Tree-thickness and caterpillar-thickness under girth constraints

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Abstract

We study extremal problems for decomposing a connected n-vertex graph G into trees or into caterpillars. The least size of such a decomposition is the *tree thickness* $\theta_{\mathbf{T}}(G)$ or *caterpillar thickness* $\theta_{\mathbf{C}}(G)$. If G has girth g with $g \geq 5$, then $\theta_{\mathbf{T}}(G) \leq \lfloor n/g \rfloor + 1$. We conjecture that the bound holds also for $g = 4$ and prove it when G contains no subdivision of $K_{2,3}$ with girth 4. For $\theta_{\mathbf{C}}$, we prove that $\theta_{\mathbf{C}}(G) \leq \lceil (n-2)/4 \rceil$ when G has girth at least 6 and is not a 6-cycle. For triangle-free graphs, we conjecture that $\theta_{\mathbf{C}}(G) \leq \lceil 3n/8 \rceil$ and prove it for outerplanar graphs. For 2-connected graphs with girth g, we conjecture that $\theta_{\mathbf{C}}(G) \leq \lfloor n/g \rfloor$ when $n \geq \max\{6, g^2/2\}$ and prove it for outerplanar graphs. All the bounds are sharp (sharpness in the $\lceil 3n/8 \rceil$ bound is shown only for $n \equiv 5 \mod 8$).

1 Introduction

A *decomposition* of a graph G is a set of pairwise edge-disjoint subgraphs with union G. We study decompositions of connected n-vertex graphs into the fewest trees or the fewest caterpillars, where a *caterpillar* is a tree of a restricted type, having a single path (the *spine*) that contains at least one endpoint of every edge.

The complete graph K_n decomposes into $\lceil n/2 \rceil$ paths and no fewer. Gallai famously conjectured that every connected n-vertex graph decomposes into $\lceil n/2 \rceil$ paths. Chung [1] proved that $\lceil n/2 \rceil$ trees suffice. In fact, her proof decomposes every connected n-vertex graph into $\lceil n/2 \rceil$ caterpillars of diameter at most 4. The connectedness condition is needed because $n/3$ disjoint triangles do not decompose into fewer than $2n/3$ trees. We consider only connected graphs, and we use n for the number of vertices.

Given a class $\mathbf{F}$ of graphs, the $\mathbf{F}$-*decomposition number* or $\mathbf{F}$-*thickness* of a graph G, written $\theta_{\mathbf{F}}(G)$, is the minimum size of a decomposition of G into subgraphs that lie in $\mathbf{F}$.

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We seek the maximum of $\theta_{\mathbf{F}}$ over graphs in some class $\mathbf{G}$. We can refine such problems by seeking tighter bounds over classes smaller than $\mathbf{G}$ or by restricting the family $\mathbf{F}$. Let $\theta_{\mathbf{T}}$ and $\theta_{\mathbf{C}}$ denote the tree-thickness and caterpillar-thickness, respectively.

For connected graphs, the maximum tree-thickness $\lceil n/2 \rceil$ is attained by K_n. Forbidding triangles excludes this example. The *girth* of a graph is the length of a shortest cycle. For $g \geq 5$, we prove in Theorem 5 that $\theta_{\mathbf{T}}(G) \leq \lfloor n/g \rfloor + 1$ when G is connected and has girth g; this is sharp for all n (Example 1). The conclusion also holds when $g = 4$ among graphs containing no subdivision of $K_{2,3}$ with girth 4. We conjecture that $\theta_{\mathbf{T}}(G) \leq \lfloor n/g \rfloor + 1$ in fact holds for all connected graphs with girth 4.

We next study caterpillar-thickness. Always $\theta_{\mathbf{T}}(G) \leq \theta_{\mathbf{C}}(G) \leq \lceil n/2 \rceil$ (by Chung's proof), with equality when $n \equiv 4 \mod 6$ for special graphs with triangles (Example 1). Forbidding triangles reduces the upper bound. We prove that girth at least 6 forces $\theta_{\mathbf{C}}(G) \leq \lceil (n-2)/4 \rceil$ when G is not a 6-cycle (Theorem 6), with equality for a special tree (Example 3).

Since $\theta_{\mathbf{C}}(G) = \lceil (n-2)/4 \rceil$ holds for special trees, the upper bound cannot be further reduced for general n-vertex graphs by enlarging the girth beyond 6. It remains to determine the best bounds for girth 4 and girth 5 and to determine the best bounds for larger girth when trees are forbidden by restricting to 2-connected graphs.

For connected graphs with girth 4, we conjecture that $\theta_{\mathbf{C}}(G) \leq \lceil 3n/8 \rceil$ for all n, and that for $n > 8$ the bound can be improved by 1 when $n \not\equiv 5 \mod 8$. The graphs in Example 2 demonstrate sharpness. In Theorem 8, we prove that $\theta_{\mathbf{C}}(G) \leq \lceil 3n/8 \rceil$ for triangle-free outerplanar graphs. A similar construction and conjecture exists for girth 5, with $\lceil 3n/10 \rceil$ as the uniform upper bound. The proof of this for outerplanar graphs is very similar to that of Theorem 8, and we omit it (see [2]).

For 2-connected graphs with girth g, we conjecture that $\theta_{\mathbf{C}}(G) \leq \lfloor n/g \rfloor$ if $n \geq g^2/2$; the graphs in Example 4 show that this is sharp. In Theorem 7, we prove the bound for outerplanar graphs. In particular, if G is a 2-connected n-vertex outerplanar graph with girth g, then the maximum possible value of $\theta_{\mathbf{C}}(G)$ is $\lfloor n/g \rfloor$ for $n \geq g^2/2$, is $\left\lfloor \frac{n-g}{g-2} \right\rfloor$ for $3g - 4 \leq n \leq g^2/2$, and is 2 for $g \leq n \leq 3g - 4$.

2 Lower Bound Constructions

In this section we present examples showing that the bounds in our later theorems are sharp. A *cactus* is a connected graph in which every edge appears in at most one cycle; equivalently, every block is an edge or a cycle. Every cactus is outerplanar. The extremal graphs presented in Example 1 are cacti and are not 2-connected.

Example 1 *Cacti with large tree-thickness.* For $k \geq 1$, let $H_{k,g}$ denote the cactus with $kg + 1$ vertices formed from k disjoint g-cycles by adding one vertex x having one neighbor in each cycle (see Figure 1). The cut-edges in $H_{k,g}$ imply that only one tree in a decomposition can extend out from each cycle. However, two trees must be used within each cycle. Hence at least k trees are confined to the cycles, and at least one more tree must be used. There is such a decomposition, so $\theta_{\mathbf{T}}(H_{k,g}) = k+1 = (n-1)/g+1$. When $n \not\equiv 1 \mod g$, we obtain a graph with $\theta_{\mathbf{T}}(G) = \lfloor (n-1)/g \rfloor + 1$ by adding pendant edges at x in $H_{k,g}$, where $k = \lfloor (n-1)/g \rfloor$.

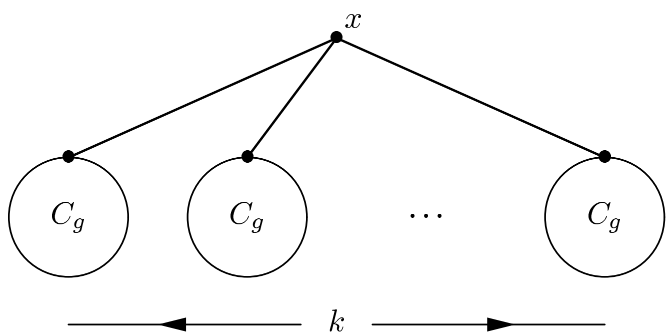

Figure 1: The graph $H_{k,g}$

When $g \mid n$ there is a better construction. Starting with n/g disjoint g-cycles $Q_1, \ldots, Q_{n/g}$, add edges to make one vertex of Q_1 adjacent to one vertex in each other Q_i; this is again a cactus. Again every tree decomposition has a member entirely contained in each of $Q_2, \ldots, Q_{n/g}$, but the graph that remains when the edges of these trees are deleted still has a cycle. Hence the tree-thickness is $n/g+1$. This yields the uniform formula $\lfloor n/g \rfloor + 1$, which is also optimal for $n < g$ among graphs with girth at least g. □

Example 2 *Cacti with large caterpillar-thickness.* The decomposition of $H_{k,g}$ in Example 1 uses trees that are not caterpillars (when $k \geq 3$). A caterpillar in $H_{k,g}$ has edges in at most two of the cycles, because a caterpillar cannot have three paths of length 2 with a common endpoint. Since only k paths can start along and depart from a cycle, the best we can do is save $\lfloor k/2 \rfloor$ by combining into pairs the paths that leave the cycles. Thus $\theta_{\mathbf{C}}(H_{k,g}) = 2k - \lfloor k/2 \rfloor = \lceil 3k/2 \rceil$. (Note that $\theta_{\mathbf{C}}(H_{k,3}) = n/2$ when $n \equiv 4 \mod 6$. For such n, the maximum value of caterpillar-thickness over n-vertex graphs is achieved not only by K_n, but also by a cactus.)

To improve the construction for other congruence classes of n, form $H'_{k,g}$ by appending a path of two edges to $H_{k,g}$ at x. There are $2k+1$ paths needed to decompose the $k+1$ components of $H'_{k,g} - x$, only one can extend from each component, and they can at best combine in pairs, so $\theta_{\mathbf{C}}(H'_{k,g}) = 2k+1-\lfloor (k+1)/2 \rfloor = \lceil (3k+1)/2 \rceil$.

Let $n = 2jg + r$, where j and r are integers with $j \geq 1$ and $1 \leq r \leq 2g$. When $r \in \{1, g+1\}$, let $G = H_{2j,g}$. When $r = 3$, let $G = H'_{2j,g}$. For all other n, append a leaf to the construction for $n-1$, without increasing $\theta_{\mathbf{C}}$. With this construction, $\theta_{\mathbf{C}}(G)$ is $3j$ when $1 \leq r \leq 2$, is $3j+1$ when $3 \leq r \leq g$, and is $3j+2$ when $g+1 \leq r \leq 2g$.

When $g = 4$ and $n > 8$, these cases combine to yield $\theta_{\mathbf{C}}(G) = \lceil 3n/8 \rceil$ when $n \equiv 5 \mod 8$ and $\theta_{\mathbf{C}}(G) = \lceil 3n/8 \rceil - 1$ otherwise. □

Example 3 When $g > 6$, a special tree has caterpillar-thickness greater than $H_{k,g}$. Form T_n by subdividing $\lfloor (n-1)/2 \rfloor$ edges in the star $K_{1,\lceil (n-1)/2 \rceil}$ (each subdivided once); this yields n vertices. At most two of the edges not containing the center can lie in a single caterpillar, so $\lceil \lfloor (n-1)/2 \rfloor /2 \rceil$ caterpillars are needed, and this many suffice. For $n > 2$, we obtain $\theta_{\mathbf{C}}(T_n) = \lceil (n-2)/4 \rceil$. For $g = 6$, this construction improves the lower bound from Example 2 in some congruence classes; for $g > 6$, it improves it for all n. □

The family $H_{k,g}$ can be excluded by restricting to 2-connected graphs, but the tree-thickness can still be almost as large as for $H_{k,g}$. Again the graphs are outerplanar.

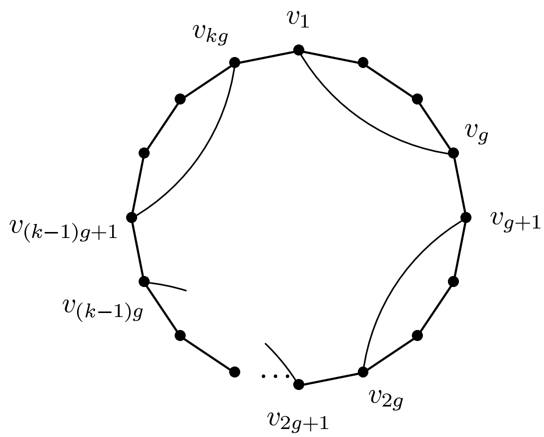

Figure 2: The graph $J_{k,g}$

Example 4 For $k \geq g/2$, let $J_{k,g}$ denote the graph formed from the n-vertex cycle C_n, where $n = kg$ and the vertices are $v_1, \dots, v_n$ in order, by adding chords of the form $v_{gi-g+1}v_{gi}$ for $1 \leq i \leq k$ (see Figure 2). Note that $J_{k,g}$ has girth g.

Each chord forms a cycle, which requires two trees in the decomposition. Only one of those two trees can continue on to the next higher cycle in the direction of increasing indices, so a new tree must start within that cycle. In traversing the full outer cycle, at least k trees must be started. Hence $\theta_{\mathbf{T}}(J_{k,g}) \geq k = n/g$, and equality holds using n/g paths.

When n is not a multiple of g, we can start with a cycle of length n and insert $\lfloor n/g \rfloor$ chords in this way while maintaining girth g (if $n \geq g \lceil g/2 \rceil$), so for $n \geq g^2/2$ we obtain examples with tree-thickness (and caterpillar-thickness and path-thickness) $\lfloor n/g \rfloor$.

When $g \leq n < g^2/2$ (or $k < g/2$), the cycle on the "inside" is too short. Instead of inserting all k chords, insert only the first m. The cycle through these chords and the remaining higher-indexed vertices has length $2m + (n - mg)$. We require $2m + (n - mg) \geq g$ and set $m = \left\lfloor \frac{n-g}{g-2} \right\rfloor$. As above, decomposing G needs m trees, so $\theta_{\mathbf{T}}(G) = \theta_{\mathbf{C}}(G) = \left\lfloor \frac{n-g}{g-2} \right\rfloor$. When $g \leq n < 3n - 4$, the existence of one chord yields $\theta_{\mathbf{T}}(G) = \theta_{\mathbf{C}}(G) = 2$. □

3 Thickness Bounds for General Graphs

We write $G[A]$ for the subgraph of G induced by a vertex set A. The tree-thickness arguments for connected graphs with girth at least 5 and for connected graphs with girth at least 4 that avoid subdivisions of $K_{2,3}$ are essentially the same, so we combine them.

Theorem 5 *Let G be an n-vertex connected graph. If* $\text{girth}(G) \geq g \geq 5$*, or if $g = 4$ and G contains no subdivision of $K_{2,3}$ with girth 4, then $\theta_{\mathbf{T}}(G) \leq \lfloor n/g \rfloor + 1$, and this is sharp.*

Proof. Sharpness was shown in Example 1. For the upper bound, we use induction on n. If $n < g$, then G has no cycle and is a tree itself. If $n = g$, then G is a cycle and decomposes into two trees. For the induction step, consider $n > g$. We may assume that G is not a tree, since then $\theta_{\mathbf{T}}(G) = 1$.

Let P be a longest path in G, with vertices $v_1, \ldots, v_m$ in order. Since $\text{girth}(G) \geq g$, we have $m \geq g$. Let $R = \{v_1, \ldots, v_g\}$. No two vertices in R have more than one common neighbor outside R, because this would create a subdivision of $K_{2,3}$ containing a 4-cycle (vertices in R with a common neighbor cannot be consecutive on R, since $\text{girth}(G) \geq 4$). The same observation holds for $R - \{v_g\}$.

Let T be a spanning tree of G that contains P. For $1 \leq i \leq m$, let S_i be the set of vertices outside P whose path to $V(P)$ in T arrives at v_i. Let $S = S_1 \cup \cdots \cup S_g$; note that $S_1 = \varnothing$. Among all the spanning trees that contain P, let T be one that minimizes $|S|$. With this choice of T, no vertex in S has a neighbor outside $S \cup R$.

Case 1: $S \neq \varnothing$.

Let $A = S \cup R - \{v_g\}$. Note that $G - A$ is connected. Also $|V(G - A)| \leq n - g$, since $S \neq \varnothing$. By the induction hypothesis, $\theta_{\mathbf{T}}(G - A) \leq \lfloor (n-g)/g \rfloor + 1 = \lfloor n/g \rfloor$. Call the trees in such a decomposition the "old" trees. We will incorporate the edges incident to A by adding some edges to old trees and creating one additional tree for the rest.

The key observation is that $G[A]$ is a forest. If there is a cycle C among the vertices of A, then it has at least g vertices. Combining a path around C with a shortest path from $V(C)$ to v_g in $G[A \cup \{v_g\}]$ contradicts the choice of P as a longest path in G.

Let $W_1, \ldots, W_t$ be the components of $G[A]$. One component contains all of $v_1, \ldots, v_{g-1}$ and $S_1, \ldots, S_{g-1}$, and the others form $G[S_g]$. By the choice of T, v_g is the only vertex outside A having neighbors in S, and v_g has a neighbor in each W_i. Use one such edge to each W_i along with $G[A]$ to form a new tree T' for the decomposition. Add the other edges from v_g to S to the tree containing $v_g v_{g+1}$.

We have now assigned all edges of G to trees in the decomposition except those from $v_1, \ldots, v_{g-1}$ to neighbors outside A. We can add these edges to T' unless two of them reach a common vertex outside A. Since G has girth at least g, the only vertices in $v_1, \ldots, v_{g-1}$ that can have such a common neighbor are v_1 and v_{g-1}. We have observed that they can have only one such common neighbor; call it x. If $x = v_g$, then we have already put one of $\{v_1 x, v_{g-1} x\}$ into T' and the other into an old tree containing an edge incident to x. If $x \neq v_g$, then we can do the same, since x has no other neighbor in the component of T' containing v_{g-1}, and v_1 has no other neighbor in the old tree.

Case 2: $S = \varnothing$.

Let $A = \{v_1, \ldots, v_g\}$. Note that $G - A$ is connected, since $S = \varnothing$; also, it has $n - g$ vertices. By the induction hypothesis, $\theta_{\mathbf{F}}(G - A) \le \lfloor n/g \rfloor$. Call the trees in such a decomposition "old" trees. An additional tree T' will contain all other edges incident to the path $G[A]$, with a few possible exceptions.

Since girth $(G) \ge g$, the only pairs in A that can have common neighbors (and only one for each pair, as noted earlier) are $\{v_1, v_g\}, \{v_1, v_{g-1}\}, \{v_2, v_g\}$. Let x, y, z denote their possible common neighbors, respectively.

If the edge v_1v_g exists, then actually $y = v_g$ and $z = v_1$, and x does not exist. In this case we add $v_{g-1}v_g$ and v_gv_{g+1} to an old tree and put all other edges incident to A in T'.

If $v_1v_g \notin E(G)$, then we can add all of $\{xv_1, yv_{g-1}, zv_2\}$ that exist into old trees and put all other edges incident to A into T'. □

As mentioned in the introduction, we conjecture that $\theta_{\mathbf{T}}(G) \le \lfloor n/4 \rfloor + 1$ whenever G has girth 4. The method in Chung's proof [1] can be strengthened to improve the upper bound from $\lceil n/2 \rceil$ to $\lceil n/3 \rceil$ when G does not have a subgraph isomorphic to the graph obtained from $K_{4,3}$ by deleting one edge. Since this argument is rather technical and does not enlarge the family where the conjecture is known to hold, we omit it; the details appear in [2].

When the girth is at least 6, an argument similar to that of Theorem 5 yields a tight upper bound for caterpillar thickness in general graphs. The bound is weaker than in Theorem 5 due to the restriction to caterpillars in the decomposition.

Theorem 6 *If G is an n-vertex graph with girth at least 6, then $\theta_{\mathbf{C}}(G) \le \lceil (n-2)/4 \rceil$ (unless $G = C_6$), and this is sharp.*

Proof. We observed in Example 3 that the bound is achieved by the tree obtained by subdividing $\lfloor (n-1)/2 \rfloor$ edges of a star that has $\lceil (n-1)/2 \rceil$ edges.

For the upper bound, we use induction on n. Every graph with at most six vertices having girth at least 6 is a caterpillar except the 6-cycle. Also, every connected edge-disjoint union of a 6-cycle and a caterpillar decomposes into two caterpillars. Hence it suffices to show for $n \ge 7$ that $V(G)$ contains a set A of size at least 4 such that $G - A$ is connected and the set of edges incident to A forms a caterpillar.

Let P be a longest path in G, with vertices $v_1, \ldots, v_m$ in order. The girth requirement yields $m \ge 6$. Let $R = \{v_1, v_2, v_3\}$. No vertex has two neighbors in R.

Let T be a spanning tree of G that contains P. For $1 \le i \le m$, let S_i be the set of vertices outside P whose path to $V(P)$ in T arrives at v_i (note that $S_1 = \varnothing$). Let $S = S_2 \cup S_3 \cup S_4$. Among all the spanning trees that contain P, consider those that minimize $|S|$, and among these choose T to maximize $|S_2 \cup S_3|$.

With this choice of T, no vertex in S has a neighbor outside $S \cup R \cup \{v_4\}$. Furthermore, every component of $G[S_3]$ is a star whose center is adjacent to v_3, and S_2 is an independent set. If $|S_2 \cup S_3| \ge 2$, then let A consist of v_1, v_2, S_2, and the vertices in a largest component of $G[S_3]$, or the vertices in two components of $G[S_3]$ if S_3 is independent and $S_2 = \varnothing$.

Except for the edges from S_3 to v_3, only v_1 and v_2 have neighbors outside A, and no two vertices of A have common neighbors. Thus A has the desired properties.

If $|S_2 \cup S_3| = 1$, then let $A = R \cup S_2 \cup S_3$. Again only vertices on the path formed by R have neighbors outside A, so A has the desired properties.

If $S_2 \cup S_3 = \varnothing$ and $S_4 \neq \varnothing$, consider a component H of $G[S_4]$; H is a tree whose vertices have distance at most 3 from v_4. If H contains a vertex with distance 3 from v_4, then H is a path, by the choice of T. Otherwise, H is a star with center adjacent to v_4. In either case, the choices of P and T prevent further edges from $V(H)$ to R. Let $A = V(H) \cup R$. Now $A \cup \{v_4\}$ induces a caterpillar, and the only edges leaving A are incident to R and reach distinct neighbors. Thus A has the desired properties. □

4 Caterpillar Thickness of Outerplanar Graphs

In studying caterpillar thickness for graphs with girth 4 and for 2-connected graphs, our proofs require outerplanarity, but we conjecture that the same bounds hold without that restriction. We first solve the extremal problem for 2-connected outerplanar graphs.

Theorem 7 *If G is a 2-connected n-vertex outplanar graph with girth at least g, then $\theta_{\mathbf{C}}(G)$ is bounded as given below, and all these bounds are sharp.*

$$\theta_{\mathbf{C}}(G) \leq \begin{cases} 2 & \text{if } g \leq n \leq 3g-4, \\ \left\lfloor \frac{n-g}{g-2} \right\rfloor & \text{if } 3g-4 \leq n \leq g^2/2, \\ \lfloor n/g \rfloor & \text{if } n \geq g^2/2 \text{ (except } n=5 \text{ when } g=3). \end{cases}$$

Proof. In Example 4, we presented 2-connected outerplanar graphs with girth g having tree-thickness and caterpillar-thickness as specified above. Note that $g^2/2 < 3g-4$ when $g = 3$; the middle "range" is empty.

For the upper bound, let C be the outer boundary in an outerplanar embedding of G. Since G is 2-connected, C is a cycle with vertices $v_1, \ldots, v_n$ in order, and G has no other vertices. A chord v_iv_j of C is *minimal* if one of the v_i, v_j-paths on C has no other endpoint of a chord as an internal vertex. Let m be the number of minimal chords.

If $m \leq 1$, then G is a cycle with at most one chord, and two caterpillars (paths) suffice. Hence we consider $m \geq 2$. Because G has girth at least g, the computation in Example 4 yields $m \leq \left\lfloor \frac{n-g}{g-2} \right\rfloor$. Therefore, to complete the proof when $n \leq g^2/2$ it suffices to show that $\theta_{\mathbf{C}}(G) \leq m$ always. For $n > g^2/2$, we will prove (inductively) that $\theta_{\mathbf{C}}(G) \leq n/g$.

Bound 1: *If $m \geq 2$, then $\theta_{\mathbf{C}}(G) \leq m$.* Decompose C into m "boundary paths" $P_1, \ldots, P_m$ such that the endpoints of each path are internal to the paths generated by the chords. In particular, if v_rv_s is the ith minimal chord, then some internal vertex of the path from v_r to v_s along C is the end of P_i and the beginning of P_{i+1}. By the minimality of v_rv_s, no chord is incident to the common vertex of P_i and P_{i+1}. We use $P_1, \ldots, P_m$ as the spines of the caterpillars in the decomposition.

Each chord of C joins vertices from two boundary paths; we assign it to one of these two paths (we have observed that each end is incident to only one boundary path). Since every chord incident to P_i is incident at its other end to exactly one other boundary path, it suffices to show that the chords joining P_i and P_j can be distributed to those two paths in such a way that the chords assigned to each have distinct endpoints in the other.

Let H be a graph consisting of paths $\langle u_1, \ldots, u_r\rangle$ and $\langle w_1, \ldots, w_s\rangle$ joined by noncrossing chords of the form $u_i w_j$. "Noncrossing" means that the chords obey a linear order L such that the indices of the vertices from each path are nondecreasing. The subgraph of G consisting of P_i and P_j and the chords joining them has this form.

Process the chords in H in the order L. If the next chord shares an endpoint with the current chord, assign it to the path containing the shared vertex; otherwise assign it to either path (this case covers the initial chord). If two vertices on one path have chords to a common neighbor on the other path, then the second chord among these two is assigned to the other path. Hence the chords assigned to each path have distinct endpoints on the other path.

Bound 2: *If $n \geq 2g$, then $\theta_{\mathbf{C}}(G) \leq n/g$.* We prove inductively that G decomposes into $\lfloor n/g \rfloor$ caterpillars whose spines cover $E(C)$. Two such caterpillars suffice when $m \leq 1$.

If no two minimal chords share an endpoint, then the minimal chords lie on m disjoint cycles, and $n \geq mg$. If $n \leq g^2/2$, then $m \leq (n-g)/(g-2) \leq n/g$. In either case, the construction for Bound 1 suffices, since the union of its spines is C. Hence we may assume that $n > g^2/2$ and that some two minimal chords share an endpoint.

If $m = 2$, then all chords have a common endpoint, and G is the edge-disjoint union of a star and a path, each of which is a caterpillar. Two edges of the star lie on C and form the spine of this caterpillar; the path is the remainder of C. Hence we may assume that $m > 2$.

Let $v_i v_j$ and $v_j v_k$ be two minimal chords with a common endpoint; we may assume that $i < j < k$. Let P be the v_i, v_k-path through v_j along C. Form a smaller 2-connected outerplanar graph G' as follows: If $g = 3$, then delete $V(P) - \{v_i, v_k\}$ from G and add the edge $v_i v_k$ (if not already present); if $g \geq 4$, then delete $V(P) - \{v_i, v_j, v_k\}$. In the first case, we deleted $k - i - 1$ vertices; in the second, we deleted $k - i - 2$.

Since $k - i - 1 \geq 2g - 3 \geq g$ if $g = 3$ and $k - i - 2 \geq 2g - 4 \geq g$ if $g \geq 4$, there are at most $n - g$ vertices in G'. We can apply the induction hypothesis unless G' has fewer than $2g$ vertices. If so, then G' is a cycle. Since G has at least three minimal chords, this case arises only if $g = 3$ and G is the union of a spanning cycle and a triangle. Such a graph decomposes into two paths; all edges lie along the spines.

Now the induction hypothesis provides a decomposition of G' into at most $\lfloor n/g \rfloor - 1$ caterpillars whose spines cover the outer edges. When $g \geq 4$, it suffices to add P to this decomposition. When $g = 3$ and $v_i v_k \notin E(G)$, the edge $v_i v_k$ lies on the outer face in G' and hence is on the spine of its caterpillar T in the decomposition of G'. Replacing $v_i v_j$ with $v_i v_j$ and $v_j v_k$ in T yields again a caterpillar. The desired decomposition is now completed by adding the caterpillar consisting of P and all other edges incident to v_j except $v_i v_j$ and $v_k v_j$.

Finally, suppose that $g = 3$ and $v_iv_k \in E(G)$. The edges $v_{i-1}v_i$, v_iv_k, and v_kv_{k+1} all lie on spines in the decomposition of G'. If they are not in the same caterpillar in the decomposition, then we add v_iv_j and v_jv_k to two different caterpillars and add P as a new caterpillar. If these three edges are in the same caterpillar T, then we break the spine of T at v_i; the piece containing $v_{i-1}v_i$ continues along P to v_j and then directly to v_k and v_{k-1}, while the piece containing $v_{k+1}v_k$ and v_kv_i continues directly to v_j and then along P to v_{k-1}. All the edges of G that are not in G' become spine edges in the resulting decomposition. □

Our final result is the most difficult. We prove the uniform upper bound of $\lceil 3n/8 \rceil$ for the caterpillar-thickness of triangle-free outerplanar graphs. As mentioned in the introduction, we believe that the bound improves to $\lceil 3n/8 \rceil - 1$ when $n \not\equiv 5 \mod 8$ and $n > 8$. Obtaining this improvement would require extensive case analysis, so we omit it. One source of difficulty is that the savings does not occur in most congruence classes until n exceeds 8; this complicates the base case. Another is that the optimal formula (with or without the floor or ceiling function) is not uniform across congruence classes. Hence we are content with a uniform formula for the bound that is sharp on one congruence class.

A *block* in a graph G is a maximal subgraph that has no cut-vertex. A *leaf block* in G is a block that contains only one cut-vertex of G. A *penultimate block* in G is a leaf block in the graph G' obtained by deleting the non-cut-vertices of leaf blocks in G.

Theorem 8 *If G is a connected triangle-free outerplanar graph with n vertices, then $\theta_{\mathbf{C}}(G) \le \lceil 3n/8 \rceil$. This bound is sharp when $n \equiv 5 \mod 8$ and is always within one of sharpness (except for $n = 3$).*

Proof. In Example 2 we presented cacti that demonstrate the sharpness results.

For the upper bound, we consider a counterexample G with fewest vertices, n. We will derive structural properties of G that eventually forbid its existence.

A subgraph H is *deletable* if it has a vertex subset S such that $G - E(H) - S$ is connected and $\theta_{\mathbf{C}}(H) \le \lfloor 3|S|/8 \rfloor$. With $a = 3|S|/8$, we have $\lceil 3(n - |S|)/8 \rceil + \theta_{\mathbf{C}}(H) \le \lceil 3n/8 - a \rceil + \lfloor a \rfloor \le \lceil 3n/8 \rceil$. Therefore, a minimal counterexample contains no deletable subgraph. When S is a set of at least three vertices, and the edges incident to S form a caterpillar H, and $G - S$ is connected, then H is deletable, so we also say that the vertex set S is deletable.

For 2-connected outerplanar graphs, Theorem 7 already provides an upper bound of $\lfloor n/4 \rfloor$, which is always at most $\lceil 3n/8 \rceil$. Thus G is not 2-connected and has at least 2 blocks. In an embedding of G with all vertices on the unbounded face, the vertices of a leaf block occur consecutively.

Step 1: *Every leaf block is an edge or a 4-cycle.* Suppose that a leaf block B has at least five vertices. Let v_1 be the cut-vertex of G in B, and let $v_1, \ldots, v_k$ be the vertices of B in order on the unbounded face. If $N(v_{i-1}) \cap N(v_{i+1}) = \{v_i\}$ for some i with $3 \le i \le k-1$, then the girth condition implies that the edges incident to S form a caterpillar, where $S = \{v_{i-1}, v_i, v_{i+1}\}$. Also, $G - S$ is connected, so S is deletable.

If $\{v_2, v_3, v_4\}$ is not deletable, then there exists $v_j \in N(v_2) \cap N(v_4) - \{v_3\}$. If $j \neq 5$, then $\{v_3, v_4, v_5\}$ is deletable. If $j = 5$, then girth 4 forces $k > 5$, and now $\{v_4, v_5, v_6\}$ is deletable.

Step 2: *Every vertex in at most one non-leaf block lies in at most one leaf block.* Let v be a vertex in leaf blocks B and B'. If B is a 4-cycle, with vertices v, w, x, y in order, and z is a neighbor of v in B' (whether B' is an edge or a 4-cycle), then $\{x, y, z\}$ is deletable (the edges incident to $\{x, y, z\}$ form a path). If three leaf blocks that are edges share v, then their leaves form a deletable triple.

Thus at most two leaf blocks can contain v, and if so both are edges. This makes it impossible that every block is a leaf block, since then $n = 3$ and G is a path. If leaf blocks B and B' are both edges containing v, and v is in at most one non-leaf block, then $V(B \cup B')$ is now deletable.

Step 3: *G has no "spear".* Define a *spear* to be a subgraph H consisting of two leaf blocks and a nontrivial path P connecting them, such that only the (possibly equal) vertices w and w' of P in the same penultimate block B^* have neighbors outside H, and $G - S$ is connected, where $S = V(H) - \{w'\}$.

If the leaf blocks B and B' are edges, then H is a path and $|S| \geq 3$, so H is deletable.

If B is a 4-cycle, then let the vertices be v, x, y, z in cyclic order, with v the cut-vertex of G. If B' is an edge, then H decomposes into the edge xv of B and a path. Thus H is deletable if $|S| \geq 6$, which fails only if P has length 1 and $w \neq w'$. In that case, delete only the path $H - xv$, with marked set S' consisting of $\{y, z\}$ and the leaf of B'.

If B and B' are 4-cycles, then H decomposes into the edge xv, one edge of B', and a path. Thus H is deletable if $|S| \geq 8$, which fails only if P has length 1 and $w \neq w'$. In that case, we may assume by symmetry that $w' = v$. Now delete only $H - xw'$ (a path plus an edge of B'), with marked 6-set S' consisting of $V(B') \cup \{y, z\}$.

Step 4: *Every penultimate block is an edge.* We observed that not all blocks are leaf blocks. Let B^* be a penultimate block. Since B^* is not a leaf block, it has a vertex v that lies in at least one leaf block and in no other non-leaf block. By Step 2, v belongs to exactly one leaf block; call it B.

Suppose that B^* is not an edge. Among the two neighbors of v along the unbounded face of B^*, we may choose x to avoid the only vertex that B^* can share with another non-leaf block. If x lies in a leaf block B', then $B \cup B' \cup xv$ is a spear. Otherwise, we find a deletable triple. It is $V(B) \cup \{x\}$ if B is an edge, and it is x together with two adjacent vertices of B other than v if B is a 4-cycle.

Step 5: *Two penultimate blocks cannot intersect.* Suppose that B_1^* and B_2^* are penultimate blocks with a common vertex w. By Step 3, each B_i^* is an edge; let v_i be the endpoint opposite w. By Step 2, each v_i lies in one leaf block B_i. Now $B_1 \cup B_2 \cup B_1^* \cup B_2^*$ is a spear.

Step 6: *A peripheral penultimate block intersects only one other non-leaf block.* A *chain* of blocks is a list of distinct blocks in which any two consecutive blocks intersect and the shared cut-vertices are all distinct. The leaf block and penultimate block at the beginning or end of a longest such chain are *peripheral* such blocks. Choose B and

B^* to start a longest chain C, so B and B^* are a peripheral leaf block and peripheral penultimate block.

By Step 3, B^* is an edge; as usual, let w be the vertex it shares with a non-leaf block. If there are two such blocks, then one of them is not in the chain of blocks starting with B and B^*. By Step 5 it is not a penultimate block. It therefore has another vertex in a non-leaf block. Thus it yields a chain of at least three blocks that can replace B and B^* in C to form a longer chain, contradicting the choice of C.

Step 7: *There is no minimal counterexample.* Let B be a peripheral leaf block, sharing v with a penultimate block B^*. Let w be the other vertex of B^*. By Step 5, w lies in only one other non-leaf block, B_0.

If w lies in a leaf block, B', then $B \cup B^* \cup B'$ is a spear, since w lies in only one other block. Hence w lies only in B_0 and B^*.

Let x be a neighbor of w in B_0, along the unbounded face, and let $S = V(B) \cup \{w, x\}$. Note that either (1) $|S| = 6$ (if B is a 4-cycle) and the subgraph of edges incident to S decomposes into two caterpillars, or (2) $|S| = 4$ (if B is an edge) and the subgraph of edges incident to S decomposes into one caterpillar. If a leaf block or a penultimate block is attached at x, then we obtain a spear. If a longer chain is attached at x, then it contradicts B' being a leaf block. Hence nothing is attached at x. Now $G-S$ is connected, and the subgraph is deletable. □

When the girth is at least 5, a proof similar to that of Theorem 8 yields the weaker bound with 8 replaced by 10 in the formulas of Theorem 8. Since the techniques are the same, we omit the details, which can be found in [2].

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