ if r is in the a_s-th part of the augmented part of $\mathcal{B}^{\mathcal{A}}$.

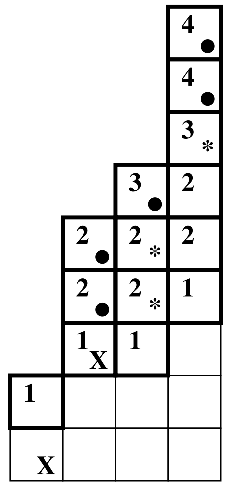

Figure 18: A Placement of Two Rooks in an Augmented Rook Board, $\mathcal{B}^{\mathcal{A}}$.

Then the goal of this section is to prove the following theorem.

Theorem 3.1. *Suppose $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$ are two sequences of nonnegative integers and $sgn : \{1, \ldots, n\} \to \{1, -1\}$ and $\overline{sgn} : \{1, \ldots, n\} \to \{1, -1\}$ are two sign functions. Then,*

$$\prod_{i=1}^{n}(x + sgn(i)(b_i)) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) \prod_{j=1}^{k}(x + \sum_{s \leq j} \overline{sgn}(s)(a_s)). \tag{3.3}$$

Proof. In order to prove Theorem 3.1, we need to define the analogue $\mathcal{B}_x^{\mathcal{A}}$ for augmented rooks boards of the board B_x. Given two sequences of nonnegative integers $\mathcal{B}$ and $\mathcal{A}$ and a nonnegative integer x, the board $\mathcal{B}_x^{\mathcal{A}}$ will have three parts. First we start with the board $\mathcal{B}^{\mathcal{A}}$ which will refer to as the *upper part of* $\mathcal{B}_x^{\mathcal{A}}$. Here the part of the upper part of $\mathcal{B}_x^{\mathcal{A}}$ that corresponds to the board $B = F(b_1, b_2, \ldots, b_n)$ will be called the *base part* of $\mathcal{B}_x^{\mathcal{A}}$ and the part which corresponds to the a_i's will be called the *upper augmented part* of $\mathcal{B}_x^{\mathcal{A}}$. Directly below $\mathcal{B}^{\mathcal{A}}$, we will attach x-rows of length n which will be referred to as the *x-part* of $\mathcal{B}_x^{\mathcal{A}}$. Finally, directly below the x-part, we will place the flip of a Ferrers board $F(A_1, \ldots, A_n)$ which will be called the *lower augmented part* of $\mathcal{B}_x^{\mathcal{A}}$. We will say that the x-part is separated from the upper part of $\mathcal{B}_x^{\mathcal{A}}$ by the *high bar* and from the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ by the *low bar*. For example, Figure 19 pictures the board $\mathcal{B}_x^{\mathcal{A}}$ where $\mathcal{B} = (1, 2, 2, 3)$, $\mathcal{A} = (1, 2, 1, 2)$, and $x = 4$ on the left. Much like we did for the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$, we will refer to the first a_1 cells of the lower augmented part of a column i, reading from top to bottom, as the a_1-st part of the i-th column of the lower augmented part, the next a_2 cells, reading from top to bottom, as the a_2-nd part of the i-th column of the lower augmented part, etc. Again, we indicate the a_s-th part of each column by placing an s in those cells.

Next we need to define the set of placements of n nonattacking rooks on $\mathcal{B}_x^{\mathcal{A}}$. First we will consider placements of n rooks on $\mathcal{B}_x^{\mathcal{A}}$ where there is exactly one rook in each column. The cancellation rules for each rook are the following:

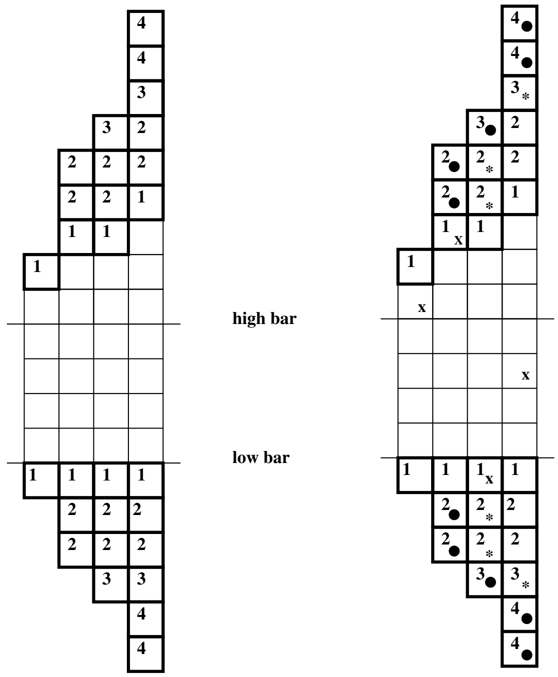

Figure 19: An Example of an Augmented General Rook Board, $B_x^{\mathcal{A}}$, with $\mathcal{B} = (1, 2, 2, 3)$, $\mathcal{A} = (1, 2, 1, 2)$, and $x = 4$, and a placement of rooks in $\mathcal{B}_x^{\mathcal{A}}$.

1. A rook placed above the high bar in the j-th column of $\mathcal{B}_x^{\mathcal{A}}$ will cancel all of the cells in columns $j+1, j+2, \ldots, n$, in both the upper and lower augmented parts, which belong to the a_i-th part of highest subscript in that column which are not canceled by a rook to the left of column j.

2. Rooks placed below the high bar do not cancel anything.

We then let $\mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ denote the set of all placements of n rooks in $\mathcal{B}_x^{\mathcal{A}}$ for which there is exactly one rook in each column and no rook lies in a cell which is canceled by a rook to its left. An example of a rook placement $\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ is pictured in Figure 19 on the right. Here we have indicated the cells canceled by the rook in the first column of upper augmented part by placing a $\bullet$ in those cells and the cells canceled by the rook in the second column of the upper augmented part by placing an $*$ in those cells. The rooks placed in the third and fourth columns do not cancel any cells since they are placed below the high bar.

First we prove two key lemmas about placements in $\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$.

Lemma 3.2. *For any placement $\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$, if there are $b_j + A_m$ uncanceled cells in the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$ in column j, then there are A_m uncanceled cells in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$.*

Proof. This lemma follows directly from our definition of cancellation for placements $\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ since it is easy to see by induction on j that for any $1 \leq s \leq j$, the a_s-th part of the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$ is canceled by a rook r to the left of column j if and only if the a_s-th part of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ is canceled by r. $\square$

Lemma 3.3. *For any placement $\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ which has k rooks above the high bar, the cells which are not canceled in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ in the i-th column from the left that does not contain a rook above the high bar are precisely the cells corresponding to the a_s-th part of that column for $s = 1, \ldots, i$. Thus the column heights of the uncanceled cells in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ in those columns which do not contain rooks above the high bar are $A_1, \ldots, A_{n-k}$, reading from left to right.*

Proof. We proceed by induction on the number of rooks k placed above the high bar in $\mathbb{P}$. Clearly, if $k = 0$, then all the rooks of $\mathbb{P}$ are placed below the high bar. Since our definitions ensure that rooks placed below the high bar do not cancel any cells, the lemma follows in this case from our definition of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$.

Now assume that the lemma holds for some $k \geq 0$ and suppose that $\mathbb{P}$ has $k+1$ rooks above the high bar such that the rightmost of these rooks, r, is placed in column j for some $k+1 \leq j \leq n$. When constructing $\mathbb{P}$, suppose that we first place the first k rooks above the high bar, from left to right. Then, by induction, in the $j-k-1$ columns available below the low bar to the left of column j, there will be, from left to right, $A_1, A_2, \ldots, A_{j-k-1}$ available cells to place a rook in each of those columns. Also from our induction hypothesis, column j will have A_{j-k} available cells, and columns $j+1, j+2, \ldots, n$ will have $A_{j-k+1}, A_{j-k+2}, \ldots, A_{n-k}$ available cells respectively. Now, when we place r in column j above the high bar, then the number of available cells to the left of r below the low bar will remain unchanged. That is, there are no longer any available cells below the low bar in column j since there is now a rook in that column. It is easy to see from our definitions that below the low bar to the right of r, the number of available cells in column $j+a$ for each $a = 1, 2, \ldots, n-j$ will be $A_{j-k+a} - a_{j-k+a} = A_{j-k+a-1}$. Thus, the number of available cells below the low bar in the columns to the right of r are respectively,

$$A_{(j-k+1)-1}, A_{(j-k+2)-1}, \ldots, A_{(n-k)-1} = A_{j-k}, A_{j-k+1}, \ldots, A_{n-(k+1)},$$

which completes the induction. $\square$

Let

$$S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}}) = \sum_{\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})} w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(\mathbb{P}) \tag{3.4}$$

where

$$w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(\mathbb{P}) = \prod_{i=1}^{n} w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) \tag{3.5}$$

and, if r_i is the rook in the i-th column of $\mathbb{P}$, then

1. $w_{sgn,\overline{sgn},,\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = \overline{sgn}(s)$ if r_i is in the a_s-th part of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$,

2. $w_{sgn,\overline{sgn},,\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = 1$ if r_i is in the x-part of $\mathcal{B}_x^{\mathcal{A}}$,

3. $w_{sgn,\overline{sgn},,\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = sgn(i)$ if r_i is in the base part of $\mathcal{B}_x^{\mathcal{A}}$, and

4. $w_{sgn,\overline{sgn},,\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = -\overline{sgn}(s)$ if r_i is in the a_s-th part of the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$.

It will be instructive to consider two special cases of (3.3). First consider the case where $sgn(i) = +1$ and $\overline{sgn}(i) = -1$ for every $1 \leq i \leq n$. In this case, we will set

$$r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) = r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}).$$

Then (3.3) reduces to

$$\prod_{i=1}^{n}(x + b_i) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})(x - A_1)(x - A_2)\cdots(x - A_k). \tag{3.6}$$

Moreover, it is easy to see that in this case,

1. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = -1$ if r_i is in the a_s-th part of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$,

2. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = 1$ if r_i is in the x-part of $\mathcal{B}_x^{\mathcal{A}}$,

3. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = 1$ if r_i is in $\mathcal{B}$-part of $\mathcal{B}_x^{\mathcal{A}}$, and

4. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = 1$ if r_i is in the a_s-th part of the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$.

Thus $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(\mathbb{P}) = (-1)^{l_{\mathbb{P}}}$ where $l_{\mathbb{P}}$ is the number of rooks in $\mathbb{P}$ which lie in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$. Similarly, $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) = |\mathcal{N}_k^{\mathcal{A}}(B)| = r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$.

Then we claim that (3.6) arises from two different ways of computing the sum $S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}})$.

If we first place the rooks starting with the leftmost column and working to the right, then we can see that in the first column there are exactly $x+b_1+2a_1$ cells in which to place the first rook, where the "$2a_1$" corresponds to the choices of placing the rook in either the upper or the lower augmented part of the first column. Since all of the rooks above the high bar are weighted with a "$+1$" and all of the rooks placed below the low bar are weighted with a "-1", we get a total weighting of $x + b_1 + a_1 + (-a_1) = x + b_1$ for the first column. When we go to place a rook in the second column, we have two cases.

<u>Case I:</u> Suppose that the rook that was placed in the first column was placed below the high bar. Then nothing was canceled in the second column, so we can place a rook in any cell of the second column. Thus we have $x + b_2 + 2(a_1 + a_2)$ choices as to where to put this rook. However, we weight the two choices which correspond to the "$2(a_1 + a_2)$" term differently, as rooks in the upper augmented part get weighted with a "$+1$" and those in the lower augmented part with a "-1". Thus, the weighting for this column is $x + b_2 + (a_1 + a_2) + (-a_1 - a_2) = x + b_2$.

Case II: If the rook placed in the first column was placed above the high bar, then the cells corresponding to a_2-nd part in both the upper and the lower augmented parts of the second column are canceled. Thus there are $x + b_2 + 2a_1$ cells left to place the rook and, hence, the weighting of the second column is $x + b_2 + a_1 + (-a_1) = x + b_2$.

In general, suppose we are placing a rook in the j-th column where we have placed s rooks above the high bar and t rooks below the high bar in the first $j-1$ columns. Then in the j-th column we have, by Lemma 3.2, $x + b_j + 2A_{t+1}$ choices as to where to place the rook in that column. Again, these placements will come with a weighting of $x + b_j + A_{t+1} + (-A_{t+1}) = x + b_j$. Thus, it follows that

$$S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}}) = \prod_{i=1}^{n}(x + b_i)$$

which gives the left-hand side of (3.6).

The second way of computing $S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}})$ is to organize the placements by how many rooks lie above the high bar. That is, suppose that we fix a placement $\mathbb{P}$ of $n-k$ nonattacking rooks in $\mathcal{B}^{\mathcal{A}}$. Then we wish to compute

$$\sum_{\substack{Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}) \\ Q \cap \mathcal{B}^{\mathcal{A}} = \mathbb{P}}} w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(Q). \tag{3.7}$$

Each such Q in the sum arises from $\mathbb{P}$ by placing one rook below the high bar in each of the k columns that do not contain a rook above the high bar. We will place the remaining rooks in these available columns starting with the leftmost one and working right. By Lemma 3.3, the number of ways we can do this will be $(x+A_1)(x+A_2)\cdots(x+A_k)$. However, as all the rooks in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ have weight "-1" and all the rooks in x-part of $\mathcal{B}_x^{\mathcal{A}}$ have weight "$+1$", we see that

$$\sum_{\substack{Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}) \\ Q \cap \mathcal{B}^{\mathcal{A}} = \mathbb{P}}} w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(Q) = w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P})(x + (-A_1))(x + (-A_2))\cdots(x + (-A_k)). \tag{3.8}$$

Thus,

$$\begin{aligned} S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}}) &= \sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{N}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} (x - A_1)(x - A_2)\cdots(x - A_k) \\ &= \sum_{k=0}^{n} r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})(x - A_1)(x - A_2)\cdots(x - A_k) \end{aligned}$$

which gives the right-hand side of (3.6).

Next consider the case where $sgn(i) = +1$ and $\overline{sgn}(i) = +1$ for every $1 \leq i \leq n$. In this case, we see that

1. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = 1$ if r_i is in the a_s-th part of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$,

2. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = 1$ if r_i is in the x-part of $\mathcal{B}_x^{\mathcal{A}}$,

3. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = 1$ if r_i is in the base part of $\mathcal{B}_x^{\mathcal{A}}$, and

4. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = -1$ if r_i is in the a_s-th part of the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$.

Thus, in this case, $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(\mathbb{P}) = (-1)^{u_{\mathbb{P}}}$ where $u_{\mathbb{P}}$ is the number of rooks in $\mathbb{P}$ which lie in the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$. Hence $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) = \sum_{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(B)}(-1)^{u_{\mathbb{P}}}$.

Now consider the two different ways of computing the sum $S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}})$. First, if we consider placing the rooks column by column, reading from left to right, then the sum of the weights of possible placements of the rook in the i-th column is still $(x + b_i)$ because that argument depended only on the fact that sum of the weights of the uncanceled cells in the upper and the lower augmented parts of the board in the i-th column equals 0. Thus,

$$S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}}) = \prod_{i=1}^{n}(x + b_i)$$

as before.

For the second way of computing $S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}})$, suppose that we fix a placement $\mathbb{P}$ of $n-k$ nonattacking rooks in $\mathcal{B}^{\mathcal{A}}$. Then we wish to compute

$$\sum_{\substack{Q\in\mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})\\ Q\cap\mathcal{B}^{\mathcal{A}}=\mathbb{P}}} w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(Q). \tag{3.9}$$

As before, each such Q in the sum arises from $\mathbb{P}$ by placing a rook below the high bar in each of remaining k empty columns. Again we can argue that Lemma 3.3 implies that the number of ways we can do this is $(x + A_1)(x + A_2)\cdots(x + A_k)$. Since the weights of all rooks below the bar is "+1", it follows that

$$\sum_{\substack{Q\in\in\mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})\\ Q\cap\mathcal{B}^{\mathcal{A}}=\mathbb{P}}} w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(Q) = w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P})(x + A_1)(x + A_2)\cdots(x + A_k). \tag{3.10}$$

Hence

$$\begin{aligned} S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}}) &= \sum_{k=0}^{n}\sum_{\mathbb{P}\in\mathcal{N}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P})(x + A_1)(x + A_2)\cdots(x + A_k) \\ &= \sum_{k=0}^{n} r_{n-k}(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}})(x + A_1)(x + A_2)\cdots(x + A_k). \end{aligned}$$

Thus, in this case, we have

$$\prod_{i=1}^{n}(x + b_i) = \sum_{k=0}^{n}\tilde{r}_{n-k}^{\mathcal{A}}(\mathcal{B})(x + A_1)(x + A_2)\cdots(x + A_k) \tag{3.11}$$

where $\tilde{r}^{\mathcal{A}}_{n-k}(\mathcal{B}) = r_{n-k}(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}})$ with $sgn(i) = \overline{sgn}(i) = 1$ for all i.

It is easy to check that these two special cases encapsulate all the $q = 1$ cases of the product formulas stated in the Section 2.

To prove the general case of (3.3), we again claim that (3.3) arises from two different ways of computing the sum $S(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}}_x)$. That is, if we first place the rooks starting with the leftmost column and working to the right, then we can see that in the first column there are exactly $x + b_1 + 2a_1$ cells in which to place the first rook, where the "$2a_1$" corresponds to placing the rook in either the upper or the lower augmented part of the first column. Since the rooks in the x-part are weighted with a "$+1$", the rooks in the i-th column of the base part are weighted with $sgn(i)$, the rooks in the lower augmented part in the a_s-part are weighted with $\overline{sgn}(s)$, and the rooks in the a_s-part of the upper augmented part are weighted with $-\overline{sgn}(s)$, the placements of the rook in the first column contributes a factor of $x + sgn(1)b_1 + (\overline{sgn}(1)a_1 - \overline{sgn}(1)a_1) = x + sgn(1)b_1$ to $S(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}}_x)$. When we go to place a rook in the second column, we have two cases.

Case I: Suppose that the rook in the first column was placed below the high bar. Then nothing was canceled in the second column, so we can place a rook in any cell of the second column. Thus we have $x + b_2 + 2(a_1 + a_2)$ choices as to where to put this rook. Note that the contribution of the weights over all placements of rooks below the low bar is $\overline{sgn}(1)a_1 + \overline{sgn}(2)a_2$ and the contributions of the weights over all placements of rooks in the upper augmented part of $\mathcal{B}^{\mathcal{A}}_x$ is $(-\overline{sgn}(1)a_1) + (-\overline{sgn}(2)a_2)$. Thus, the placements of rooks the second column contribute a factor of $x + sgn(2)b_2 + (\overline{sgn}(1)a_1 + \overline{sgn}(2)a_2) - (\overline{sgn}(1)a_1 + \overline{sgn}(2)a_2) = x + sgn(2)b_2$ to $S(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}}_x)$.

Case II: If the rook in the first column was placed above the high bar, then the cells corresponding to a_2-nd part in both the upper and the lower augmented parts of the second column are canceled. Hence, there are $x + b_2 + 2a_1$ cells left to place the rook. Thus in this case, the placements of rooks in the second column contributes a factor of $x + sgn(2)b_2 + \overline{sgn}(1)a_1 - \overline{sgn}(1)a_1 = x + sgn(2)b_2$ to $S(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}}_x)$.

In general, suppose we are placing a rook in the j-th column where we have placed s rooks above the high bar and t rooks below the high bar in the first $j - 1$ columns. Then in the j-th column we have, by Lemma 3.2, $x + b_j + 2A_{t+1}$ choices as to where to place the rook in that column. Since the weight of the cells in the a_s-th part of the upper augmented board in this column is $-\overline{sgn}(s)$ and the weight of the cells in the a_s-th part of the lower augmented board in this column is $\overline{sgn}(s)$, it follows that the placements in column j contribute a factor of $x + sgn(j)b_j + \sum_{i=1}^{t+1}(\overline{sgn}(i)a_i - \overline{sgn}(i)a_i) = x + sgn(j)b_j$ to $S(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}}_x)$. Thus, it follows that

$$S(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}}_x) = \prod_{i=1}^{n}(x + sgn(i)b_i),$$

which gives the left hand side of (3.3).

The second way of computing $S(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}}_x)$ is to organize the placements by how many rooks lie above the high bar. Suppose that we fix a placement $\mathbb{P}$ of $n - k$ nonattacking rooks in $\mathcal{B}^{\mathcal{A}}$. Then we wish to compute

$$\sum_{\substack{Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}) \\ Q \cap \mathcal{B}^{\mathcal{A}} = \mathbb{P}}} w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(Q). \tag{3.12}$$

Again, each such Q in the sum arises from $\mathbb{P}$ by placing rooks below the high bar in the remaining columns. Thus there are k columns left that need to have rooks placed in them, below the high bar. We will place the remaining rooks in these available columns starting with the leftmost one and working right. By Lemma 3.3, the number of ways we can do this will be $(x+A_1)(x+A_2)\cdots(x+A_k)$. However, as all the rooks in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ have weight $\overline{sgn}(i)$ if they are in the a_i-th part of the column in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ and all the rooks in x-part of $\mathcal{B}_x^{\mathcal{A}}$ have weight "$+1$", we see that

$$\sum_{\substack{Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}) \\ Q \cap \mathcal{B}^{\mathcal{A}} = \mathbb{P}}} w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(Q) = w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \prod_{j=1}^{k}(x + \sum_{s \le j} \overline{sgn}(s)(a_s)). \tag{3.13}$$

Thus,

$$\begin{aligned} S(sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}) &= \sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{N}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \prod_{j=1}^{k}(x + \sum_{s \le j} \overline{sgn}(s)(a_s)) \\ &= \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) \prod_{j=1}^{k}(x + \sum_{s \le j} \overline{sgn}(s)(a_s)) \end{aligned}$$

which gives the right hand side of (3.3). □

3.1 Comparisons With Other Rook Models

In this section, we shall compare our rook model to the j-attacking rook model of Remmel-Wachs [9] and the j-creation model of Goldman-Haglund [5]. As noted in the introduction, the special cases of the Remmel-Wachs model when $j = 2$ correspond to the product formulas in the Haglund-Remmel model. The product formula in the Remmel-Wachs model also cover the product formulas in Briggs-Remmel model. In particular, we want to compare the rook numbers that correspond to a given product formula in our model versus these two models.

3.1.1 The Remmel-Wachs j-Attacking Model

We start with the Remmel-Wachs model. Suppose that we are given a j-attacking board $D = F(d_1, \ldots, d_n)$. Then in the Remmel-Wachs model, D gives rise to the following

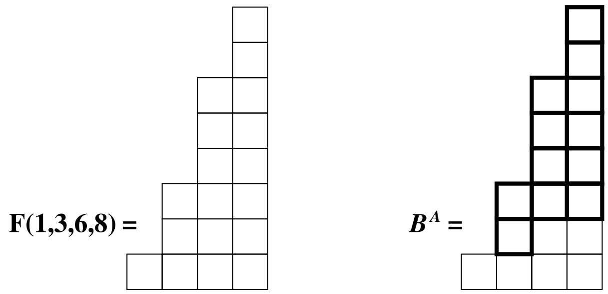

Figure 20: $F(1,3,6,8)$ versus $\mathcal{B}^{\mathcal{A}}$ where $\mathcal{B} = (1,1,2,2)$ and $\mathcal{A} = (0,2,4,6)$.

product formula:

$$\prod_{i=1}^{n}(x + d_i - j(i-1)) = \sum_{k=0}^{n} \tilde{r}^{j}_{n-k,D}\, x(x-j)\cdots(x-(k-1)j) \tag{3.14}$$

where $\tilde{r}^{j}_{n-k,D} = \tilde{r}^{j}_{n-k,D}(1,1) = |\mathcal{N}^{j}_{k}(D)|$. Now if we want to obtain the same product in our model, we must start with the sequences $\mathcal{B} = (d_1, |d_2 - j|, \ldots, |d_n - (n-1)j|)$ and $\mathcal{A} = (0, j, j, \ldots, j)$. We also must define the sign functions, sgn and $\overline{sgn}$ so that for all $i = 1, \ldots, n$,

$$\begin{aligned} \overline{sgn}(i) &= -1 \text{ and} \\ sgn(i) &= \begin{cases} 1 & \text{if } d_i - j(i-1) \geq 0 \text{ and} \\ -1 & \text{if } d_i - j(i-1) < 0. \end{cases} \end{aligned}$$

Then our general product formula will take the form

$$\prod_{i=1}^{n}(d_i - j(i-1)) = \sum_{k=0}^{n} r^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})\, x(x-j)\cdots(x-(k-1)j). \tag{3.15}$$

Because $\{(x)\downarrow_{j,n}\}_{n\geq 0}$ is a basis for the polynomial ring $\mathbb{Q}[x]$, it immediately follows from (3.14) and (3.15) that $\tilde{r}^{j}_{n-k,D} = r^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$ for all $0 \leq k \leq n$. We shall show that we can give a completely combinatorial proof of this fact. This result is best explained through some examples.

In the simplest case, when $d_i \geq j(i-1)$ for $i = 1, \ldots, n$, then it will be the case that the boards D and $\mathcal{B}^{\mathcal{A}}$ are identical. For example, if $j = 2$ and $D = (1,3,6,8)$, then $B = (1,1,2,2)$ and the board $\mathcal{B}^{\mathcal{A}}$ is just the Ferrers board $F(1,3,6,8)$ as pictured in Figure 20. In this case, it is easy to see that for any $\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{k}(\mathcal{B}^{\mathcal{A}})$, $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = 1$ so that $r^{\mathcal{A}}_{k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) = |\mathcal{N}^{\mathcal{A}}_{k}(\mathcal{B}^{\mathcal{A}})|$. Thus to prove that $\tilde{r}^{j}_{n-k,D} = r^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$, we need only find a bijection between $\mathcal{N}^{j}_{n-k}(D)$ and $\mathcal{N}^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}})$. It is easy to see that in the j-attacking Remmel-Wachs model, each rook r in a placement $\mathbb{P} \in \mathcal{N}^{j}_{k}(D)$ cancels exactly j cells in each column to its right. Similarly since $a_1 = 0$ and $a_i = j$ for $i \geq 2$, it easy to see that each rook placement $Q \in \mathcal{N}^{\mathcal{A}}_{k}(\mathcal{B}^{\mathcal{A}})$ also cancels j cells in each column

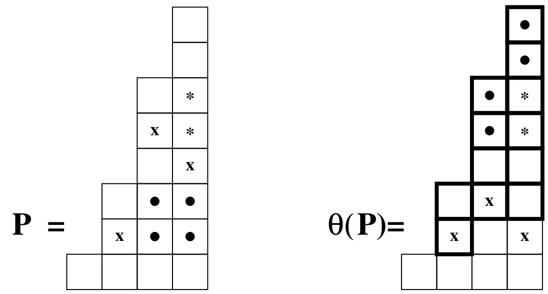

Figure 21: An example of $\Theta^{(2)}$ in the case where $d_i \geq 2(i-1)$ all i.

to its right. Thus the only real difference between the two types of rook placements in this case is the exact cells that get canceled. This suggests a very simple bijection $\Theta^{(j)} : \mathcal{N}_k^j(D) \to \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$. Namely, if $\mathbb{P} \in \mathcal{N}_k^j(D)$ has rooks $r_1, \ldots, r_k$ in columns $1 \leq i_1 < \cdots < i_k \leq n$ respectively, then $\Theta^{(j)}(\mathbb{P})$ should be the placement of rooks $\tilde{r}_1, \ldots, \tilde{r}_k$ in columns $1 \leq i_1 < \cdots < i_k \leq n$ such that for all u, if r_u is in the s_u-th cell in column i_u which is not canceled by a rook to the left of r_u, reading from bottom to top, then $\tilde{r}_u$ is in the s_u-th cell in column i_u which is not canceled by a rook to the left of $\tilde{r}_u$, reading from bottom to top. For example, if $\mathbb{P} \in \mathcal{N}_k^2(F(1,3,6,8))$ is the placement pictured on the left in Figure 21, then its image $\Theta^{(2)}(\mathbb{P})$ is pictured on the right in Figure 21. That is, the leftmost rook of $\mathbb{P}$ is in row 2 of column 2 so that the leftmost rook of $\Theta^{(2)}(\mathbb{P})$ must be placed in row 2 of column 2. We then put a $\bullet$ in those cells canceled by the leftmost rook in each case. Then we see that the rook in column 3 of $\mathbb{P}$ is in the third available cell, reading from bottom to top, so that the rook in column 3 of $\Theta^{(2)}(\mathbb{P})$ must be in the third available cell, reading from bottom to top. We then put a $*$ in those cells canceled by the rook in column 3 in each case. Finally, the rook in column 4 in $\mathbb{P}$ is in the second available cell, reading from bottom to top, so that the rook in column 4 in $\Theta^{(2)}(\mathbb{P})$ is in the second available cell in that column, reading from bottom to top.

In the general case, it may not be the case that $d_i \geq j(i-1)$. If $d_i < j(i-1)$, then $d_i - j(i-1)$ is negative and hence $sgn(i)$ must be negative. It follows that the rooks in the base part of $\mathcal{B}^{\mathcal{A}}$ in the i-th column will contribute a factor of "-1" to the weight of a placement. We will call such columns in $\mathcal{B}^{\mathcal{A}}$ the *negative columns* of $\mathcal{B}^{\mathcal{A}}$. If the i-th column of $\mathcal{B}^{\mathcal{A}}$ is a negative column so that $d_i - j(i-1) < 0$, then clearly $|d_i - j(i-1)| \leq j(i-1)$. In such a case, we will call the first $|d_i - j(i-1)|$ cells in the augmented part of $\mathcal{B}^{\mathcal{A}}$ in column i, the *mirror image* of the base part of column i. For example, suppose that $j = 2$ and D is the 2-attacking board $F(0,0,1,3,6,7)$. Hence, the product formula for this 2-attacking board is

$$x(x-2)(x+1-4)(x+3-6)(x+6-8)(x+7-10) = \sum_{k=0}^{6} \tilde{r}_{6-k}^{(2)}(x)\downarrow_{2,k} \tag{3.16}$$

Thus, the corresponding product formula for the board $\mathcal{B}^{\mathcal{A}}$ will be produced by defining $\mathcal{B} = (0,2,3,3,2,3)$, $\mathcal{A} = (0,2,2,2,2,2)$, $\overline{sgn}(i) = 1$ for $i = 1, \ldots, 6$ and $sgn(1) = 1$ and $sgn(i) = -1$ for $i > 1$. Then in Figure 22, we have pictured the board D and $\mathcal{B}^{\mathcal{A}}$ and

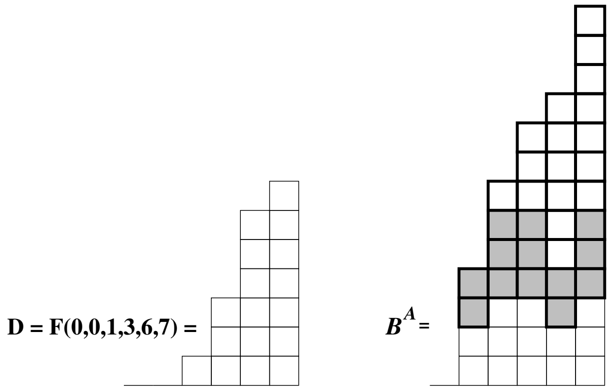

Figure 22: An example with negative columns and their mirror images.

have shaded the squares in mirror images of negative columns. In this case, the negative columns are columns 2 through 6. Note that if column i is negative, then the first $j(i-1)-d_i$ cells of the augmented part of the board $\mathcal{B}^{\mathcal{A}}$ in column i will be in the mirror image of the base part of column i and hence the number of squares in the augmented part of column i which are not in the mirror image is $j(i-1)-(j(i-1)-d_i)=d_i$. Of course, if column i is not negative, then the total number of squares in column i in $\mathcal{B}^{\mathcal{A}}$ is $d_i - j(i-1) + j(i-1) = d_i$. Thus $(d_1, \ldots, d_n)$ represents the column heights, reading from left to right, of either (i) all cells of $\mathcal{B}^{\mathcal{A}}$ in positive column or (ii) all cells in the augmented part of a column that do not lie in the mirror image of the base part of a negative column.

In the case where there are negative columns, we can define a simple sign-reversing involution I on $\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ which reduces ourselves to considering only the class of placements $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ in which no rook lies in either in a negative column or the mirror image of a negative column. That is, suppose $\mathbb{P}$ is a placement which contains a rook in a negative column or its mirror image. Let r be the leftmost rook of $\mathbb{P}$ with this property. If r is in the s-th row of the base part of the column, we let $I(\mathbb{P})$ denote the placement which results in moving r to the s-th row of its mirror image and leaving all other rooks in the same place. Note that in this case, $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},\mathbb{P}}(r) = -1$ and $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},I(\mathbb{P})}(r) = 1$ so that $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = -w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(I(\mathbb{P}))$. If r is in the s-th row of the mirror image of the base part of the column, we let $I(\mathbb{P})$ denote the placement which results in moving r to the s-th row of the negative part of the column and leaving all other rooks in the same place. Note that in this case, $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},\mathbb{P}}(r) = 1$ and $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},I(\mathbb{P})}(r) = -1$ so that once again $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = -w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(I(\mathbb{P}))$. Finally, if $\mathbb{P}$ does not have any rooks in either a negative column or its mirror image, then we let $I(\mathbb{P}) = \mathbb{P}$. An example of the involution I, when D is the 2-attacking board $F(0,0,1,3,6,7)$, $\mathcal{B} = (0,2,3,3,2,3)$, $\mathcal{A} = (0,2,2,2,2,2)$, $\overline{sgn}(i) = -1$ for $i = 1, \ldots, 6$ and $sgn(1) = 1$ and $sgn(i) = -1$ for $i > 1$, is given in Figure 23.

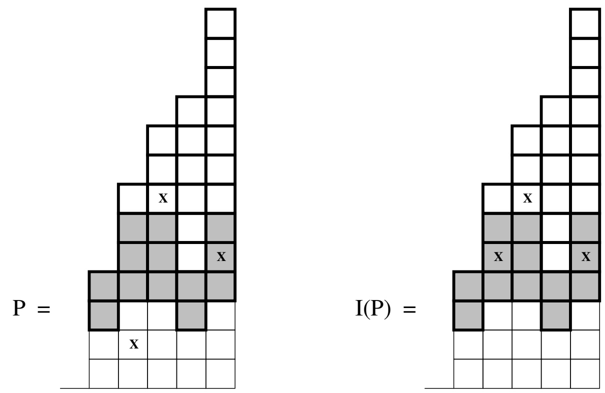

Figure 23: An example of the involution I.

Clearly, $I(I(\mathbb{P})) = \mathbb{P}$ so that I shows that

$$\begin{aligned} r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) &= \sum_{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \\ &= \sum_{\substack{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}) \\ I(\mathbb{P})=\mathbb{P}}} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \end{aligned}$$

Since the weight of any $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ such that $I(\mathbb{P}) = \mathbb{P}$ is "+1", then we need only show that there is a bijection $\Theta^{(j)}$ from $\mathcal{N}_k^j(D)$ to the fixed points of I. But we have already shown that the fixed points of I lies in a region of $\mathcal{B}^{\mathcal{A}}$ whose column heights are $(d_1, \ldots, d_n)$, reading from left to right. Since in both the Remmel-Wachs model and our model, each rook cancels j cells in each column to its right, we can use the same bijection $\Theta^{(j)}$ described previously to give the desired bijection between these two sets of rook placements.

3.1.2 Goldman-Haglund j-Creation Boards

Next we consider the j-creation model of Goldman and Haglund [5]. If we fix j and start with a Ferrers board $D = F(d_1, \ldots, d_n)$, then the product formula that arises out of the j-creation model in this case is

$$\prod_{i=1}^{n}(x + d_i + (j-1)(i-1)) = \sum_{k=0}^{n} r_{n-k}^{(j)}(D)(x) \uparrow_{k,j-1} \tag{3.17}$$

In this case, to get an equivalent product formula in our model, we must let $\mathcal{B} = (d_1, d_2 + j - 1, d_3 + 2(j-1), \ldots, d_n + (n-1)(j-1))$, $\mathcal{A} = (0, j-1, j-1, \ldots, j-1)$ and $sgn(i) = \overline{sgn}(i) = 1$ for all i so that our product formula will become

$$\prod_{i=1}^{n}(x + d_i + (j-1)(i-1)) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})(x) \uparrow_{k,j-1} . \tag{3.18}$$

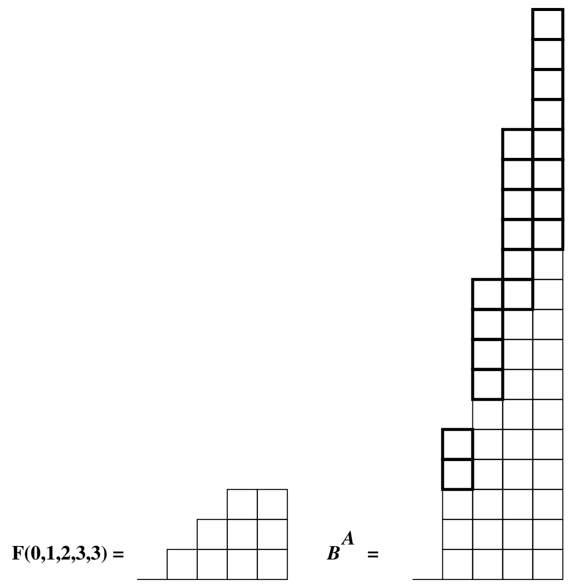

Figure 24: An example of the difference of shapes between the 3-creation board $B^{(3)}$ with $B = F(0,1,2,3,3)$ and the corresponding augmented rook board.

Since $\{(x)\uparrow_{n,j}\}_{n\geq 0}$ is a basis for the polynomial ring $\mathbb{Q}[x]$, it immediately follows from (3.17) and (3.18) that $r^{(j)}_{n-k}(D) = r^{\mathcal{A}}_{n-k}(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}})$ in this case.

Again our goal is to give a completely combinatorial proof of this fact. To do this, we will follow the same steps that we did for the general case of the Remmel-Wachs model, namely, we will first define an involution J on $\mathcal{N}^{\mathcal{A}}_k(\mathcal{B}^{\mathcal{A}})$ and then give a bijection between $\mathcal{N}^{(j)}(D)$ and the fixed points of J. As before, these steps are best illustrated through an example. We will let $j = 3$ and consider the 3-creation board $B^{(3)}$ with $B = F(0,1,2,3,3)$. Thus the product formula for $B^{(3)}_x$ in this case is

$$x(x+3)(x+6)(x+9)(x+11) = \sum_{k=0}^{5} r^{(3)}_{5-k}(B)x\uparrow_{k,2}. \tag{3.19}$$

To generate the same product formula in our model, we must set $\mathcal{B} = (0,3,6,9,11)$, $\mathcal{A} = (0,2,2,2,2)$, and $sgn(i) = \overline{sgn}(i) = 1$ for every i. If we then construct $\mathcal{B}^{\mathcal{A}}$, we can see the vast difference in shape between these two boards which are pictured in Figure 24.

Note also that our weighting ensures that each rook in the augmented part of the board has weight "-1" and each rook in base part of the board has weight "$+1$". For each rook placement $\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_k(\mathcal{B}^{\mathcal{A}})$, we define the *mirror image of the augmented part of* $\mathcal{B}^{\mathcal{A}}$ *in the i-th column relative to* $\mathbb{P}$ to consist of the top s cells in the base part of column i if there are s cells in the augmented part of the column i which are not canceled by the rooks to the left of column i. In Figure 25, we have pictured a placement $\mathbb{P}$ which has rooks in columns 2 and 4. We have placed a $\bullet$ in the cells canceled by the rook in column 2 and we have placed an $*$ in the cells canceled by the rooks in column 4. We

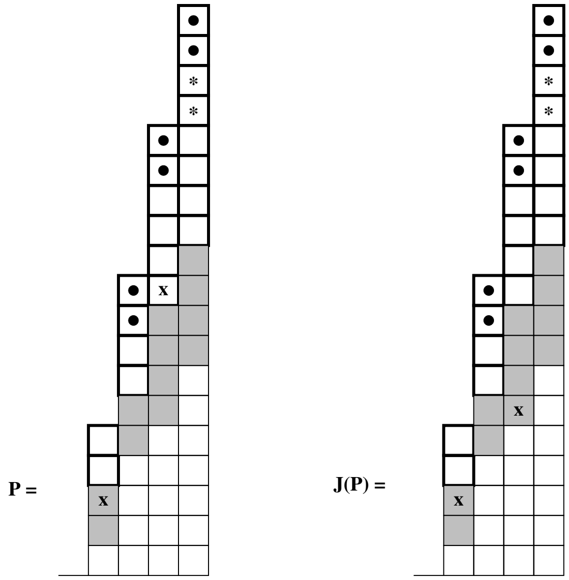

Figure 25: An example of the involution J.

have also shaded, in the base part of the board, each cell which is in the mirror image of the uncanceled cells in the augmented part of its column. Now the involution J is very simple. That is, suppose $\mathbb{P}$ is a placement which contains a rook in an cell in the augmented part of $\mathcal{B}^{\mathcal{A}}$ or its mirror image. Let r be the rightmost rook of $\mathbb{P}$ with this property. If r is in the s-th row, reading from bottom to top, in the augmented part of the column, we let $J(\mathbb{P})$ denote the placement which results by moving r to the s-th row of its mirror image, reading from bottom to top, and leaving all other rooks in the same place. Note that in this case, $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},\mathbb{P}}(r) = -1$ and $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},J(\mathbb{P})}(r) = 1$ so that $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = -w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(J(\mathbb{P}))$. If r is in the s-th row, reading bottom to top, of the mirror image of the uncanceled cells in the augmented part of its column, we let $J(\mathbb{P})$ denote the placement which results by moving r to the s-th row of the uncanceled cells in the augmented part of its column, reading bottom to top, and leaving all other rooks in the same place. Note that in this case, $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},\mathbb{P}}(r) = 1$ and $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},J(\mathbb{P})}(r) = -1$ so that once again $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = -w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(J(\mathbb{P}))$. Finally, if $\mathbb{P}$ does not have any rooks in either an uncanceled cell in the augmented part or its mirror image in any column, then we let $J(\mathbb{P}) = \mathbb{P}$. In Figure 25, we have pictured a placement $\mathbb{P} \in \mathcal{N}_2^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ on the left, where $\mathcal{B} = (0,3,6,9,11)$, $\mathcal{A} = (0,2,2,2,2)$, and $sgn(i) = \overline{sgn}(i) = +1$ for every j, and the placement $J(\mathbb{P})$ on the right.

Clearly, $J(J(\mathbb{P})) = \mathbb{P}$, so J shows that

$$\begin{aligned} r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) &= \sum_{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \\ &= \sum_{\substack{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}) \\ J(\mathbb{P})=\mathbb{P}}} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \end{aligned}$$

The weight of any $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ such that $J(\mathbb{P}) = \mathbb{P}$ is 1 since the only way that $J(\mathbb{P}) = \mathbb{P}$ is to have all the rooks of $\mathbb{P}$ lie in the base part of $\mathcal{B}^{\mathcal{A}}$. Thus we need only show that there is a bijection Δ from the fixed points of J to $\mathcal{N}_k^{(j)}(D)$. In this case, the bijection Δ can be constructed by recursion. That is, suppose that $D = F(d_1, \ldots, d_n,)$ is a Ferrers board where $n \geq 2$. Then we claim that for $0 \leq k \leq n$, we have the following recursion among rook numbers in the j-creation model of Goldman and Haglund [5].

$$\begin{aligned} & r_k^{(j)}(F(d_1, \ldots, d_n)) = \\ & r_k^{(j)}(F(d_1, \ldots, d_{n-1})) + (d_n + (j-1)(k-1)) r_{k-1}^{(j)}(F(d_1, \ldots, d_{n-1})). \end{aligned} \tag{3.20}$$

The recursion given in (3.20) is the result of classifying the rook placements in $\mathcal{N}_k^{(j)}(F(d_1, \ldots, d_n))$ according to whether there is a rook in the last column or not. If $\mathbb{P} \in \mathcal{N}_k^{(j)}(F(d_1, \ldots, d_n))$ does not have a rook in the last column, then we can also regard $\mathbb{P}$ as a placement in $\mathcal{N}_k^{(j)}(F(d_1, \ldots, d_{n-1}))$. However if $\mathbb{P} \in \mathcal{N}_k^{(j)}(F(d_1, \ldots, d_n))$ has a rook in the last column, then we let $Q \in \mathcal{N}_{k-1}^{(j)}(F(d_1, \ldots, d_{n-1}))$ be the rook placement which consists of the rooks of $\mathbb{P}$ in the first $n-1$ columns. For each such Q, there will be a total of $d_n + (j-1)(k-1)$ rows in which to place a rook in the last column since each of the $k-1$ rooks in Q create $j-1$ new rows in which to place a rook in the last column.

Next consider the two sequences of length n,

$$\mathcal{B}_n = (d_1, d_2 + (j-1), \ldots, d_n + (n-1)(j-1)) \text{ and } \mathcal{A}_n = (0, j-1, \ldots, j-1)$$

and the two sequences of length $n-1$,

$$\mathcal{B}_{n-1} = (d_1, d_2 + (j-1), \ldots, d_{n-1} + (n-2)(j-1)) \text{ and } \mathcal{A}_{n-1} = (0, j-1, \ldots, j-1).$$

We claim that if

$$\tilde{r}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) = |\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P}) = \mathbb{P}\}|, \tag{3.21}$$

then,

$$\tilde{r}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) = \tilde{r}_k^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}) + (d_n + (j-1)(k-1)) \tilde{r}_{k-1}^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}). \tag{3.22}$$

Again, this recursion is the result of classifying the rook placements in $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n})$ such that $J(\mathbb{P}) = \mathbb{P}$ according to whether or not there is a rook in the last column. That is, if $R \in \{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P}) = \mathbb{P}\}$ has no rook in the last column, then R can be viewed as an element of $\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}) : J(\mathbb{P}) = \mathbb{P}\}$. On the other hand, if $\mathbb{P}$ does have a rook in its last column, then let Q be the placement that is the restriction of $\mathbb{P}$ to the first $n-1$ columns. It is easy to check that our definition of the involution J for $\mathcal{N}_{k-1}^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}})$ ensures that $Q \in \mathcal{N}_{k-1}^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}})$ and $J(Q) = Q$. Moreover, we claim that number of ways to extend Q to placement in $\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P}) = \mathbb{P}\}$ is $d_n + (j-1)(k-1)$. Note that the n-th column of $\mathcal{B}_n^{\mathcal{A}_n}$ has height $d_n + (j-1)(n-1) + (j-1)(n-1)$ with the top $(j-1)(n-1)$ cells being in the augmented part of the board. Each of the rooks in Q cancels $j-1$ cells in the augmented part of $\mathcal{B}_n^{\mathcal{A}_n}$ in the

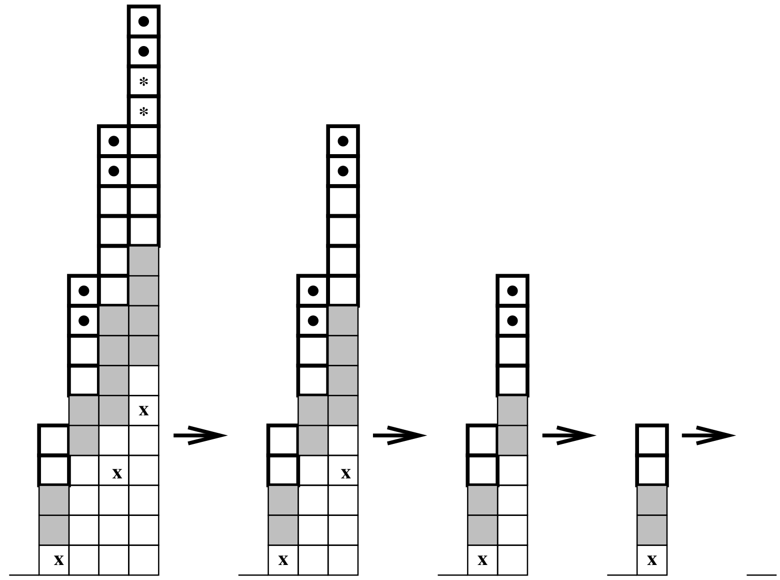

Figure 26: The recursive deconstruction of $\mathbb{P} \in \mathcal{N}_3^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$.

n-th column. Thus there are $(j-1)(n-1)-(j-1)(k-1)=(j-1)(n-k)$ cells in the augmented part of $\mathcal{B}_n^{\mathcal{A}_n}$ in the n-th column which are not canceled so that the mirror image of these cells is the top $(j-1)(n-k)$ cells in the base part of $\mathcal{B}_n^{\mathcal{A}_n}$. Since for a fixed point of J, we are not allowed to place a rook in either the augmented part of the n-th column or the mirror image of its uncanceled cells, it follows that we have a total of $d_n+(j-1)(n-1)-(j-1)(n-k)=d_n+(j-1)(k-1)$ ways to extend Q to a placement $\mathbb{P} \in \{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P})=\mathbb{P}\}$.

Our proofs of the recursions (3.20) and (3.22) naturally lead to a recursive way to define our desired bijection Δ from $\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P})=\mathbb{P}\}$ to $\mathcal{N}_k^{(j)}(D)$. That is, suppose we are given a $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n})$ such that $J(\mathbb{P})=\mathbb{P}$. Then consider the sequence of j-creation rook placements $\mathbb{P}=\mathbb{P}_n, \mathbb{P}_{n-1}, \ldots \mathbb{P}_1$ where $\mathbb{P}_i$ is just the restriction of $\mathbb{P}$ to the first i columns for $i=1,\ldots,n$. For example, if we start with the Ferrers board $D = F(0,1,3,3,3)$ and we consider the 3-creation version of the Goldman-Haglund model, then in our model we must take $\mathcal{B}=(0,3,6,9,11)$, $\mathcal{A}=(0,2,2,2,2)$, and $sgn(i) = \overline{sgn}(i)=+1$ for every j. Then such a sequence $\mathbb{P}=\mathbb{P}_5,\mathbb{P}_4,\mathbb{P}_3,\mathbb{P}_2,\mathbb{P}_1$ is pictured in Figure 26.

Now, the image $\Delta(\mathbb{P})$ will be obtained by constructing an element of $\mathcal{N}_k^{(j)}(D)$ by using the analogous steps to build up $\Delta(\mathbb{P})_1,\ldots,\Delta(\mathbb{P})_n=\Delta(\mathbb{P})$. That is, at step 1, if $\mathbb{P}_1$ has no rook in the first column, then $\Delta(\mathbb{P})_1$ has no rook in the first column. Otherwise, the length of the first column in both the restriction of D to the first column and $\mathcal{B}^{\mathcal{A}}$ is the same, namely d_1, so that if the rook in $\mathbb{P}_1$ is in the r-th row, reading from bottom to top, then we place the rook in $\Delta(\mathbb{P})_1$ in the r-th row, reading from bottom to top. In general, having constructed $\Delta(\mathbb{P})_1,\ldots,\Delta(\mathbb{P})_i$, then if $\mathbb{P}_{i+1}$ has no rook in the column $i+1$, we define $\Delta(\mathbb{P})_{i+1}$ so that it has no rook in column $i+1$ and the restriction of $\Delta(\mathbb{P})_{i+1}$ to the first i columns is just $\Delta(\mathbb{P}_i)$. Otherwise, if the rook in the $(i+1)$-st

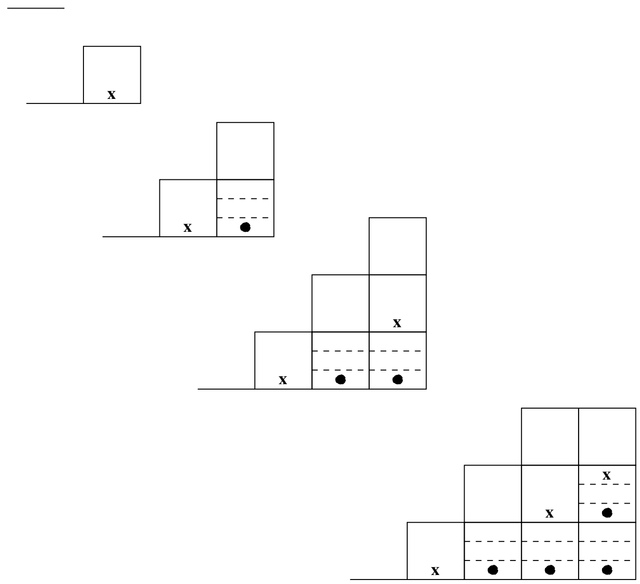

Figure 27: The recursive construction of $\Delta(\mathbb{P}) \in \mathcal{N}_3^{(j)}(D)$.

column of $\mathbb{P}_{i+1}$ is in the r-th available cell, reading from bottom to top, then we set $\Delta(\mathbb{P})_{i+1}$ to be the rook placement which results by extending $\Delta(\mathbb{P})_i$ by adding a rook in the column $i+1$ which is in the r-th available cell, reading from bottom to top. that is not in either the augmented part of the n-th column or the mirror image of its uncanceled cells. An example of the sequence $\Delta(\mathbb{P})_1, \ldots, \Delta(\mathbb{P})_5$ for the $\mathbb{P}$ pictured in Figure 26 is given in Figure 27.

Thus, in both the j-cancellation model of Remmel and Wachs and j-creation model of Goldman and Haglund, one can give direct combinatorial proofs of the fact that the rooks numbers that appear in the product formulas for those models are the same as the rook numbers that appear in the corresponding product formulas in our model.

Here we should point out that our model gives rise to a much wider class of product formulas than can be derived in either of those two models. That is, in the j-cancellation model of Remmel-Wachs, the product formulas holds only for j-attacking Ferrers boards. Similarly, in the j-creation model of Goldman and Haglund, the product formula holds only for Ferrers boards. However, the boards in our model that give rise to our product formulas can be arbitrary skyline boards.

4 Q-Analogues of the General Product Formula

In this section, we shall describe how we can derive several q-analogues of Theorem 3.1 by q-counting rook placements in our model. For any $n \in \mathbb{N}$, define $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and $[-n]_q := -[n]_q$.

Now fix two sequences $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$ and two sign functions $sgn, \overline{sgn} : \{1, \ldots, n\} \to \{1, -1\}$. Let $\overline{A}_i := \sum_{s=1}^{i} \overline{sgn}(s) a_s$. First we shall define the

q-weight, $\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(c)$, of each cell c in $\mathcal{B}_x^{\mathcal{A}}$ as follows.

1. For each i, the q-weights, $\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(c)$, of the cells c in the i-th column of the x-part of $\mathcal{B}_x^{\mathcal{A}}$ are $1, q, q^2, \ldots, q^{x-1}$, reading from bottom to top.

2. For each i, the q-weights, $\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(c)$, of the cells c in the i-th column of the base part of $\mathcal{B}_x^{\mathcal{A}}$ are $sgn(i)$, $sgn(i)q$, $sgn(i)q^2$, $\ldots$, $sgn(i)q^{b_i-1}$, reading from bottom to top.

3. For each i, we assign the q-weights, $\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(c)$, to the cells c in the i-th column of the lower augmented part as follows. First, we assign the q-weights $\overline{sgn}(i)1$, $\overline{sgn}(i)q$, $\overline{sgn}(i)q^2$, ..., $\overline{sgn}(i)q^{a_1-1}$ to the cells in a_1-st part of column i in the lower augmented board reading from top to bottom. Thus the sum of the q-weights of the cells in a_1-st part of column i in the lower augmented board is $[\overline{sgn}(i)a_1]_q$. Next suppose that we have assigned the q-weights to the cells in a_j-th part of column i in the lower augmented part for $j = 1, \ldots, s$ so that the sum of the q-weights of the cells that lie in a_j-th part of column i in the lower augmented board for $j \leq s$ is $[\overline{A}_s]_q$. Then we assign the q-weights to the cells in a_{s+1}-st part of column i in the lower augmented part according to the following cases.

   Case 1: $0 \leq \overline{A}_s \leq \overline{A}_{s+1}$**.** In this case, we assign the q-weights of the cells in the a_{s+1}-st part to be $q^{\overline{A}_s}, q^{\overline{A}_s+1}, \ldots, q^{\overline{A}_{s+1}-1}$, reading from top to bottom.

   Case 2: $0 \leq \overline{A}_{s+1} < \overline{A}_s$**.** In this case, we assign the q-weights of the cells in the a_{s+1}-st part to be $-q^{\overline{A}_s-1}, -q^{\overline{A}_s-2}, \ldots, -q^{\overline{A}_{s+1}}$, reading from top to bottom.

   Case 3: $\overline{A}_{s+1} < 0 \leq \overline{A}_s$**.** In this case, we assign the q-weights of the cells in the a_{s+1}-st part to be $-q^{\overline{A}_s-1}, -q^{\overline{A}_s-2}, \ldots, -q, -1, -1, -q, \ldots - q^{|\overline{A}_{s+1}|-1}$, reading from top to bottom.

   Case 4: $0 \geq \overline{A}_s \geq \overline{A}_{s+1}$**.** In this case, we assign the q-weights of cells in the a_{s+1}-st part to be $-q^{|\overline{A}_s|}, -q^{|\overline{A}_s|+1}, \ldots, -q^{|\overline{A}_{s+1}|-1}$, reading from top to bottom.

   Case 5: $0 \geq \overline{A}_{s+1} > \overline{A}_s$**.** In this case, we assign the q-weights of the cells in the a_{s+1}-st part to be $q^{|\overline{A}_s|-1}, q^{|\overline{A}_s|-2}, \ldots, q^{|\overline{A}_{s+1}|}$, reading from top to bottom.

   Case 6: $\overline{A}_{s+1} > 0 \geq \overline{A}_s$**.** In this case, we assign the q-weights of the cells in the a_{s+1}-st part to be $q^{|\overline{A}_s|-1}, q^{|\overline{A}_s|-2}, \ldots, q, 1, 1, q^1, \ldots q^{\overline{A}_{s+1}-1}$, reading from top to bottom.

4. For each i, the cell in the r-th row of the i-th column of the upper augmented part, reading from bottom to top, is equal to -1 times the weight of the cell in the r-th row of i-th column of the lower augmented board, reading from top to bottom. That is, in the upper augmented part of column i, the q-weight of a cell in the i-th column is just the negative of the q-weight of its corresponding cell in the lower augmented part.

The key property of our weighting of the cells in $\mathcal{B}_x^{\mathcal{A}}$, which is easily established by induction, is that the sum of the q-weights of cells that lie in a_j-th part of column i in the lower augmented board for $j \leq s$ is $[\overline{A}_s]_q$ and the sum of the q-weights of cells that lie in a_j-th part of column i in the upper augmented board for $j \leq s$ is $[-\overline{A}_s]_q$.

This given, we define the q-weight of a placement $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ which has rooks in cells $c_1, \ldots, c_k$ by

$$\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = \prod_{i=1}^{k} \mu_{q,\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{4.1}$$

Similarly, we define the weight of placement $Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ which has rooks in cells $c_1, \ldots, c_n$ by

$$\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(Q) = \prod_{i=1}^{n} \mu_{q,\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{4.2}$$

Finally we define

$$r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) = \sum_{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} \mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \tag{4.3}$$

An example of the q-weights of the cells in $\mathcal{B}_x^{\mathcal{A}}$ is given on the left-hand side of Figure 28 in the case where $x = 4$, $\mathcal{B} = (1, 2, 2, 4)$, $\mathcal{A} = (2, 1, 2, 1)$,

$$sgn(i) = \begin{cases} +1 & \text{if } i = 1, 2, 4, \\ -1 & \text{if } i = 3 \end{cases}$$

and

$$\overline{sgn}(i) = \begin{cases} +1 & \text{if } i = 2, \\ -1 & \text{if } i = 1, 3, 4. \end{cases}$$

If we consider the placement, $\mathbb{P}$, pictured on the right-hand side of Figure 28 , then the rooks are placed in cells with q-weights $1, q^2, q, -1$, reading from left to right. Thus

$$\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(\mathbb{P}) = (1)(q^2)(q)(-1) = -q^3.$$

Our first q-analogue of Theorem 3.1 is the following.

Theorem 4.1. *Suppose $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$ are two sequences of nonnegative integers and $sgn : \{1, \ldots, n\} \to \{1, -1\}$ and $\overline{sgn} : \{1, \ldots, n\} \to \{1, -1\}$ are two sign functions. Then,*

$$\prod_{i=1}^{n} ([x]_q + sgn(i)[b_i]_q) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) \prod_{s=1}^{k} ([x]_q + [\overline{A}_s]_q). \tag{4.4}$$

Proof. We would like to compute the polynomial

$$G(q) := \sum_{\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})} \mu_{q,\mathcal{B}_x^{\mathcal{A}}}(\mathbb{P}) \tag{4.5}$$

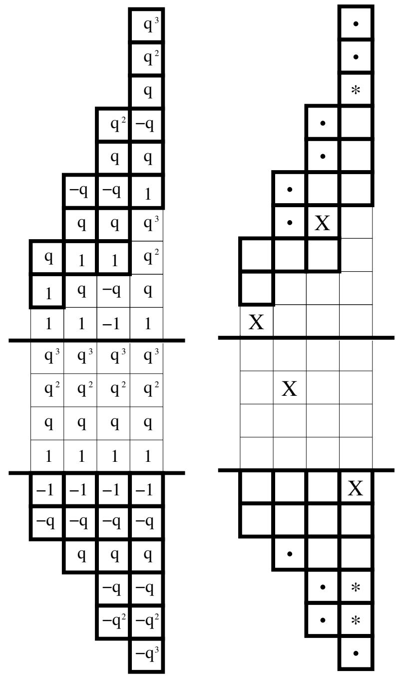

Figure 28: The weighting of cells in placements in $\mathcal{B}_x^{\mathcal{A}}$

in two different ways.

First, we consider placing the rooks starting with the leftmost column and working to the right. Then the sum of the q-weights of the cells in the first column is $[x]_q + [sgn(1)b_1]_q + [\overline{sgn}(1)a_1]_q + [-\overline{sgn}(1)a_1]_q = [x]_q + [sgn(1)b_1]_q$. When we go to place a rook in the second column, we have two cases.

Case I: Suppose that the rook that was placed in the first column was placed below the high bar. Then nothing was canceled in the second column so that we can place a rook in any cell of the second column. It follows that our weighting ensures that the sum of the q-weights of the cells in the second column is $[x]_q + [sgn(2)b_2]_q + [\overline{A}_2]_q + [-\overline{A}_2]_q = [x]_q + [sgn(2)b_2]_q$.

Case II: If the rook placed in the first column was placed above the high bar, then the cells corresponding to a_2-nd part of the second column in both the upper and lower augmented parts are canceled. It follows that our weighting ensures that the sum of the q-weights of the cells in the second column is $[x]_q + [sgn(2)b_2]_q + [\overline{A}_1]_q + [-\overline{A}_1]_q = [x]_q + [sgn(2)b_2]_q$.

In general, suppose we are placing a rook in the j-th column where we have placed s rooks above the high bar and t rooks below the high bar in the first $j-1$ columns. Then in the j-th column we have, by Lemma 3.2, $x + b_j + 2A_{t+1}$ choices as to where to place the rook in that column. Again, it follows that our weighting ensures that the sum of the q-weights of the cells in the j-th column is $[x]_q + [sgn(j)b_j]_q + [\overline{A}_{t+1}]_q + [-\overline{A}_{t+1}]_q = [x]_q + [sgn(j)b_j]_q$.

Thus, it follows that

$$G(q)=\prod_{i=1}^{n}([x]_q+[sgn(i)b_i]_q)$$

which gives the left-hand side of (4.4).

The second way of computing $G(q)$ is to organize the placements by how many rooks lie above the high bar. Suppose that we fix a placement $\mathbb{P}$ of $n-k$ nonattacking rooks in $\mathcal{B}^{\mathcal{A}}$. Then we wish to compute

$$\sum_{\substack{Q\in\mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})\\ Q\cap\mathcal{B}^{\mathcal{A}}=\mathbb{P}}}\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(Q). \tag{4.6}$$

Clearly each such Q in the sum arises from $\mathbb{P}$ by placing k rooks below the high bar in the remaining columns. We will place the remaining rooks in these available columns starting with the leftmost one and working right. By Lemma 3.3, the number of ways we can do this will be $(x+A_1)(x+A_2)\cdots(x+A_k)$. However, it follows by the properties of our assignment of weights to the cells that

$$\sum_{\substack{Q\in\mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})\\ Q\cap\mathcal{B}^{\mathcal{A}}=\mathbb{P}}}\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(Q)=\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})([x]_q+[\overline{A}_1]_q)([x]_q+[\overline{A}_2]_q)\cdots([x]_q+[\overline{A}_k]_q). \tag{4.7}$$

Thus

$$\begin{aligned} G(q) &= \sum_{k=0}^{n}\sum_{\mathbb{P}\in\mathcal{N}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})([x]_q+[\overline{A}_1]_q)([x]_q+[\overline{A}_2]_q)\cdots([x]_q+[\overline{A}_k]_q)\\ &= \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},sgn,\overline{sgn},q)([x]_q+[\overline{A}_1]_q)([x]_q+[\overline{A}_2]_q)\cdots([x]_q+[\overline{A}_k]_q)\end{aligned}$$

which gives the right-hand side of (4.4). ☐

4.1 Modified q-analogues of the general product formula

Note that in (4.4), we have taken the q-analogue of $(x+a)$ to be $[x]_q+[a]_q$ and the q-analogue of $(x-a)$ to be $[x]_q-[a]_q$ if x and a are nonnegative integers. This does not agree with the q-analogues of the models discussed in section 2. For example, in the Garsia and Remmel model, the q-analogue of $x-a$ is $[x-a]_q$. Similarly, in Goldman and Haglund's j-creation model, the q-analogue of $(x+a)$ is $[x+a]_q$. It turns out that we can easily modify (4.4) to produce q-analogues of our general product formula formulas where we take the q-analogue of $(x+a)$ to be $[x+a]_q$ and the q-analogue of $(x-a)$ to be $[x-a]_q$ by using some simple transformations of our formulas and q-rook numbers. That is, consider the following simple identities which hold when x and a are nonnegative integers with $x\geq a$:

$$[x]_q-[a]_q=q^a[x-a]_q \tag{4.8}$$

and

$$[x]_q + q^x[a]_q = [x+a]_q. \tag{4.9}$$

We shall first explore how these transformations force us to modify our q-rook numbers to prove q-analogues of our general product formula in four special cases. Throughout this section, we shall fix two sequences of nonnegative integers
$\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$.

Case I: $sgn(i) = \overline{sgn}(i) = -1$ for all i.

In this case, it is easy to see that $\overline{A}_i = \sum_{j=1}^{i} \overline{sgn}(i)a_i = -A_i$ where $A_i = a_1 + \cdots a_i$. Thus (4.4) becomes

$$\prod_{i=1}^{n}([x]_q - [b_i]_q) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) \prod_{s=1}^{k}([x]_q - [A_s]_q). \tag{4.10}$$

Now if $x \geq max(\{b_i, A_i : i = 1, \ldots, n\})$, then (4.10) becomes

$$\prod_{i=1}^{n} q^{b_i}[x-b_i]_q = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) \prod_{s=1}^{k}(q^{A_s}[x-A_s]_q). \tag{4.11}$$

If we now replace $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$ with $\hat{r}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q)$ where

$$\hat{r}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) := q^{(A_1+A_2+\cdots+A_{n-k})-(b_1+\cdots+b_n)} r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q),$$

we obtain the following q-analogue of Equation (3.3):

$$\prod_{i=1}^{n}[x-b_i]_q = \sum_{k=0}^{n} \hat{r}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q)[x-A_1]_q[x-A_2]_q \cdots [x-A_k]_q. \tag{4.12}$$

We note that in this case, our weighting of cells in $\mathcal{B}_x^{\mathcal{A}}$ is very simple.

1. For each i, the q-weights of the cells in the i-th column of the x-part of $\mathcal{B}_x^{\mathcal{A}}$ are $1, q, \ldots, q^{x-1}$, reading from bottom to top.
2. For each i, the q-weights of the cells in the i-th column of the base part of $\mathcal{B}_x^{\mathcal{A}}$ are $-1, -q, \ldots, -q^{b_i-1}$, reading from bottom to top.
3. For each i, the q-weights of the cells in the i-th column of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ are $-1, -q, \ldots, -q^{A_i-1}$, reading from top to bottom.
4. For each i, the q-weights of the cells in the i-th column of the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$ are $1, q, \ldots, q^{A_i-1}$, reading from bottom to top.

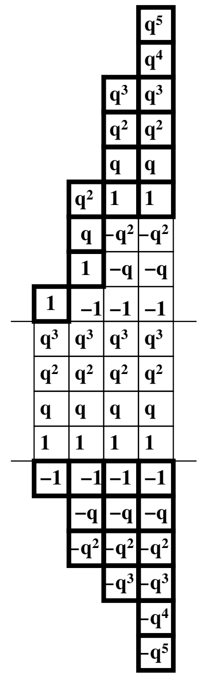

Figure 29: The weighting of cells in placements in $\mathcal{B}_x^{\mathcal{A}}$ when $sgn(i) = \overline{sgn}(i) = -1$.

An example of such a weighting of the cells of $\mathcal{B}_x^{\mathcal{A}}$ is pictured in Figure 28 in the case where $\mathcal{B} = (0, 1, 3, 3)$ and $\mathcal{A} = (1, 2, 1, 2)$.

Case II: $sgn(i) = +1, \overline{sgn}(i) = -1$ for all i

If $sgn(i) = 1$, then the left-hand side of (4.4) is $\prod_{i=1}^k([x]_q + [b_i]_q)$. Since $[x + b_i]_q = [x]_q + q^x[b_i]_q$, we can can replace by $\prod_{i=1}^k([x]_q + [b_i]_q)$ by $\prod_{i=1}^k[x + b_i]_q$ on the left-hand side of (4.4) by simply weighting each rook that appears in the base part of the board $\mathcal{B}^{\mathcal{A}}$ with an extra factor of q^x. In this case, for any $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$, we set

$$\overline{\mu}_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = q^{base(\mathbb{P})x}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \tag{4.13}$$

where $base(\mathbb{P})$ is the number of rooks of $\mathbb{P}$ which lie in the base part of the board $\mathcal{B}^{\mathcal{A}}$. Then if we define

$$\overline{r}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = \sum_{\mathbb{P}\in\mathcal{N}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} q^{A_1+\cdots+A_k}\overline{\mu}_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}), \tag{4.14}$$

one can show that

$$\prod_{i=1}^{n}[x + b_i]_q = \sum_{k=0}^{n} \overline{r}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q)[x - A_1]_q[x - A_2]_q \cdots [x - A_k]_q \tag{4.15}$$

Case III: $sgn(i) = -1, \overline{sgn}(i) = +1$ for all i.

If we want to replace $[x]_q + [A_i]_q$ by $[x + A_i]_q = [x]_q + q^x[A_i]$, then we should weight each rook that lies in the lower augmented part of $\mathcal{B}^{\mathcal{A}}$ by an extra factor of q^x. This means that when we consider placements in $\mathcal{B}_x^{\mathcal{A}}$, then we must also weight each rook that lies in the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$ with an extra factor of q^x so that for any given column the weights of possible placements in the lower and upper augmented parts cancel each other as in the proof of (4.4). In this case, for any $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$, we set

$$\overline{\mu}_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = q^{aug(\mathbb{P})x}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \tag{4.16}$$

where $aug(\mathbb{P})$ is the number of rooks of $\mathbb{P}$ which lie in the augmented part of the board $\mathcal{B}^{\mathcal{A}}$. Then if we define

$$\tilde{\tilde{r}}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = q^{-(b_1+b_2+\cdots+b_n)} \sum_{\mathbb{P}\in\mathcal{N}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} \overline{\mu}_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}), \tag{4.17}$$

one can show that

$$\prod_{i=1}^{n}[x-b_i]_q = \sum_{k=0}^{n} \tilde{\tilde{r}}_{n-k}^{\mathcal{A}}(\mathcal{B}, q)[x+A_1]_q[x+A_2]_q\cdots[x+A_k]_q. \tag{4.18}$$

Case IV: $sgn(i) = \overline{sgn}(i) = +1$ for all i.

In this case, we need to weight each rook in base part of the board by an extra factor of q^x and weight each rook in both the lower and the upper augmented part of the board $\mathcal{B}_x^{\mathcal{A}}$ by an extra factor of q^x to obtain new q-rook numbers $\overline{\overline{r}}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q)$. Then it will be the case that

$$\prod_{i=1}^{n}([x+b_i]_q) = \sum_{k=0}^{n} \overline{\overline{r}}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q)([x+A_1]_q)([x+A_2]_q)\cdots([x+A_k]_q) \tag{4.19}$$

We end this subsection by making a few remarks about how one can modify our q-analogue of the general product formulas to obtain the following q-analogue of our general product formula

$$\prod_{i=1}^{n}[x+sgn(i)b_i]_q = \sum_{k=0} R_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) \prod_{j=1}^{k}([x+\sum_{s\leq j}\overline{sgn}(s)a_s]_q). \tag{4.20}$$

The basic idea to obtain the left-hand side is simple. That is, if $sgn(i) = 1$, then we can obtain a factor of $[x]_q + q^x[b_i]_q = [x+b_i]_q$ weighting each cell in the base part of column i with an extra factor of q^x. If $sgn(i) = -1$, then we weight each rook in the base part of $\mathcal{B}_x^{\mathcal{A}}$ as we did in the proof of (4.4) so that the cells in the base part and the x-part of $\mathcal{B}_x^{\mathcal{A}}$ in the i-th column will contribute a factor of $[x]_q - [b_i]_q = q^{b_i}[x-b_i]_q$.

The basic idea to obtain the right-hand side is to ensure that we weight the cells in the lower augmented part of the board so that the sum of the weights of the cells that lie in the a_s-th part of the lower augmented board for $s \leq i$ is $q^x[\sum_{s=0}^{i}\overline{sgn}(s)a_s]_q$ if $\sum_{s=0}^{i}\overline{sgn}(s)a_s \geq 0$ and is $[\sum_{s=0}^{i}\overline{sgn}(s)a_s]_q$ if $\sum_{s=0}^{i}\overline{sgn}(s)a_s < 0$. We can accomplish this by defining a new weight, $M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c)$, for the cells $c \in \mathcal{B}_x^{\mathcal{A}}$ according the following scheme.

1. For each i, the weights, $M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c)$, of the cells c in the i-th column of the x-part of $\mathcal{B}_x^{\mathcal{A}}$ are $1, q, q^2, \ldots, q^{x-1}$, reading from bottom to top.

2. For each i, the weights, $M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c)$, of the cells c in the i-th column of the base part of $\mathcal{B}_x^{\mathcal{A}}$ are -1,$-q$, $-q^2, \ldots, -q^{b_i-1}$, reading from bottom to top, if $sgn(i) = -1$.

3. For each i, the weights, $M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c)$, of the cells c in the i-th column of the base part of $\mathcal{B}_x^{\mathcal{A}}$ are q^x,q^{x+1}, $q^{x+2}, \ldots, q^{x+b_i-1}$, reading from bottom to top, if $sgn(i) = 1$.

4. For each i, we assign the weights, $M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c)$, to the cells c in the i-th column of the lower augmented part as follows. First, if $\overline{sgn}(1) = -1$, we assign the weights $-1, -q, -q^2, \ldots, -q^{a_1-1}$ to cells in a_1-st part of column i in the lower augmented board, reading from top to bottom. If $\overline{sgn}(1) = 1$, then we assign the weights $q^x, q^{x+1}, q^{x+2}, \ldots, q^{x+a_1-1}$ to cells in a_1-st part of column i in the lower augmented board, reading from top to bottom. Thus, the sum of the weights of cells in a_1-st part of column i in the lower augmented board is $-[a_1]_q$ if $sgn(1) = -1$ and $q^x[a_1]_q$ if $sgn(1) = 1$.

   Next suppose that we have assigned the weights to cells in the a_j-th part of column i in the lower augmented part for $j = 1, \ldots, s$ so that the sum of the weights of cells that lie in the a_j-th part of column i in the lower augmented board for $j \leq s$ is $-[\sum_{r=0}^{s} \overline{sgn}(r)a_r]_q$ if $\sum_{r=0}^{s} \overline{sgn}(r)a_r < 0$ and is $q^x[\sum_{r=0}^{s} \overline{sgn}(r)a_r]_q$ if $\sum_{r=0}^{s} \overline{sgn}(r)a_r \geq 0$. Then we define the weights to the cells in the a_{s+1}-st part of column i in the lower augmented part according to the following cases.

   Case 1: $0 \leq \sum_{r=0}^{s} \overline{sgn}(r)a_r \leq \sum_{r=0}^{s+1} \overline{sgn}(r)a_r$.
   In this case, the weights of the cells in the a_{s+1}-st part are
   $q^x q^{\sum_{r=0}^{s} \overline{sgn}(r)a_r}, q^x q^{1+\sum_{r=0}^{s} \overline{sgn}(r)a_r}, \ldots, q^x q^{(\sum_{r=0}^{s+1} \overline{sgn}(r)a_r)-1}$,
   reading from top to bottom.

   Case 2: $0 \leq \sum_{r=0}^{s+1} \overline{sgn}(r)a_r < \sum_{r=0}^{s} \overline{sgn}(r)a_r$.
   In this case, the weights of the cells in the a_{s+1}-st part are
   $-q^x q^{(\sum_{r=0}^{s} \overline{sgn}(r)a_r)-1}, -q^x q^{(\sum_{r=0}^{s} \overline{sgn}(r)a_r)-2}, \ldots, -q^x q^{(\sum_{r=0}^{s+1} \overline{sgn}(r)a_r)}$,
   reading from top to bottom.

   Case 3: $(\sum_{r=0}^{s+1} \overline{sgn}(r)a_r) < 0 \leq (\sum_{r=0}^{s} \overline{sgn}(r)a_r)$.
   In this case, the weights of the cells in the a_{s+1}-st part are
   $-q^x q^{\overline{A}_s-1}, -q^x q^{\overline{A}_s-2}, \ldots, -q^x, -q^0, -q^1, \ldots, -q^{|\sum_{r=0}^{s+1} \overline{sgn}(r)a_r|-1}$,
   reading from top to bottom.

   Case 4: $0 \geq (\sum_{r=0}^{s} \overline{sgn}(r)a_r) \geq (\sum_{r=0}^{s+1} \overline{sgn}(r)a_r)$.
   In this case, the weights of the cells in the a_{s+1}-st part are
   $-q^{|\sum_{r=0}^{s} \overline{sgn}(r)a_r|}, -q^{|\sum_{r=0}^{s} \overline{sgn}(r)a_r|+1}, \ldots, -q^{|\sum_{r=0}^{s+1} \overline{sgn}(r)a_r|-1}$,
   reading from top to bottom.

   Case 5: $0 \geq (\sum_{r=0}^{s+1} \overline{sgn}(r)a_r) > (\sum_{r=0}^{s} \overline{sgn}(r)a_r)$.
   In this case, the weights of the cells in the a_{s+1}-st part are

$q^{|\sum_{r=0}^{s}\overline{sgn}(r)a_r|-1}, q^{|\sum_{r=0}^{s}\overline{sgn}(r)a_r|-2}, \ldots, q^{|\sum_{r=0}^{s}\overline{sgn}(r)a_r|}$,
reading from top to bottom.

Case 6: $(\sum_{r=0}^{s+1}\overline{sgn}(r)a_r) > 0 \geq (\sum_{r=0}^{s}\overline{sgn}(r)a_r)$**.**
In this case, the weights of the cells in the a_{s+1}-st part are
$q^{|\sum_{r=0}^{s}\overline{sgn}(r)a_r|-1}, q^{|\sum_{r=0}^{s}\overline{sgn}(r)a_r|-2}, \ldots, q, 1, q^x, q^{x+1}, \ldots, q^x q^{(\sum_{r=0}^{s+1}\overline{sgn}(r)a_r)-1}$,
reading from top to bottom.

5. For each i, the cell in the r-th row of the i-th column of the upper augmented part, reading from bottom to top, is equal to -1 times the weight of the cell in the r-th row of i-th column of the lower augmented board, reading from top to bottom. That is, in the upper augmented part of column i, the q-weight of a cell in the i-th column is just the negative of the q-weight of its corresponding cell in the lower augmented part.

An example of this kind of q-weighting can be seen in Figure 30 where $x = 4$, $\mathcal{B} = (1,2,3,4)$, $\mathcal{A} = (2,1,2,1)$,

$$sgn(i) = \begin{cases} +1 & \text{if } i = 1,2,4, \\ -1 & \text{if } i = 3 \end{cases}$$

and

$$\overline{sgn}(i) = \begin{cases} +1 & \text{if } i = 2,3,4, \\ -1 & \text{if } i = 1. \end{cases}$$

Now suppose that $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ has rooks in cells $c_1, \ldots, c_k$. Then we set

$$M_{q,sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = \prod_{i=1}^{k} M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{4.21}$$

Similarly, if $Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ has rooks in cells $c_1, \ldots, c_n$, we set

$$M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(Q) = \prod_{i=1}^{n} M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{4.22}$$

Then we define

$$MR_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) = \sum_{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} M_{q,sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \tag{4.23}$$

By computing the sum

$$H(q) = \sum_{Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})} M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(Q) \tag{4.24}$$

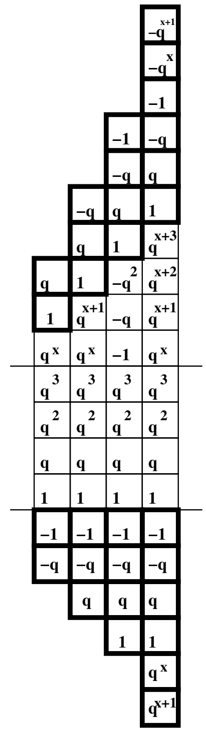

Figure 30: The modified weighting of cells in placements in $\mathcal{B}_x^{\mathcal{A}}$.

in two different ways as we did in the proof of (4.4), we can prove the following:

$$\left(\prod_{i:sgn(i)=1}([x]_q+q^x[b_i]_q)\right)\left(\prod_{i:sgn(i)=-1}([x]_q-[b_i]_q)\right) \tag{4.25}$$
$$=\sum_{k=0}^{n} MR_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},sgn,\overline{sgn},q)\prod_{i=1}^{k}\phi(i)$$

where

$$\phi(s)=\begin{cases}[x]_q+q^x[\sum_{r=0}^{s}\overline{sgn}(r)a_r]_q & \text{if } (\sum_{r=0}^{s}\overline{sgn}(r)a_r)\geq 0\text{, and}\\ [x]_q-[\sum_{r=0}^{s}\overline{sgn}(r)a_r]_q & \text{if } (\sum_{r=0}^{s}\overline{sgn}(r)a_r)<0.\end{cases} \tag{4.26}$$

Thus, if we set

$$R_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},sgn,\overline{sgn},q)= \tag{4.27}$$
$$\left(\prod_{i:sgn(i)=-1}q^{-b_i}\right)\left(\prod_{\substack{r\leq k\\ \sum_{r=0}^{s}\overline{sgn}(r)a_r<0}}q^{|\sum_{r=0}^{s}\overline{sgn}(r)a_r|}\right)MR_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},sgn,\overline{sgn},q),$$

then we will have the following theorem.

Theorem 4.2. *Suppose $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$ are two sequences of nonnegative integers and $sgn : \{1, \ldots, n\} \to \{1, -1\}$ and $\overline{sgn} : \{1, \ldots, n\} \to \{1, -1\}$ are two sign functions. Then,*

$$\prod_{i=1}^{n} [x + sgn(i)b_i]_q = \sum_{k=0}^{n} R_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) \prod_{j=1}^{k} ([x + \sum_{s \le j} \overline{sgn}(s)a_s]_q). \tag{4.28}$$

4.2 Comparisons with q-analogues of other rook models.

It is no longer the case that the q-rook numbers $R_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q)$ that occur in (4.28) are the same as the q-rook numbers that appear in the other rook models described in section 2.

For example, consider the j-attacking Remmel-Wachs model where we have a j-attacking Ferrers board $D = F(d_1, \ldots, d_n)$ such that $d_i \ge j(i-1)$ for all i. In this case, we would be led to the following product formula in the Remmel-Wachs model.

$$\prod_{i=1}^{n} [x + d_i - j(i-1)]_q = \sum_{k=0}^{n} \tilde{r}_{n-k,D}^{(j)}(1, q)[x]_q\downarrow_{k,j}. \tag{4.29}$$

We can obtain the same product formula from (4.15) by setting $\mathcal{A} = (0, j, \ldots, j)$, $\mathcal{B} = (d_1, d_2 - j, \ldots, d_n - j(n-1))$, $sgn(i) = 1$, and $\overline{sgn}(i) = -1$ for all i. In that case, (4.15) becomes

$$\prod_{i=1}^{n} [x + d_i - j(i-1)]_q = \sum_{k=0}^{n} \overline{r}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q)[x]_q\downarrow_{k,j}. \tag{4.30}$$

It is not necessarily the case that $\overline{r}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = \tilde{r}_{n-k,D}^{(j)}(1, q)$ as the following example will show. Let $j = 2$ and $D = F(1, 2)$. Then in our model, we must set $\mathcal{B} = (1, 0)$, $\mathcal{A} = (0, 2)$, and $sgn(i) = 1$ and $\overline{sgn}(i) = -1$ for all i. In Figure 31, we have pictured all the rook placements in $\mathcal{N}_k^{(2)}(D)$ for $k = 1, 2$ and their corresponding weights $\tilde{W}_{1,q,D}^{(2)}(\mathbb{P})$ at the top of the figure. Note that $\mathcal{N}_2^{(2)}(D)$ is empty since the rook in first column 2-attacks both cells in the second column and hence $\tilde{r}_{2,D}^{(2)} = 0$. Similarly, at the bottom of the figure, we have pictured all the weights of $\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ for $k = 1, 2$ and their corresponding weights $q^{A_1 + \cdots + A_k}\overline{\mu}_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})$. Again $\mathcal{N}_2^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ is empty so that $\overline{r}_2^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = 0$. It follows that $\tilde{r}_{0,D}^{(2)} = q^3$, $\tilde{r}_{1,D}^{(2)} = 1 + q + q^2$, and $\tilde{r}_{2,D}^{(2)} = 0$ so that (4.15) becomes

$$[x+1]_q[x]_q = q^3[x]_q[x-2]_q + (1 + q + q^2)[x]_q. \tag{4.31}$$

Similarly $\overline{r}_0^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = q^2$, $\overline{r}_1^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = 1 + q + q^x$, and $\overline{r}_2^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = 0$ so that (4.30) becomes

$$[x+1]_q[x]_q = q^2[x]_q[x-2]_q + (1 + q + q^x)[x]_q. \tag{4.32}$$

Note that in this case, these two identities hold because there are two ways to write $[x+1]_q$ when $x \in \mathbb{N}$, namely,

$$[x+1]_q = q^3[x-2]_q + (1 + q + q^2) = q^2[x-2]_q + (1 + q + q^x). \tag{4.33}$$

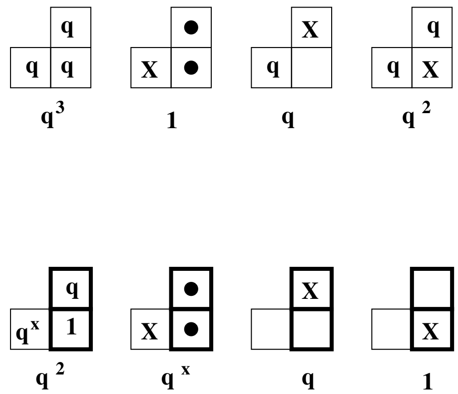

Figure 31: Rook placements in $\mathcal{N}_k^{(2)}(F(1,2))$ and $\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ for $k=1,2$.

However, there are many cases where the q-rook numbers in the j-attacking rook model of Remmel and Wachs do coincide with the q-rooks numbers in our model. For example, consider the case of the Remmel-Wachs model where we start with a j-attacking Ferrers board $D=F(d_1,\ldots,d_n)$ such that $d_i \leq j(i-1)$ for all i. In particular, this forces $d_1=0$. In such a case, the q-analogue of the product formula, which is just the $p=1$ case of the (2.9), is

$$[x]_q[x-(j-d_2)]_q\cdots[x-((n-1)j-d_n)]_q = \sum_{k=0}^{n} \tilde{r}_{k,B}^{j}(1,q)[x]_q\downarrow_{n-k,j} \tag{4.34}$$

We can produce an equivalent formula from (4.11) if we set $\mathcal{A}=(0,j,\ldots,j)$, $\mathcal{B}=(0,j-d_2,\ldots,j(n-1)-d_n)$, and $sgn(i)=\overline{sgn}(i)=-1$ for all i. In that case, (4.11) becomes

$$[x]_q[x-(j-d_2)]_q\cdots[x-((n-1)j-d_n)]_q = \sum_{k=0}^{n} \hat{r}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},q)[x]_q\downarrow_{n-k,j} \tag{4.35}$$

If one thinks of $[x-a]$ as $\frac{q^xq^{-a}-1}{q-1}$, then one can think of (4.11) and (4.35) as formulas involving the variable q^x with coefficients which are in the field $\mathbb{Q}(q)$. That is, we can think of (4.11) and (4.35) as identities in $\mathbb{Q}(q)[q^x]$. Thus, since $\{[x_q]\downarrow_{n,j}\}_{n\geq 0}$ is basis for $\mathbb{Q}(q)[q^x]$, it automatically follows that $\hat{r}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},q)=\tilde{r}_{k,B}^{j}(1,q)$ for all k in this case. As in the $q=1$ case, we can give a completely combinatorial proof of this fact.

Recall the involution I that we used to show that the rook numbers in our model and the j-attacking Remmel-Wachs model were equivalent in the $q=1$ case. In the current case, we are assuming that every non-empty column is negative. Thus our q-weighting of cells in this case ensures that the q-weight of the cells in r-th row of a negative column is $-q^{r-1}$ while in its mirror image in the augmented part, the q-weight of the cell in the r-th row is q^{r-1}, reading from bottom to top. It follows that our sign reversing involution I has the property that $\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})=-\mu_{q,\mathcal{B}^{\mathcal{A}}}(I(\mathbb{P}))$ if $I(\mathbb{P})\neq\mathbb{P}$. Thus

I shows that

$$\hat{r}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) = \sum_{\substack{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}) \\ I(\mathbb{P})=\mathbb{P}}} \mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \tag{4.36}$$

Moreover, if $I(\mathbb{P}) = \mathbb{P}$, then all the rooks $\mathbb{P}$ must lie in the augmented part of $\mathcal{B}^{\mathcal{A}}$. Since the weights of the cells in augmented part of $\mathcal{B}^{\mathcal{A}}$ are just powers of q with no negative signs, it follows that in order to prove that $\hat{r}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = \tilde{r}_{k,B}^{j}(1, q)$, we need only construct a weight preserving bijection from $\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}) : I(\mathbb{P}) = \mathbb{P}\}$ onto $\mathcal{N}^{(j)}(D)$. We can construct such a map by recursion in essentially the same way that we constructed a map to show that rook numbers in our model and the rook numbers in j-creation model are equivalent.

First suppose that $D_n = F(d_1, \ldots, d_n)$ is a j-attacking Ferrers board. Then we have that

$$\begin{aligned} \tilde{r}^{j}_{k,F(d_1,\ldots,d_n)}(1,q) &= q^{d_n-jk}\tilde{r}^{j}_{k,F(d_1,\ldots,d_{n-1})}(1,q) \\ &\quad + [d_n - j(k-1)]_q \tilde{r}^{j}_{k-1,F(d_1,\ldots,d_{n-1})}(1,q). \end{aligned} \tag{4.37}$$

This recursion is proved by partitioning the placements $\mathbb{P} \in \mathcal{N}_k^j(F(d_1, \ldots, d_n))$ based on whether or not $\mathbb{P}$ has a rook in column n. That is, if $\mathbb{P}$ does not have a rook in column n, then $\mathbb{P}$ can be regarded as a placement in $\mathcal{N}_k^j(F(d_1, \ldots, d_{n-1}))$. Then our definition of the weight function $\tilde{W}_{1,q,F(d_1,\ldots,d_n)}$ ensures that

$$\tilde{W}_{1,q,F(d_1,\ldots,d_n)}(\mathbb{P}) = q^{d_n-jk}\tilde{W}_{1,q,F(d_1,\ldots,d_{n-1})}(\mathbb{P}), \tag{4.38}$$

since there are a total of $d_n - jk$ uncanceled cells in the last column and each to these cells contributes a factor of q to $\tilde{W}_{1,q,F(d_1,\ldots,d_n)}(\mathbb{P})$. Thus the sum of $\tilde{W}_{1,q,F(d_1,\ldots,d_n)}(\mathbb{P})$ over all placements $\mathbb{P} \in \mathcal{N}_k^j(F(d_1, \ldots, d_n))$ which do not have a rook in last column is $q^{d_n-jk}\tilde{r}^{j}_{k,F(d_1,\ldots,d_{n-1})}(1, q)$. Now if $\mathbb{P}$ does have a rook in the last column and Q is the restriction of $\mathbb{P}$ to the first $n - 1$ columns of D, then Q can be regarded as a placement in $\mathcal{N}_{k-1}^j(F(d_1, \ldots, d_{n-1}))$. Moreover, if we want to extend Q to a placement $\mathbb{P}^* \in \mathcal{N}_k^j(F(d_1, \ldots, d_n))$, then we can place the rook in any of the $d_n - j(k-1)$ cells in the last row since there are exactly $d_n - j(k-1)$ cells in column n which are not canceled by the rooks in Q. The rook in the last column will contribute a factor of $1, q, q^2, \ldots, q^{d_n-j(k-1)-1}$ to weight $\tilde{W}_{1,q,F(d_1,\ldots,d_n)}(\mathbb{P}^*)$ depending on whether it is placed in highest row with an uncanceled cell, second highest with an uncanceled cell, etc.. It follows that the sum of $\tilde{W}_{1,q,F(d_1,\ldots,d_n)}(\mathbb{P})$ over all placements $\mathbb{P} \in \mathcal{N}_k^j(F(d_1, \ldots, d_n))$ which do have a rook in last column is $[d_n - j(k-1)]_q\tilde{r}^{j}_{k-1,F(d_1,\ldots,d_{n-1})}(1, q)$.

Now assume that $d_i \leq j(i-1)$ for all $i \leq n$, $\mathcal{B} = (d_1, j - d_2, \ldots, j(n-1) - d_n)$, $\mathcal{A} = (0, j, \ldots, j)$, and $sgn(i) = \overline{sgn}(i) = -1$ for all i. The sequence of partial sums of the a_i's is $A_1 = 0$ and $A_i = j(i-1)$ for $i \geq 0$. Now consider the sequences of length $n - 1$,

$\mathcal{B}_{n-1}$ and $\mathcal{A}_{n-1}$, which result by removing the last elements of $\mathcal{B}$ and $\mathcal{A}$ respectively. Let

$$\begin{aligned}\hat{\hat{r}}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},q) &= \sum_{\substack{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})\\ I(\mathbb{P})=\mathbb{P}}} \mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \text{ and}\\ \hat{\hat{r}}_k^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}},q) &= \sum_{\substack{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}})\\ I(\mathbb{P})=\mathbb{P}}} \mu_{q,\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}}(\mathbb{P}).\end{aligned}$$

Then we claim that

$$\hat{\hat{r}}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},q) = q^{d_n-jk}\hat{\hat{r}}_k^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}},q) + [d_n-j(k-1)]_q\hat{\hat{r}}_{k-1}^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}},q). \tag{4.39}$$

That is, we can partition the placements $\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ into two sets depending on whether there is a rook in the last column or not. If $\mathbb{P}$ does not have a rook in the last column, then $\mathbb{P}$ can be regarded as a placement in $\mathcal{N}_k^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}})$ such that $I(\mathbb{P})=\mathbb{P}$. Then our definitions ensure that

$$\begin{aligned}& q^{A_1+\cdots+A_{n-k}-\sum_{i=1}^{n}(j(i-1)-d_i)}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})\\ =\ & q^{A_{n-k}-(j(n-1)-d_n)}\left(q^{A_1+\cdots+A_{n-1-k}-\sum_{i=1}^{n-1}(j(i-1)-d_i)}\mu_{q,\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}}(\mathbb{P})\right)\\ =\ & q^{j(n-k-1)-(j(n-1)-d_n)}\left(q^{A_1+\cdots+A_{n-1-k}-\sum_{i=1}^{n-1}(j(i-1)-d_i)}\mu_{q,\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}}(\mathbb{P})\right)\\ =\ & q^{d_n-kj}\left(q^{A_1+\cdots+A_{n-1-k}-\sum_{i=1}^{n-1}(j(i-1)-d_i)}\mu_{q,\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}}(\mathbb{P})\right).\end{aligned}$$

Hence it follows that the sum of the $q^{A_1+\cdots+A_{n-k}-\sum_{i=1}^{n}(j(i-1)-d_i)}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})$ over all placement $\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ which have no rook in the last column and $I(\mathbb{P})=\mathbb{P}$ is equal to $q^{d_n-kj}\hat{\hat{r}}_k^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}},q)$. Now if $\mathbb{P}$ does have a rook in its last column, then let Q be the restriction of $\mathbb{P}$ to first $n-1$ columns. Then Q can be regarded as a placement in $\mathcal{N}_{k-1}^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}})$ for which $I(Q)=Q$. We can extend Q to placement $\mathbb{P}^*\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ such that $I(\mathbb{P}^*)=\mathbb{P}^*$ in $d_n-j(k-1)$ ways. That is, there are a total of $j(n-1)$ cells in the augmented part of the last column. Of those cells in the augmented part, the first $j(n-1)-d_n$ are in the mirror image of the base part of the n-th column so that we have a total of $j(n-1)-(j(n-1)-d_n)=d_n$ cells at the top of the column where we can place a rook. Then each rook in Q will cause the top $j(k-1)$ cells in the augmented part of column n to be canceled. Thus we can only have $d_n-j(k-1)$ cells the augmented part of column n in which to place a rook to obtain such a $\mathbb{P}^*$. Those cells are in rows $(j(n-1)-d_n)+1,\ldots,j(n-1)-d_n+(d_n-j(k-1))$ which have weights $q^{(j(n-1)-d_n)},\ldots,q^{j(n-1)-j(k-1)-1}$ respectively. Now whenever we place rook in the cell with weight $q^{(j(n-1)-d_n)+i}$ for $i=0,\ldots,d_n-j(k-1)-1$ to obtain a rook placement $\mathbb{P}^*$, then we have that

$$\begin{aligned}
& q^{A_1+\cdots+A_{n-k}-\sum_{i=1}^{n}(j(i-1)-d_i)}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}^*) \\
= \; & q^{(j(n-1)-d_n)+i}q^{-(j(n-1)-d_n)} \\
& \times q^{A_1+\cdots+A_{n-1-(k-1)}-\sum_{i=1}^{n-1}(j(i-1)-d_i)}\mu_{q,\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}}(Q) \\
= \; & q^i\left(q^{A_1+\cdots+A_{n-1-(k-1)}-\sum_{i=1}^{n-1}(j(i-1)-d_i)}\mu_{q,\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}}(Q)\right).
\end{aligned}$$

It then follows that the sum of the $q^{A_1+\cdots+A_{n-k}-\sum_{i=1}^{n}(j(i-1)-d_i)}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})$ over all placements $\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ which have a rook in the last column and $I(\mathbb{P})=\mathbb{P}$ is

$$\begin{aligned}
& (1+q+\cdots q^{d_n-j(k-1)-1})\hat{\hat{r}}_{k-1}^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}},q) \\
= \; & [d_n-j(k-1)]_q\hat{\hat{r}}_{k-1}^{\mathcal{A}_{n-1}}(,q).
\end{aligned}$$

Our proofs of the recursions (4.37) and (4.39) show that we can recursively construct a weight preserving bijection Θ from $\mathcal{N}_k^j(D)$ onto $\{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}):I(\mathbb{P})=\mathbb{P}\}$. That is, given a placement $Q\in\mathcal{N}_k^j(D)$, let $Q=Q_n,\ldots,Q_1$ be the sequence of placements that results by letting Q_i be the restriction of Q to the first i columns. Then we can construct a sequence $\mathbb{P}_1,\ldots,\mathbb{P}_n$ such that $\mathbb{P}_i\in\mathcal{N}_{k_i}^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}})$ where k_i is the number of rooks in Q_i and $I(\mathbb{P}_i)=\mathbb{P}_i$ as follows.

Case 1: If Q_1 is the empty placement, then $\mathbb{P}_1$ is the empty placement.

Case 2: If Q_1 has a rook in row i so that its weight is q^{d_1-1-i}, then $\mathbb{P}_1$ has rook in row d_1-i so that its weight is q^{d_1-1-i}.

Assuming that we have constructed $\mathbb{P}_i$ so that

$$\tilde{W}_{1,q,F(d_1,\ldots,d_i)}^{j}(Q_i)=q^{A_1+\cdots+A_{i-k_i}-\sum_{s=0}^{i}j(s-1)-d_s}\mu_{q,(d_1,j-d_2,\ldots,,j(i-1)-d_i)^{(0,j,\ldots,j)}}(\mathbb{P}_i),$$

we define $\mathbb{P}_{i+1}$ as follows.

Case 3: If Q_{i+1} has no rook in the last column, then $\mathbb{P}_{i+1}$ is the placement that results by using $\mathbb{P}_i$ in the first i-columns and leaving column $i+1$ empty.

Case 4: If Q_{i+1} has a rook in the s-th row from the top which contains a cell which is not canceled so that $\tilde{W}_{1,q,F(d_1,\ldots,d_i,d_{i+1})}^{j}(Q_{i+1})=q^{s-1}$, then $\mathbb{P}_{i+1}$ is the result of starting with $\mathbb{P}_i$ and placing rook in row $(j(i-1)-d_i)+s$ of the augmented part of column i so that weight of $\mathbb{P}_{i+1}$ is just q^{s-1} times the corresponding weight of $\mathbb{P}_i$.

Then we let $\Theta(Q)=\mathbb{P}_n$.

For example, in Figure 32, we have pictured $Q=Q_5,\ldots,Q_1$ for a $Q\in\mathcal{N}_3^{(2)}(D)$ where D is the 2-attacking board $F(0,1,2,3,6)$ at the top of the figure. In this case, $\mathcal{B}=$

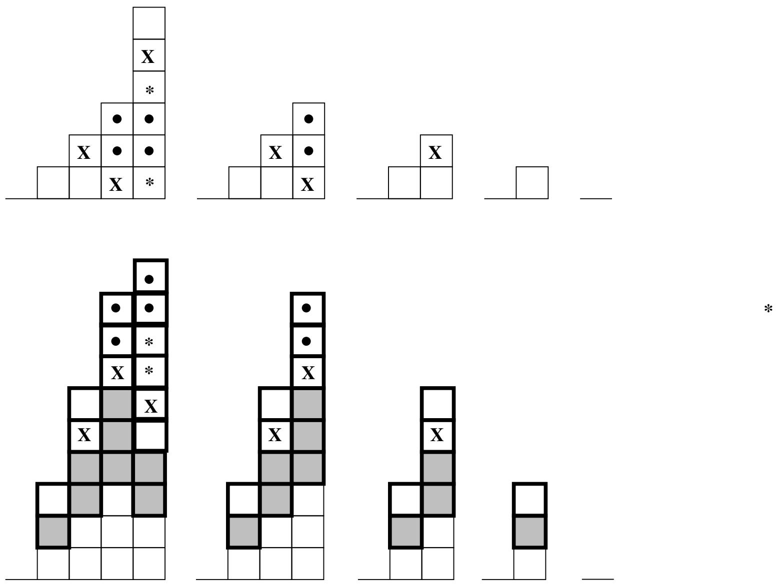

Figure 32: An example of the bijection Θ between a placement of 3 rooks in a 2-attacking Ferrers board $B = F(0,1,2,3,6)$ and a placement in the corresponding board $\mathcal{B}^{\mathcal{A}}$.

$(0,1,2,3,2)$ and $\mathcal{A} = (0,2,2,2,2)$. Since we are assuming that $sgn(i) = \overline{sgn}(i) = -1$ for all i, all columns are negative so that we have shaded the cells in the upper augmented board of $\mathcal{B}^{\mathcal{A}}$ which are in the mirror image of its column. Then we have pictured the corresponding sequence of rook placements $P_1, \ldots, P_5$ at the bottom. In each case, we have used a $\bullet$ to indicate the cells canceled by the rook in column 3 and an $*$ to indicate the cells canceled by the rook in column 4.

5 A (P,Q)-Analogue of the General Product Formula

In this section, we will define an appropriate (p,q)-analogue $r^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, p, q)$ of the rook numbers $r^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$ so that we can prove the following theorem.

Theorem 5.1. *Suppose $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$ are two sequences of nonnegative integers and $sgn : \{1, \ldots, n\} \to \{1, -1\}$ and $\overline{sgn} : \{1, \ldots, n\} \to \{1, -1\}$ are two sign functions. Then,*

$$\prod_{i=1}^{n}([x]_{p,q} + sgn(i)[b_i]_{p,q}) = \sum_{k=0}^{n} r^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, p, q) \prod_{s=1}^{k}([x]_{p,q} + [\overline{A}_s]_{p,q}). \tag{5.1}$$

If n is positive integer, then $[n]_{p,q} = p^{n-1} + qp^{n-2} + \cdots pq^{n-2} + q^{n-1}$ and $[-n]_{p,q} = -[n]_{p,q}$. We will refer to equation (4.4) as the (p,q)-*general product formula*.

The proof of this theorem is essentially the same as the proof of Theorem 4.1 with the exception that we have to use a different weighting function on the cells. However, this case is a bit harder because we cannot just have the (p,q)-weight of the cell be of the form $\pm p^a q^b$ in the lower augmented part of the board. That is, the key property of our weighting of cells in the proof of Theorem (4.1) is that the sum of the weights of the cells that lie in the a_j-th part of column i in the lower augmented part of the board for $j \leq s$ was $[\overline{A}_s]_q$ and the sum of weights of the cells that lie in the a_j-th part of column i in the upper augmented part of the board for $j \leq s$ was $-[\overline{A}_s]_q$. We would like to define the (p,q)-weights of the cells so that the sum of the (p,q)-weights of the cells that lie in the a_j-th part of column i in the lower augmented part for $j \leq s$ is $[\overline{A}_s]_{p,q}$ and the sum of the (p,q)-weights of the cells that lie in the a_j-th part of column i in the upper augmented part for $j \leq s$ is $-[\overline{A}_s]_{p,q}$. Now suppose that $sgn(1) = sgn(2) = 1$ and $a_1 = a_2 = 3$. Then the most natural thing to do would be to assign the (p,q)-weights to the cells in the a_1-st part of the lower augmented board to be p^2, pq, q^2 reading from top to bottom. However, at that point, we want the sum of the (p,q)-weights of the cells that lie in the a_1-st part plus a_2-nd part of column i to be $[6]_{p,q} = p^5 + p^4 q + p^3 q^2 + p^2 q^3 + pq^4 + p^5$. But there is no way to weight the cells of the a_2-nd part with weights of the form $p^a q^b$ to transform $[3]_{p,q}$ to $[6]_{p,q}$. Thus we have to allow the (p,q)-weights of cells to be polynomials in p and q if we are going to be able to make such a transformation. Our idea is quite simple. Namely, we shall just weight the lowest cell of the a_2-nd part with $[6]_{p,q} - [3]_{p,q}$ and the other cells with 0. Extending this idea will allow us to define the (p,q)-weights of the cells so that the sum of the (p,q)-weights of the cells that lie in the a_j-th part of column i in the lower augmented board for $j \leq s$ is $[\overline{A}_s]_{p,q}$ and the sum of the (p,q)-weights of the cells that lie in the a_j-th part of column i in the upper augmented board for $j \leq s$ is $-[\overline{A}_s]_{p,q}$.

Fix the two sequences $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$ and the two sign functions $sgn : \{1, \ldots, n\} \rightarrow \{1, -1\}$ and $\overline{sgn} : \{1, \ldots, n\} \rightarrow \{1, -1\}$. The first step in proving equation (5.1) is to define the (p,q)-weight, $\mu_{p,q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})$, of each placement $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ and the (p,q)-weight, $\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(Q)$, of each placement $Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$. To do this, we shall define the (p,q)-weight, $\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c)$, of each cell c in $\mathcal{B}_x^{\mathcal{A}}$. Then if $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ has rooks in cells $c_1, \ldots, c_k$, we set

$$\mu_{p,q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = \prod_{i=1}^{k} \mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{5.2}$$

Similarly, if $Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ has rooks in cells $c_1, \ldots, c_n$, then

$$\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(Q) = \prod_{i=1}^{n} \mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{5.3}$$

Then we define

$$r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, p, q) = \sum_{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} \mu_{p,q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \tag{5.4}$$

To define $\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c)$, we proceed as follows.

1. For each i, the (p,q)-weights, $\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c)$, of the cells c in the i-th column of the x-part of $\mathcal{B}_x^{\mathcal{A}}$ are $p^{x-1}, p^{x-2}q, p^{x-3}q^2, \ldots, pq^{x-2}, q^{x-1}$, reading from bottom to top.

2. For each i, the (p,q)-weights, $\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c)$, of the cells c in the i-th column of the base part of $\mathcal{B}_x^{\mathcal{A}}$ are $sgn(i)p^{b_i-1}$, $sgn(i)p^{b_i-2}q$, $sqn(i)p^{b_i-3}q^2, \ldots, sgn(i)pq^{b_i-2}, sgn(i)q^{b_i-1}$, reading from bottom to top.

3. For each i, we assign (p,q)-weights, $\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c)$, to the cells c in the i-th column of the lower augmented part as follows. First, we assign the (p,q)-weights $\overline{sgn}(i)p^{a_1-1}$, $\overline{sgn}(i)p^{a_1-2}q$, $\overline{sgn}(i)p^{a_1-3}q^2$, $\ldots$, $\overline{sgn}(i)pq^{a_1-2}$, $\overline{sgn}(i)q^{a_1-1}$ to the cells in the a_1-st part of column i in the lower augmented part of the board reading from top to bottom. Thus the sum of the (p,q)-weights of cells in the a_1-st part of column i in the lower augmented part is $[\overline{sgn}(i)a_1]_{p,q}$. Next suppose that we have assigned the (p,q)-weights to cells in the a_j-th part of column i in the lower augmented part for $j = 1, \ldots, s$ so that the sum of the (p,q)-weights of the cells that lie in the a_j-th part of column i in the lower augmented part for $j \leq s$ is $[\overline{A}_s]_{p,q}$. Then we define the (p,q)-weights to the cells in the a_{s+1}-st part of column i in the lower augmented part according to the following cases.

   Case 1: $0 \leq \overline{A}_s \leq \overline{A}_{s+1}$. In this case, we assign the (p,q)-weights of cells in the a_{s+1}-st part to be $[\overline{A}_{s+1}]_{p,q} - [\overline{A}_s]_{p,q}, 0, \ldots, 0$, reading from top to bottom.

   Case 2: $0 \leq \overline{A}_{s+1} < \overline{A}_s$. In this case, we assign the (p,q)-weights of cells in the a_{s+1}-st part to be $[\overline{A}_{s+1}]_{p,q} - [\overline{A}_s]_{p,q}, 0, \ldots, 0$, reading from top to bottom.

   Case 3: $\overline{A}_{s+1} < 0 \leq \overline{A}_s$. In this case, we assign the (p,q)-weights of the cells in the a_{s+1}-st part to be $-[\overline{A}_s]_{p,q}, 0, \ldots, 0, -p^{|\overline{A}_{s+1}|-1}$, $-qp^{|\overline{A}_{s+1}|-2}, \ldots, -pq^{|\overline{A}_{s+1}|-2}$, $-\,q^{|\overline{A}_{s+1}|-1}$, reading from top to bottom.

   Case 4: $0 \geq \overline{A}_s \geq \overline{A}_{s+1}$. In this case, we assign the (p,q)-weights of the cells in the a_{s+1}-st part to be $[\overline{A}_{s+1}]_{p,q} - [\overline{A}_s]_{p,q}, 0, \ldots, 0$, reading from top to bottom.

   Case 5: $0 \geq \overline{A}_{s+1} > \overline{A}_s$. In this case, we assign the (p,q)-weights of the cells in the a_{s+1}-st part to be $[\overline{A}_{s+1}]_{p,q} - [\overline{A}_s]_{p,q}, 0, \ldots, 0$, reading from top to bottom.

   Case 6: $\overline{A}_{s+1} > 0 \geq \overline{A}_s$. In this case, we assign the (p,q)-weights of the cells in the a_{s+1}-st part to be $-[\overline{A}_s]_{p,q}, 0, \ldots, 0, p^{|\overline{A}_{s+1}|-1}, qp^{|\overline{A}_{s+1}|-2}, \ldots, pq^{|\overline{A}_{s+1}|-2}, q^{|\overline{A}_{s+1}|-1}$, reading from top to bottom.

4. For each i, the cell in the r-th row of the i-th column of the upper augmented part of the board, reading from bottom to top, is equal to -1 times the weight of the cell in the r-th row of the i-th column of the lower augmented part of the board, reading from top to bottom.

Then we can prove Theorem 5.1 by computing the sum

$$L(q) = \sum_{Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})} \mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(Q) \tag{5.5}$$

in two different ways as before.

We can also obtain variations of the (p,q)-analogue Theorem 5.1 where we replace $[x]_{p,q} - [a]_{p,q}$ by $[x-a]_{p,q}$ and replace $[x]_{p,q} + [a]_{p,q}$ by $[x+a]_{p,q}$ much as we did in the q-analogue case by using the transformations

$$[x]_{p,q} - p^{x-a}[a]_{p,q} = q^a[x-a]_{p,q} \tag{5.6}$$

and

$$p^a[x]_{p,q} + q^x[a]_{p,q} = q^a[x+a]_{p,q} \tag{5.7}$$

where $x \geq a \geq 0$.

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