Dihedral F-Tilings of the Sphere by Equilateral and Scalene Triangles - II

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Abstract

The study of dihedral f-tilings of the Euclidean sphere S^2 by triangles and r-sided regular polygons was initiated in 2004 where the case $r = 4$ was considered [5]. In a subsequent paper [1], the study of all spherical f-tilings by triangles and r-sided regular polygons, for any $r \geq 5$, was described. Later on, in [3], the classification of all f-tilings of S^2 whose prototiles are an equilateral triangle and an isosceles triangle is obtained. The algebraic and combinatorial description of spherical f-tilings by equilateral triangles and scalene triangles of angles β, γ and δ ($\beta > \gamma > \delta$) whose edge adjacency is performed by the side opposite to β was done in [4]. In this paper we extend these results considering the edge adjacency performed by the side opposite to δ.

Keywords: dihedral f-tilings, combinatorial properties, symmetry groups

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1 Introduction

Spherical folding tilings or *f-tilings* for short, are edge-to-edge decompositions of the sphere by geodesic polygons, such that all vertices are of even valency and the sum of alternate angles around each vertex is π. A f-tiling τ is said to be *monohedral* if it is composed by congruent cells, and *dihedral* if every tile of τ is congruent to one of two fixed sets X and Y (prototiles of τ). We shall denote by $\Omega(X, Y)$ the set, up to isomorphism, of all dihedral f-tilings of S^2 whose prototiles are X and Y.

The classification of all spherical folding tilings by rhombi and triangles was obtained in 2005 [6]. However the corresponding study considering two triangular (non- isomorphic) prototiles is not yet completed. This is not surprising, since it is much harder.

At this moment, the cases are known in which the prototiles are:

- an equilateral triangle and an isosceles triangle, [3];
- an equilateral triangle of side a and a scalene triangle of sides $b > c > d$, with adjacency of type I, that is, $a = b$, [4].

Here our interest is focused on spherical triangular dihedral f-tilings whose prototiles are an equilateral triangle and a scalene triangle with adjacency of type II (Figure 1).

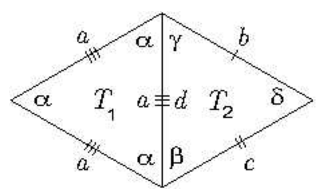

Figure 1: Adjacency of type II (performed by the side opposite to δ, i.e., $a = d$).

From now on T_1 denotes an equilateral spherical triangle of angle α $\left(\alpha > \frac{\pi}{3}\right)$ and side a and T_2 a scalene spherical triangle of angles δ, γ, β, with the order relation $\delta < \gamma < \beta$ $(\beta + \gamma + \delta > \pi)$ and with sides b (opposite to β), c (opposite to γ) and a (opposite to δ). The type II edge-adjacency condition can be analytically described by the equation

$$\frac{\cos\alpha(1+\cos\alpha)}{\sin^2\alpha} = \frac{\cos\delta + \cos\gamma\cos\beta}{\sin\gamma\sin\beta} \tag{1.1}$$

In order to get any dihedral f-tiling $\tau \in \Omega(T_1, T_2)$, we find useful to start by considering one of its *planar representations*, beginning with a common vertex to an equilateral triangle and a scalene triangle in adjacent positions. In the diagrams that follows it is convenient to label the tiles according to the following procedures:

(i) The tiles by which we begin the planar representation of a tiling $\tau \in \Omega(T_1, T_2)$ are labelled by 1 and 2, respectively;

(ii) For $j \geq 2$, the location of tile j can be deduced from the configuration of tiles $(1, 2, \ldots, j-1)$ and from the hypothesis that the configuration is part of a complete planar representation of a f-tiling (except in the cases indicated).

2 Triangular Dihedral F-Tilings with Adjacency of Type II

Starting a planar representation of $\tau \in \Omega(T_1, T_2)$ with two adjacent cells congruent to T_1 and T_2 respectively, see Figure 2, a choice for angle $x \in \{\gamma, \delta\}$ must be made. We shall consider separately each one of these situations.

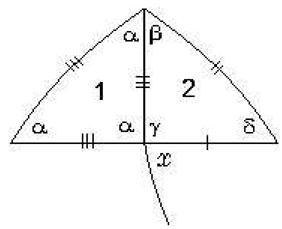

Figure 2: Planar representation.

With the above terminology one has:

Proposition 2.1. *If $x = \gamma$, then $\Omega(T_1, T_2)$ consists of three discrete families of isolated dihedral triangles f-tilings $(\mathcal{D}^p)_{p\geq 4}$, $(\mathcal{F}^p)_{p\geq 4}$ and $(\mathcal{E}^m)_{m\geq 5}$, such that the sums of alternate angles around vertices are respectively of the form:*

$$\alpha + \beta = \pi,\ 2\alpha + \gamma = \pi \text{ and } p\delta = \pi,\ \text{for } \mathcal{D}^p,\ p \geq 4;$$
$$\alpha + \beta = \pi,\ 2\gamma + \alpha = \pi \text{ and } p\delta = \pi,\ \text{for } \mathcal{F}^p,\ p \geq 4;$$
$$\alpha + \beta = \pi,\ \alpha + 2\gamma + \delta = \pi \text{ and } m\delta = \pi,\ \text{for } \mathcal{E}^m,\ m \geq 5.$$

3D representations of $\mathcal{D}^4, \mathcal{F}^4$ and $\mathcal{E}^5$ are given, respectively, in Figures 11,14 and 22.

Proof. In order to have $\Omega(T_1, T_2) \neq \emptyset$, necessarily $\alpha + x \leq \pi$.
1. Let us assume that $\alpha + x = \pi$ and $x = \gamma$.
In this case, $\alpha + \beta > \pi$ and so expanding the configuration illustrated in Figure 2, we obtain the following one, and consequently $\delta + \beta \leq \pi$.

Let us assume that $\beta + \delta = \pi$. As $\alpha + \gamma = \pi$, then by the adjacency condition (1.1), we conclude that $\cot \alpha = -\cot \beta$. Therefore $\alpha < \dfrac{\pi}{2}$ and so $\gamma > \dfrac{\pi}{2}$.

The local configuration started in Figure 3 can be extended to the one given in Figure 4. However at vertex v_1, the alternate angle sum which contains 2γ does not satisfy the angle folding relation.

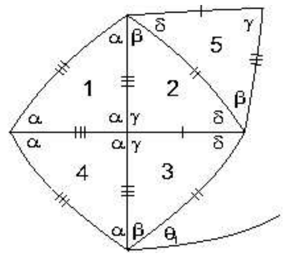

Figure 3: Planar representation.

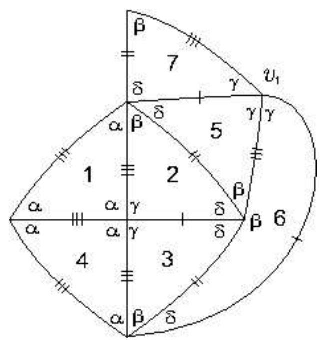

Figure 4: Planar representation.

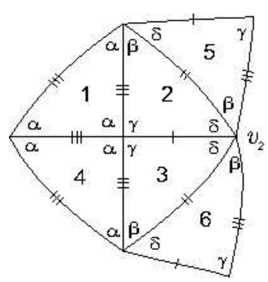

Figure 5: Planar representation.

In case $\beta + \delta < \pi$, the angle labelled θ_1 in Figure 3 is δ, otherwise we would have $\alpha + \beta > \pi$, violating the angle folding relation. Therefore, the configuration gives rise to the one illustrated in Figure 5.

Looking at the angles surrounding vertex v_2, one has $\beta + \delta + \lambda > \pi$, for $\lambda \in \{\alpha, \gamma, \beta\}$. The angle folding relation is once again, not satisfied.

2. Now, let us assume that $\alpha + x < \pi$ and $x = \gamma$.

Starting from the configuration in Figure 2, we end up with the one given in Figure 6, with $\theta_2 \in \{\beta, \delta\}$.

2.1 If $\theta_2 = \beta$ and $\alpha + \beta = \pi$, then $\gamma + \delta > \dfrac{\pi}{3}$ and by (1.1) we conclude that $\alpha < \dfrac{\pi}{2} < \beta$.

Now, the sums of the alternate sequence of angles at vertices containing α and γ must

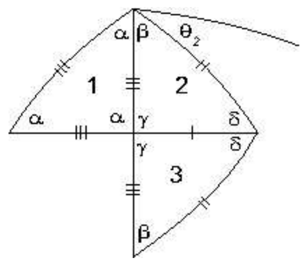

Figure 6: Planar representation.

be $\alpha + \gamma + \lambda = \pi$, (Figure 7), where the parameter λ cannot be β and being a sum of angles (α, γ, δ). The angle α will appear at most once.

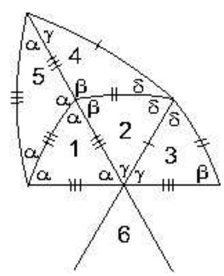

Figure 7: Planar representation.

2.1.1 Suppose that λ is a sum of angles with one angle α. Then, $2\alpha + \gamma \leq \pi$, but having in account that $2\alpha + \gamma + \mu > \pi$, for any $\mu \in \{\alpha, \delta, \gamma, \beta\}$ one has $2\alpha + \gamma = \pi$.

Adding some new cells to the configuration illustrated in Figure 6 we obtain the one shown in Figure 8.

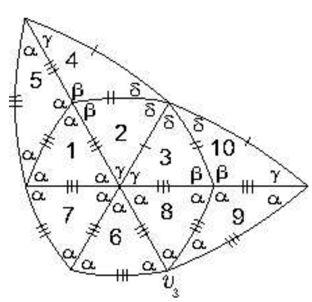

Figure 8: Planar representation.

Observe that tile 9 must be an equilateral triangle, otherwise the angle folding relation will be not fulfilled.

One of the alternate angle sums at vertex v_3 is $2\alpha + k\delta = \pi, k \geq 1$ or $2\alpha + \gamma = \pi$.

Suppose that $2\alpha + k\delta = \pi$, for some $k \geq 2$. Then, expanding the local configuration illustrated in Figure 8 we get the one below (Figure 9).

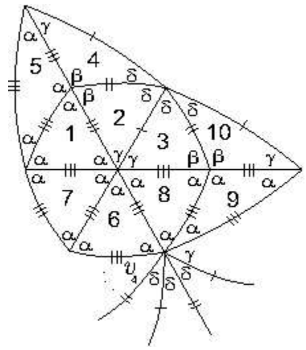

Figure 9: Planar representation.

At vertex v_4 we observe that the other alternate sum is $\alpha + \gamma + (k-1)\delta + \beta$ which is impossible since it is bigger than π.

Assume now that $2\alpha + \gamma = \pi$. Choosing one of the possible positions for tile 11, the extended local configuration started in Figure 8 is the following one.

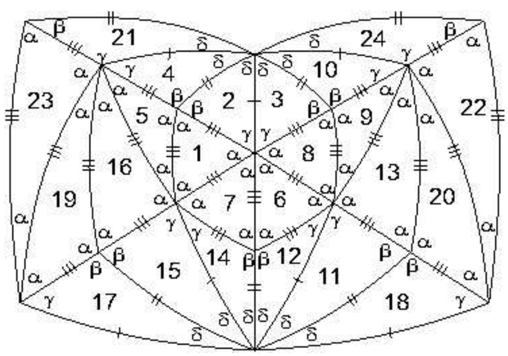

Figure 10: Planar representation.

The construction of this configuration follows a symmetric pattern with three type of vertices: the vertices of valency four whose alternate sums are ruled by the equation $\alpha + \beta = \pi$, the vertices of valency 6 surrounded by the angular sequence $(\alpha, \alpha, \alpha, \alpha, \gamma, \gamma)$, and the vertices of valency $2p$ whose alternate sums are $p\delta = \pi$. The parameter p must be greater or equal to 4, since $\delta = \dfrac{\pi}{2}$ or $\delta = \dfrac{\pi}{3}$ contradicts the adjacency condition. For $p = 4$, we obtain a global configuration (Figure 11) of a tiling $\tau \in \Omega(T_1, T_2)$, which will be denoted by $\mathcal{D}^4$.

The corresponding f-tiling is composed of 16 equilateral triangles and 16 scalene triangles; and the angles are $\delta = \dfrac{\pi}{4}, \alpha = \arccos\left(\dfrac{-1+\sqrt{1+4\sqrt{2}}}{4}\right) (\alpha \approx 66.7^\circ), \gamma = \pi - 2\alpha\, (\gamma \approx 46.5^\circ)$ and $\beta = \pi - \alpha\, (\beta \approx 113^\circ)$.

The other possible position for tile 11 gives rise to a similar global configuration of the tiling $\mathcal{D}^4$.

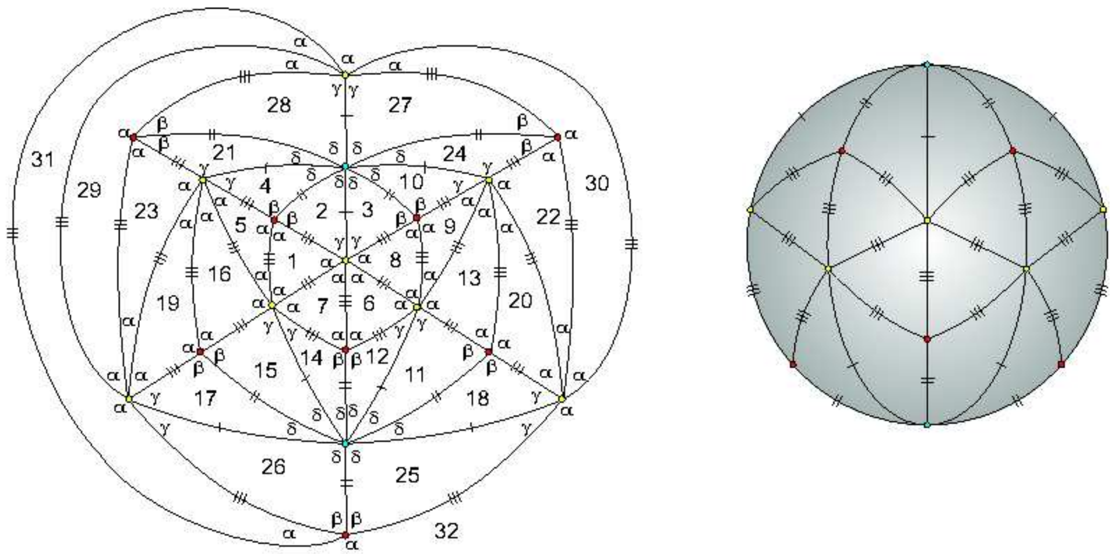

Figure 11: Global configuration and 3D representation of $\mathcal{D}^4$.

For each $p \geq 4$, we obtain a global configuration of a tiling $\tau \in \Omega(T_1, T_2)$ with vertices of valency 4, 6 and $2p$ composed by $4p$ equilateral and $4p$ scalene triangles which, will be denoted by $\mathcal{D}^p, p \geq 4$.

2.1.2 Suppose that λ is a sum of angles containing at least one angle γ. Then, $\alpha+2\gamma \leq \pi$.

2.1.2.1 If $\alpha + 2\gamma = \pi$, then $\gamma < \alpha$ and we can expand the local planar representation illustrated in Figure 6; we obtain one of the configurations illustrated in Figure 12.

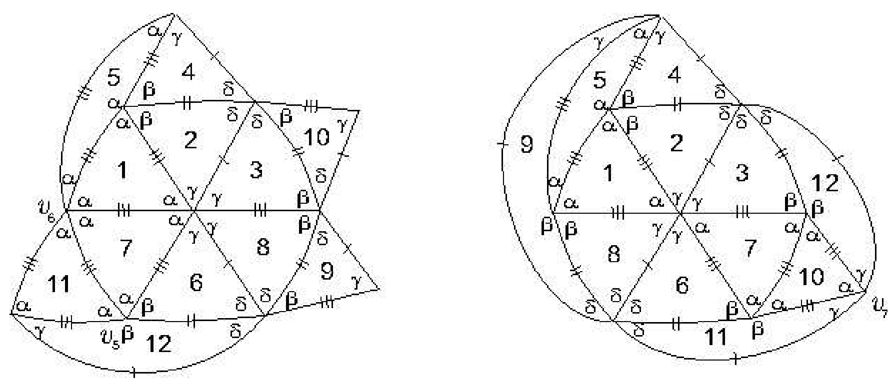

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Figure 12: Planar representations.

Observe that for tile 6 there are two possibilities for the position of its sides (see Figure 12-I and II). In configuration I, tile 11 is necessarily an equilateral triangle (otherwise, one of the alternate angle sums at vertex v_5 would be $\gamma + \beta = \pi$, so that $\gamma = \alpha > \dfrac{\pi}{3}$, contradicting $\alpha+2\gamma = \pi$) and so vertex v_6 is surrounded by an angular sequence containing four adjacent angles α. Accordingly, at this vertex we should have $2\alpha + k\delta = \pi, k \geq 1$ (otherwise we would have $\gamma = \alpha$, contradicting $\alpha + 2\gamma = \pi$).

However, the cyclic sequence of angles $(\alpha, \alpha, \alpha, \alpha, \delta, ..., \delta)$ around vertex v_6 violates edge compatibility.

Concerning to the configuration II and taking into account that $\alpha + \beta = \pi$, $\alpha + 2\gamma = \pi$ $(\gamma < \frac{\pi}{3} < \alpha)$ and $\beta + \gamma + \delta > \pi$ $(\beta > \gamma > \delta)$ we conclude that the alternate sum containing two angles γ at vertex v_7 must be $2\gamma + m\delta = \pi,\ \ m \geq 2$ or $2\gamma + \alpha = \pi$.

2.1.2.1.1 If $2\gamma + m\delta = \pi$ for some $m \geq 2$, the sides arrangement emanating from vertex v_7 require the other alternate sum to contain one angle α and $1+m$ angles δ, which is impossible as illustrated in Figure 13.

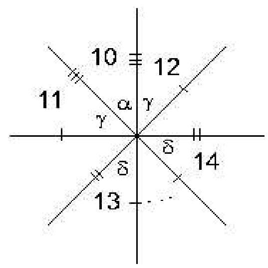

Figure 13: Angles arrangement around vertex v_7.

2.1.2.1.2 If $2\gamma + \alpha = \pi$, the configuration in Figure 12-II expands globally, and in a symmetric way, if and only if $p\delta = \pi$ with $p \geq 4$. Observe that $\delta = \frac{\pi}{2}$ or $\delta = \frac{\pi}{3}$ violates the adjacency condition.

For $p = 4$, we get a tiling, $\mathcal{F}^4$, with 8 equilateral triangles and 16 scalene triangles; and the angles are:

$$\delta = \frac{\pi}{4},\ \gamma = \arccos\left(\frac{-\sqrt{2}+\sqrt{34}}{8}\right) \approx 56.4^\circ,\ \ \beta = \pi - \alpha \approx 113^\circ \text{ and } \alpha = \pi - 2\gamma \approx 67^\circ.$$

For each $p \geq 4$, we get a tiling $\tau \in \Omega(T_1, T_2)$ with vertices of valency 4, 6 and $2p$, composed by $2p$ equilateral and $4p$ scalene triangles, which will be denoted by $\mathcal{F}^p, p \geq 4$.

2.1.2.2 If $\alpha + 2\gamma < \pi$ $\left(\gamma < \frac{\pi}{3} < \alpha\right)$, then $\alpha + 3\gamma = \pi$ or $\alpha + 2\gamma + \delta = \pi$, since from $\alpha + \beta = \pi$ and $\delta + \gamma + \beta > \pi$ $(\beta > \gamma > \delta)$, one has $\gamma + \delta > \alpha > \frac{\pi}{3}$.

2.1.2.2.1 Suppose $\alpha + 3\gamma = \pi$ (Figure 6). Then $\gamma < \frac{2\pi}{9}$ and $\beta = 3\gamma$. As $\delta + \gamma + \beta > \pi$, we conclude that $\delta > \frac{\pi}{9}$.

Extending the configuration illustrated in Figure 6, we may add some new cells ending up with the one illustrated in Figure 15.

Note that there are two possible positions for the sides of tile 6. If we make the choice shown in Figure 16, the angle θ_3 at vertex v_8 may be α, γ or β. Whichever we choose $\beta + \delta + \theta_3 > \pi$ and we cannot expand this configuration to a planar representation of an f-tiling.

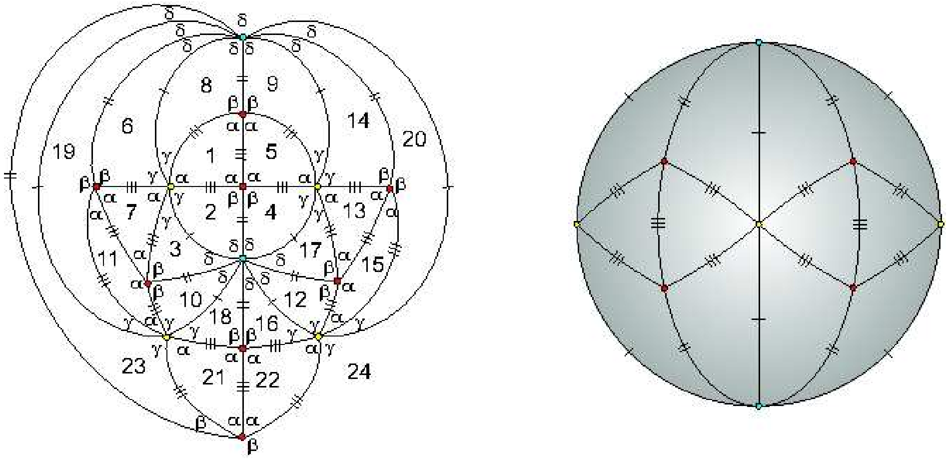

Figure 14: Global configuration and 3D representation of $\mathcal{F}^4$.

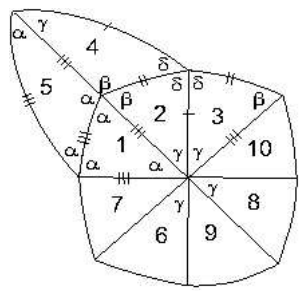

Figure 15: Planar representation.

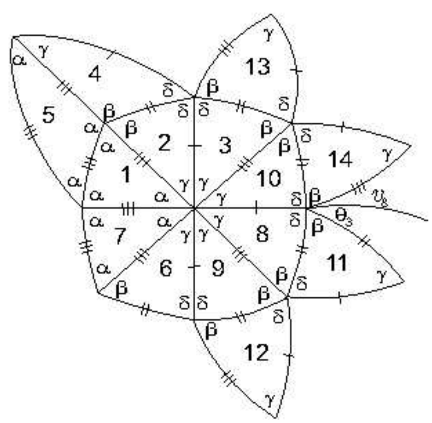

Figure 16: Planar representation.

The other choice on the sides of tile 6 forces the configuration below (Figure 17).

Tile 8 of Figure 17 is forced in order to avoid the same situation of incompatibility as the one shown in Figure 16.

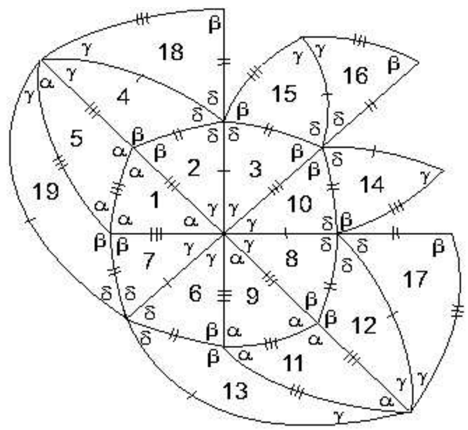

Figure 17: Planar representation.

We conclude that the vertices surrounded by alternate angles β and δ must have at most four angles δ, since $\beta > \dfrac{\pi}{2}$ and $\delta > \dfrac{\pi}{9}$. By the adjacency condition we have

$$\frac{\cos k\delta(1+\cos k\delta)}{\sin^2 k\delta} = \frac{\cos\delta - \cos k\delta \cos\left(\dfrac{\pi}{3} - k\dfrac{\delta}{3}\right)}{\sin k\delta \sin\left(\dfrac{\pi}{3} - k\dfrac{\delta}{3}\right)}.$$

As $\delta > \dfrac{\pi}{9}$, then $k = 2$ and $\delta \approx 30.9°$ (for $k = 3, 4$, we get, respectively, $\delta \approx 19.481°, 14.324°$, contradicting $\delta > \dfrac{\pi}{9}$). Consequently, $\beta \approx 118.2°, \gamma \approx 39.4°$ and $\alpha \approx 61.8°$. The configuration can be expanded ending up at a vertex, v_9, whose alternate angle sum does not satisfied $\beta + 2\delta = \pi$ (see Figure 18).

2.1.2.2.2 Suppose now that $\alpha + 2\gamma + \delta = \pi$ (Figure 7). Tile 6 can be either a scalene triangle or an equilateral one.

2.1.2.2.2.1 Assume first that tile 6 is a scalene triangle, as is illustrated in Figure 19.

At vertex v_{10}, the alternate sum containing α and δ is

$$\alpha + k\delta = \pi \ (k \geq 4), \ \ \alpha + t\delta + \gamma = \pi \ (t \geq 3), \ \ \alpha + \delta + 2\gamma = \pi \ \text{ or } \ 2\alpha + q\delta = \pi \ (q \geq 1).$$

The other alternate sum at vertex v_{10} containing γ and δ is

$$\gamma + m\delta = \pi \ (m \geq 4), \ \ \gamma + \alpha + n\delta = \pi \ (n \geq 3), \ \ \alpha + \delta + 2\gamma = \pi \ \text{ or } \ 2\gamma + p\delta = \pi \ (p \geq 1).$$

Taking into account:

- the angular order relation, $\dfrac{\pi}{3} < \alpha < \dfrac{\pi}{2}, \ \ \delta < \gamma < \beta, \ \ \gamma > \dfrac{\pi}{6}, \ \ \beta > \dfrac{\pi}{2}$,
- $\alpha + \beta = \pi$,
- $\alpha + 2\gamma + \delta = \pi$ and
- the adjacency condition,

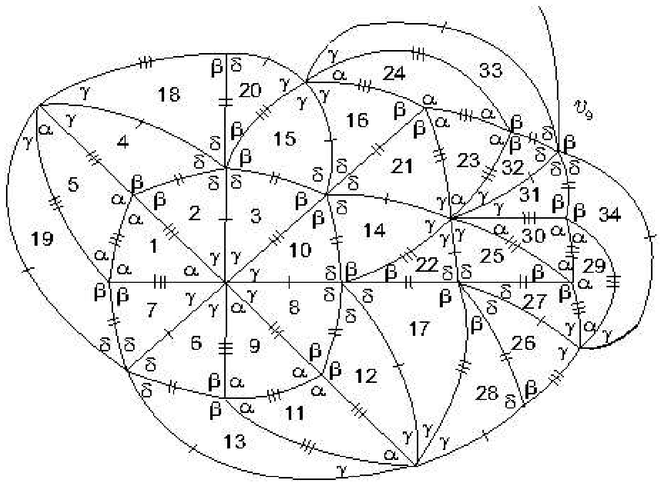

Figure 18: Planar representation.

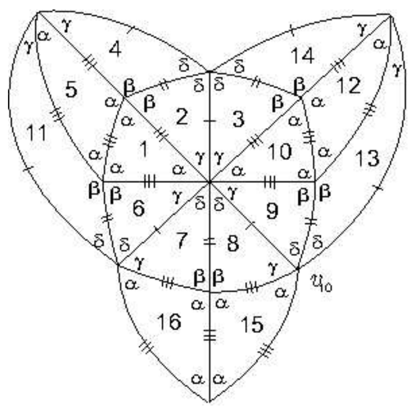

Figure 19: Planar representation.

we conclude that the alternate angle sums at vertex v_{10} are $\alpha+\gamma+t\delta=\pi,\ t\geq 3$ or $\alpha+2\gamma+\delta=\pi$, but not both.

2.1.2.2.2.1.1 If $\alpha+\gamma+t\delta=\pi$, for some $t\geq 3$, the possible positions of γ are the ones illustrated in Figures 20-I and 20-II.

In angular sequence I, the alternate angle sums are $\alpha+\gamma+t\delta=\pi$ and $\gamma+t\delta+\beta=\pi$, which is impossible, as $\beta+\gamma+\delta>\pi$. Thus Figure 20-II illustrates the way vertex v_{10} must be surrounded.

Expanding the configuration in Figure 19, we end up with a contradiction at vertex v_{11}, see below (Figure 21).

2.1.2.2.2.1.2 Suppose now that one of the alternate angle sums at vertex v_{10} is $\alpha+2\gamma+\delta=\pi$. Expanding the configuration in Figure 19 we may deduce the existence of vertices surrounded uniquely by angles δ and so $\delta=\dfrac{\pi}{m}$. Since $\delta<\gamma<\alpha$, we conclude that $m\geq 5$.

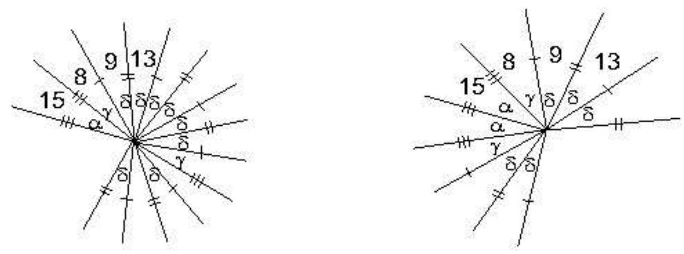

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Figure 20: Angles arrangement around vertex v_{10}.

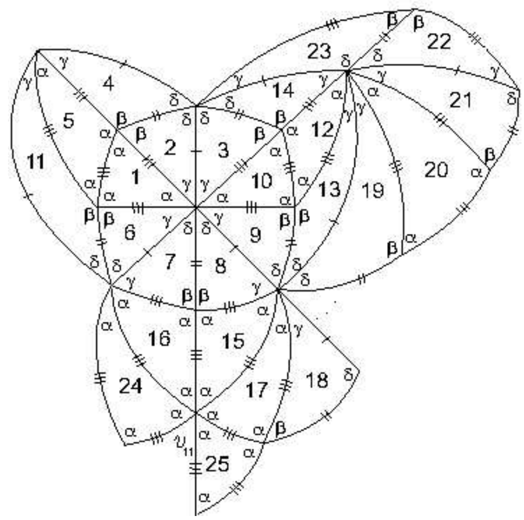


Figure 21: Planar representation.

For any $m \geq 5$, we obtain an f-tiling $\mathcal{E}^m$ that has one class of vertices of valency 4, one of valency 8, and one of valency 2m, being composed of $4m$ equilateral triangles and $8m$ scalene triangles. A 3D representation for $m = 5$ is illustrated in Figure 22.

2.1.2.2.2.2 Suppose now, that tile 6 (Figure 7) is an equilateral triangle. Adding some new cells to the illustrated configuration, we get a vertex, v_{12}, surrounded by the angular sequence $(\alpha, \gamma, \delta, \beta)$, which does not satisfy the angle folding relation (Figure 23).

2.1.3 Suppose that λ is a sum of angles containing δ. Then, $\alpha + \gamma + \delta \leq \pi$.

If $\alpha + \gamma + \delta = \pi$, from the configuration in Figure 7, tile 6 has two possible positions, see below (Figure 24).

In any of theses cases, we get the alternate angle sum $\beta + 2\gamma = \pi$, which is impossible since $\beta + 2\gamma > \beta + \gamma + \delta > \pi$.

Therefore, $\alpha + \gamma + \delta < \pi$. As the case $\alpha + \gamma + \delta + \gamma = \pi$ was just studied, we assume that $\alpha + \gamma + t\delta = \pi$, for some $t > 2$. Pursuing the expansion of the configuration given in Figure 7, we end up with a vertex v_{13} surrounded by a cyclic sequence of angles of the form $(\gamma, \gamma, \alpha, \gamma, \delta, \delta, ..., \delta, \beta)$, which is a contradiction since $\beta + \gamma + \delta > \pi$, see Figure 25.

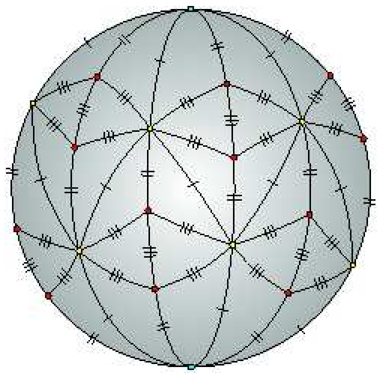


Figure 22: 3D representation of $\mathcal{E}^5$

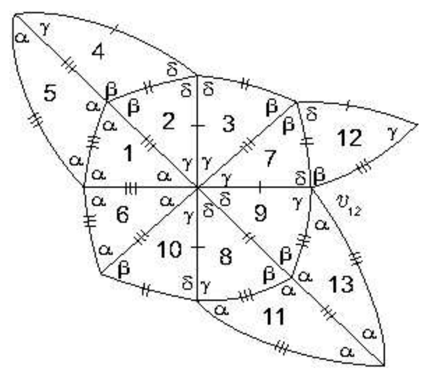


Figure 23: Planar representation.

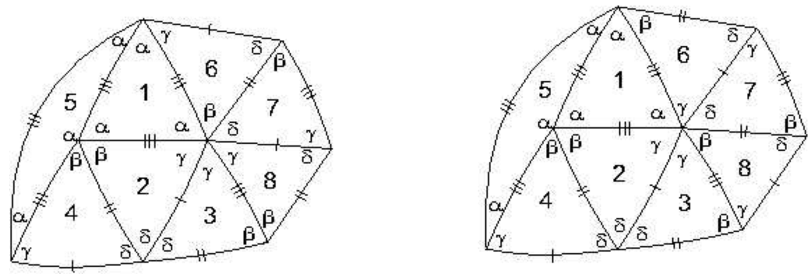


Figure 24: Planar representations.

2.2 If $\theta_2 = \beta$ and $\alpha + \beta < \pi$ (Figure 6), then $\alpha + \beta + \gamma = \pi$ or $\alpha + \beta + k\delta = \pi$, $k \geq 1$, since $\alpha, \beta > \dfrac{\pi}{3}$ and $\gamma > \dfrac{\pi}{6}$.

Suppose that $\alpha + \beta + \gamma = \pi$. Then $\delta > \alpha > \dfrac{\pi}{3}$, which is an impossibility.

If $\alpha + \beta + k\delta = \pi, k \geq 1$, the configuration illustrated in Figure 6 can be expanded to the one below (Figure 26), according to a choice for the edge position of tile 6. This choice ends up in a contradiction, since $2\beta + \rho > \pi$, for any $\rho \in \{\alpha, \delta, \gamma, \beta\}$.

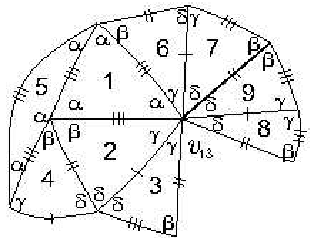


Figure 25: Planar representation.

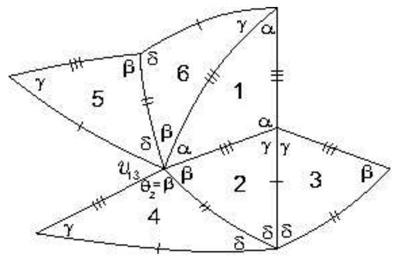


Figure 26: Planar representation.

Observe that the other possible position for tile 6 leads to an alternate angle sum of the form $\gamma + \beta + \mu = \pi$, for some μ. Nevertheless $\gamma + \beta + \mu > \pi$, for each $\mu \in \{\alpha, \gamma, \delta, \beta\}$.
2.3 If $\theta_2 = \delta$ (Figure 6), then $\alpha + \delta < \pi$, since $\alpha + \gamma < \pi$ and $\delta < \gamma$. Consequently, $\beta + \delta \leq \pi$. If $\beta + \delta = \pi$, then $\beta + \gamma > \pi$ and the configuration in Figure 27 exhibits a contradiction at the vertex surrounded by the sequence of angles $\alpha, \beta, \delta, \gamma$.

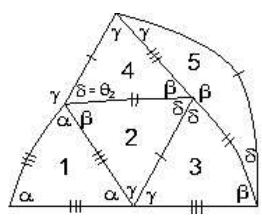


Figure 27: Planar representation.

Therefore, $\beta + \delta < \pi$. The configuration is now (Figure 28):

A decision about the angle labelled $\theta_3 \in \{\gamma, \delta\}$ must be taken.
2.3.1 If $\theta_3 = \gamma$, then the alternate sum containing β and γ at vertex v_{14} is $\beta + \gamma = \pi$ or $\beta + \gamma + \alpha = \pi$. However, if $\beta + \gamma = \pi$, the other alternate sum would be $\alpha + \delta = \pi$, which is a contradiction. Therefore, $\beta + \gamma + \alpha = \pi$, but since $\beta + \gamma + \delta > \pi$, then $\alpha < \delta < \gamma$, which is an impossibility.
2.3.2 If $\theta_3 = \delta$, then looking at vertex v_{15} in Figure 29, the alternate sum containing β must be of the form $\beta + \alpha + n\delta = \pi, n \geq 1$.

Observe that $\beta \geq \dfrac{\pi}{2}$, otherwise, $\delta < \gamma < \beta < \dfrac{\pi}{2}$ and taking into account the adjacency

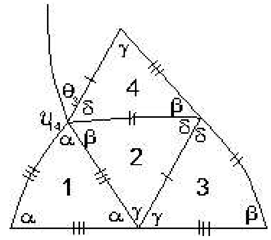


Figure 28: Planar representation

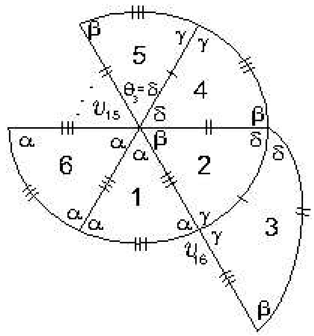


Figure 29: Planar representation.

condition, we would have $\alpha < \frac{\pi}{2}$, which is impossible, since vertices of valency four must occur. In fact, any f-tiling $\tau \in \Omega(T_1, T_2)$ has at least six vertices of valency four as established in [5].

2.3.2.1 Considering $\beta = \frac{\pi}{2}$, then $\delta < \gamma < \frac{\pi}{2}$, $\gamma > \frac{\pi}{4}$ and once again by the adjacency condition $\alpha < \frac{\pi}{2}$.

On the other hand, the sequence of alternate angles containing α and γ, at vertex v_{16}, satisfy $2\alpha + \gamma = \pi$ or $\alpha + 2\gamma = \pi$ or $\alpha + \gamma + t\delta = \pi, t \geq 2$.

If $2\alpha + \gamma = \pi$, then $\gamma < \frac{\pi}{3}$ and $\delta > \frac{\pi}{6}$. But from $\beta + \alpha + n\delta = \pi$, for some $n \geq 1$, which is a contradiction.

If $\alpha + 2\gamma = \pi$, the same argument is valid. Therefore, $\alpha + \gamma + t\delta = \pi, t \geq 2$ and the configuration illustrated in Figure 29 can be extended to the one shown in Figure 30.

As we also have $\beta + \gamma + \mu > \pi$ for any $\mu \in \{\alpha, \delta, \gamma, \beta\}$, there is only one way to arrange the sides of the tile numbered 7.

Looking at vertex v_{16}, we conclude that the other sequence of alternate angles must have one angle β, which is impossible.

2.3.2.2 Considering now that $\beta > \frac{\pi}{2}$, since $\beta + \delta < \pi, \beta + \alpha < \pi$ and $\alpha + \gamma < \pi$, the alternate angle sums at vertices of valency four are $\beta + \gamma = \pi$ or $2\gamma = \pi$.

If $\gamma = \frac{\pi}{2}$, then $\beta > \gamma > \alpha > \delta$ and so $\alpha + \gamma + m\delta = \pi$, for some $m \geq 2$ at vertex v_{16} (see Figure 29). The edge length compatibility forces the existence of an angle β at

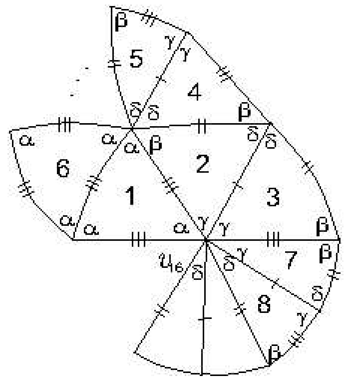


Figure 30: Planar representation.

vertex v_{16}, which is a contradiction since $\beta+\gamma>\pi$.

Summarizing, we have $\beta+\gamma=\pi$ and $\beta+\alpha+n\delta=\pi;(n\geq 1)$ and so $\gamma>\alpha>\dfrac{\pi}{3}$, which means that a sequence of alternate angles at vertex v_{17} is $\alpha,\gamma,\delta,...,\delta$. Adding some new cells to the configuration in Figure 29, we conclude that there is an angle β surrounding vertex v_{17}, which is an impossibility since we have $\beta+\gamma+\gamma>\pi$ (see Figure 31).

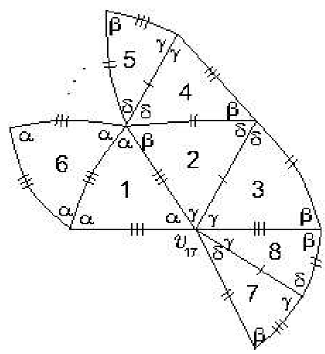


Figure 31: Planar representation.

$\square$

Proposition 2.2. *If $x=\delta$ (Figure 2) , then $\Omega(T_1,T_2)=\emptyset$.*

Proof. Assume first that:
1. $\alpha+x=\pi$ and $x=\delta$.

If $\alpha\leq\dfrac{\pi}{2}$, then $\delta\geq\dfrac{\pi}{2}$ and consequently $\beta>\gamma>\dfrac{\pi}{2}$, turning impossible any expansion of the configuration shown in Figure 2.

Therefore, $\alpha>\dfrac{\pi}{2},\delta<\dfrac{\pi}{2}$ and by the adjacency condition $\gamma<\dfrac{\pi}{2}$ and $\beta>\dfrac{\pi}{2}$.

The configuration illustrated in Figure 2 can be extended to the following one (Figure 32).

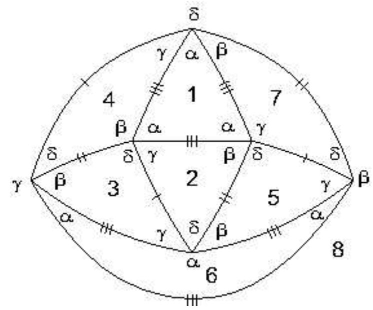


Figure 32: Planar representation.

As $\alpha+\delta=\pi=\beta+\gamma$ and $\gamma>\delta$, then $\alpha>\beta$. Accordingly, $\alpha>\beta>\dfrac{\pi}{2}>\gamma>\delta$ and by the adjacency condition we conclude that $-\cos\alpha-\cos^2\beta<0$. However, $0<\cos^2\beta<-\cos\beta<-\cos\alpha$ and consequently $-\cos\alpha-\cos^2\beta>0$, which is a contradiction. Therefore, the configuration illustrated in Figure 32 does not extend to an f-tiling $\tau\in\Omega(T_1,T_2)$.

2. If $\alpha+x<\pi$ and $x=\delta$, a decision must be taken about the angle $\theta_1\in\{\beta,\delta\}$ in Figure 33.

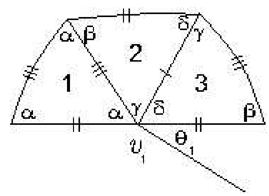


Figure 33: Planar representation.

2.1 If $\theta_1=\beta$, then the alternate sum containing β and γ at vertex v_1 is $\beta+\gamma+\alpha=\pi$, but since $\beta+\gamma+\delta>\pi$, then $\dfrac{\pi}{3}<\alpha<\delta<\gamma$, which is an impossibility.

2.2 If $\theta_1=\delta$, we can add a new cell to the configuration and obtain the one in Figure 34.

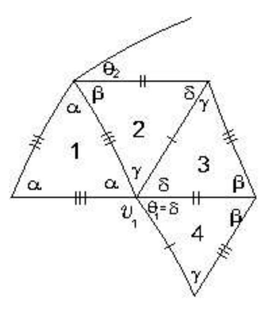


Figure 34: Planar representation.

A decision about the angle $\theta_2\in\{\delta,\beta\}$ must be taken.

2.2.1 If $\theta_2 = \delta$, the configuration illustrated in Figure 34 expands and gives rise to the one in Figure 35.

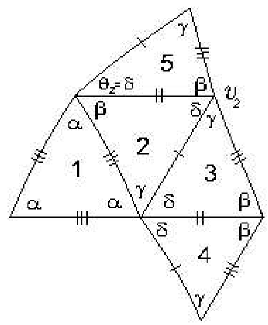


Figure 35: Planar representation.

Consider the alternate angle sum containing β and γ at vertex v_2. This sum must be $\beta + \gamma = \pi$. We have at the same vertex another alternate angle sum which is $\alpha + \delta = \pi$, contradicting our assumption.

2.2.2 If $\theta_2 = \beta$, then $\alpha + \beta \leq \pi$ and consequently $\gamma + \delta > \dfrac{\pi}{3}$.

2.2.2.1 Let us first analyze the case $\alpha + \beta = \pi$. We shall consider separately the cases $\alpha + \beta = \pi$ and $\alpha + \beta < \pi$. By the adjacency condition, we have $\alpha < \dfrac{\pi}{2} < \beta$.

Consider the alternate angle sum containing γ and δ at vertex v_1 (Figure 34). Taking into account the relation between angles and the edge lengths compatibility it can be seen that this sum must be of the form $\alpha + \gamma + n\delta = \pi, n \geq 1$ or $\alpha + 2\gamma + \delta = \pi$.

2.2.2.1.1 In the first case, the configuration in Figure 34 extends to the following one (Figure 36).

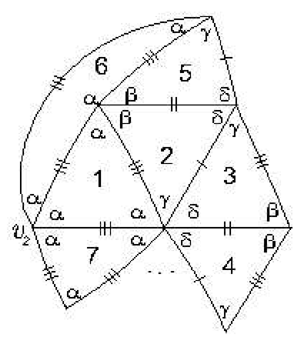


Figure 36: Planar representation.

Looking at vertex v_2, the alternate sum containing two angles α is $2\alpha + \gamma = \pi$ or $2\alpha + p\delta = \pi, p \geq 1$. In case, $2\alpha + \gamma = \pi$ the configuration ends up in to the one illustrated in Figure 37.

However, the sequence of alternate angles containing γ and β, at vertices v_3 and $\widetilde{v}_3$, must satisfy $\beta + \gamma = \pi$, which is a contradiction since $\beta + \alpha = \pi$ and $2\alpha + \gamma = \pi$. Observe

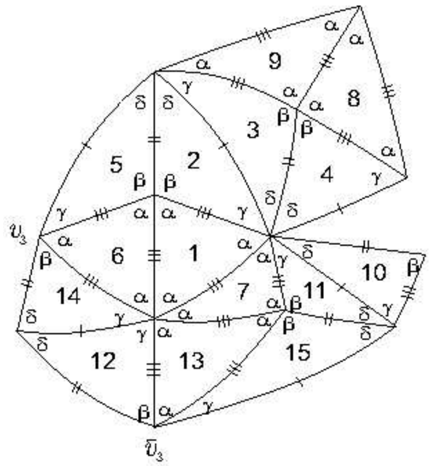


Figure 37: Planar representation.

that due to tile 11, tile 13 must be an equilateral triangle in order to avoid two alternate angles β.

In case $2\alpha + p\delta = \pi, p \geq 1$, the sequence of angles at vertex v_2 must be the one in Figure 38 and the alternate sequence of angles containing α and γ must contain one angle β, which is an impossibility since $\alpha + \gamma + \beta > \pi$.

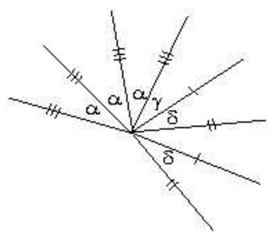


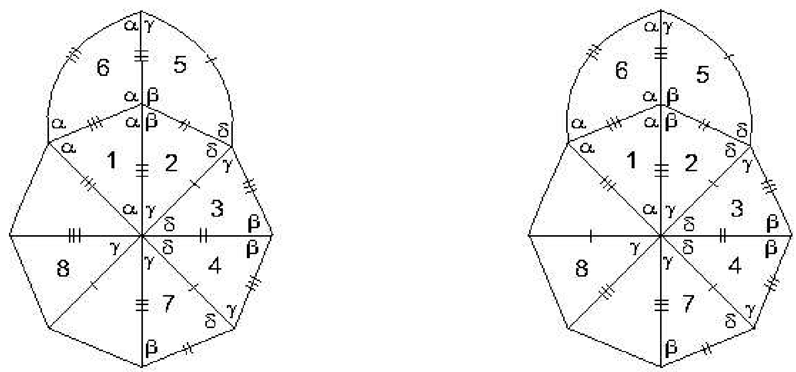
Figure 38: Angles arrangement around vertex v_2

2.2.2.1.2 If $\alpha + 2\gamma + \delta = \pi$ (vertex v_1, Figure 34), then the local planar representation extends to one of the configurations illustrated in Figure 39, accordingly to the edge position for tile label by 8.

In the first situation a vertex surrounded by three consecutive angles α takes place. As $\alpha > \gamma$, then $\alpha + m\delta = \pi, m \geq 1$ must be an alternate angle sum at this vertex. Taking in account the edge length, there is a contribution of an angle β surrounding such vertex leading us to a contradiction.

In the second case, the local configuration is uniquely extended to a global representation of the f-tilings $\mathcal{E}^m$, $m \geq 5$, described before.

2.2.2.2 Consider now that $\alpha + \theta_2 < \pi$, with $\theta_2 = \beta$ (see Figure 34).
Then, necessarily $\alpha + \beta + t\delta = \pi, t \geq 1$. The configuration in Figure 34, ends up, according to a choice of the edge position of tile 7, to the one illustrated below (Figure 40). It reveals one alternate angle sum at vertex v_4 containing β, γ and δ, which is impossible.



I II

Figure 39: Angle arrangement around vertex v_2

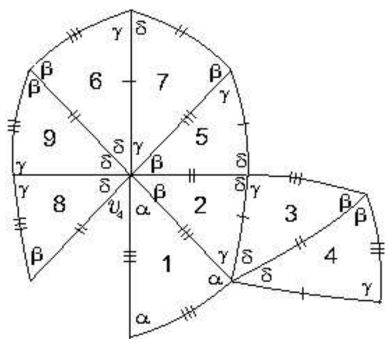


Figure 40: Planar representation.

Observe that the other choice for the position of tile 7 implies that one of the alternate angle sums at vertex v_4 is $2\beta + t\delta = \pi, t \geq 1$, which is an impossibility. $\square$

3 Symmetry Groups

Here we present the group of symmetries of the spherical f-tilings obtained: $\mathcal{D}^p, \mathcal{F}^p$ $(p \geq 4)$ and $\mathcal{E}^m$ $(m \geq 5)$. We also indicate the transitivity classes of isogonality and isohedrality.

In **Table 1** it is shown a complete list of all spherical dihedral f-tilings, whose prototiles are an equilateral triangle T_1 of angle α and a scalene triangle T_2 of angles δ, γ, β, $(\delta < \gamma < \beta)$. We have used the following notation.

- M and N are, respectively, the number of triangles congruent to T_1 and the number of triangles congruent to T_2 used in such dihedral f-tilings;

- $G(\tau)$ is the symmetry group of the f-tiling τ. The numbers of isohedrality-classes and isogonality-classes for the symmetry group are denoted, respectively, by # isoh. and # isog.;

- By C_n and D_n we denote, respectively, the cyclic group of order n and the dihedral group of order $2n$.
- $\alpha = \alpha_1^p$, $p \geq 4$ is the solution of (1.1) with $p\,\delta = \pi, \beta = \pi - \alpha$ and $\gamma = \pi - 2\alpha$;
- $\alpha = \alpha_2^p$, $p \geq 4$ is the solution of (1.1) with $p\,\delta = \pi, \beta = \pi - \alpha$ and $\gamma = \dfrac{\pi}{2} - \dfrac{\alpha}{2}$;
- $\alpha = \alpha_3^m$, $m \geq 5$ is the solution of (1.1) with $m\,\delta = \pi, \beta = \pi - \alpha$ and $\gamma = \dfrac{\pi}{2} - \dfrac{\alpha}{2} - \dfrac{\pi}{2m}$.

f-tiling	α	δ	γ	β	M	N	$G(\tau)$	# isoh.	# isog.
$\mathcal{D}^p, p \geq 4$	α_1^p	$\frac{\pi}{p}$	$\pi - 2\alpha$	$\pi - \alpha$	$4p$	$4p$	D_{2p}	3	3
$\mathcal{F}^p, p \geq 4$	α_2^p	$\frac{\pi}{p}$	$\frac{\pi}{2} - \frac{\alpha}{2}$	$\pi - \alpha$	$2p$	$4p$	$C_2 \times D_p$	2	3
$\mathcal{E}^m, m \geq 5$	α_3^m	$\frac{\pi}{m}$	$\frac{\pi}{2} - \frac{\alpha}{2} - \frac{\pi}{2m}$	$\pi - \alpha$	$4m$	$8m$	D_{2m}	4	3

Table 1: The Combinatorial Structure of the Dihedral F-Tilings of the Sphere by Equilateral and Scalene Triangles with adjacency of type II

References

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