A character on the quasi-symmetric functions coming from multiple zeta values

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Abstract

We define a homomorphism ζ from the algebra of quasi-symmetric functions to the reals which involves the Euler constant and multiple zeta values. Besides advancing the study of multiple zeta values, the homomorphism ζ appears in connection with two Hirzebruch genera of almost complex manifolds: the Γ-genus (related to mirror symmetry) and the $\hat{\Gamma}$-genus (related to an S^1-equivariant Euler class). We decompose ζ into its even and odd factors in the sense of Aguiar, Bergeron, and Sottille, and demonstrate the usefulness of this decomposition in computing ζ on the subalgebra of symmetric functions (which suffices for computations of the Γ- and $\hat{\Gamma}$-genera).

1 Introduction

Let $x_1, x_2, \dots$ be a countably infinite sequence of indeterminates, each having degree 1, and let $\mathfrak{P} \subset \mathbf{R}[[x_1, x_2, \dots]]$ be the set of formal power series in the x_i having bounded degree. Then $\mathfrak{P}$ is a graded algebra over the reals. An element $f \in \mathfrak{P}$ is called a symmetric function if

$$\text{coefficient of } x_{n_1}^{i_1} x_{n_2}^{i_2} \cdots x_{n_k}^{i_k} \text{ in } f = \text{coefficient of } x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} \text{ in } f \tag{1}$$

for any k-tuple $(n_1, \dots, n_k)$ of distinct positive integers, and f is called a quasi-symmetric function if equation (1) holds whenever

$$n_1 < n_2 < \cdots < n_k.$$

The vector spaces Sym and QSym of symmetric and quasi-symmetric functions respectively are both subalgebras of $\mathfrak{P}$, with Sym $\subset$ QSym. Of course Sym is a familiar object, for which the first chapter of Macdonald [20] is a convenient reference. The algebra QSym was introduced by Gessel [9], and in recent years has become increasingly important in combinatorics; see, e.g., [24].

A vector space basis for QSym is given by the monomial quasi-symmetric functions, which are indexed by compositions (ordered partitions). The monomial quasi-symmetric function M_I corresponding to the composition $I = (i_1, i_2, \dots, i_k)$ is

$$M_I = \sum_{n_1 < n_2 < \cdots < n_k} x_{n_1}^{i_1} x_{n_2}^{i_2} \cdots x_{n_k}^{i_k}. \tag{2}$$

For a composition $I = (i_1, \dots, i_k)$ we write $\ell(I) = k$ for the number of parts of I, and $|I| = i_1 + \cdots + i_k$ for the sum of the parts of I. If $|I| = n$, we say I is a composition of n and write $I \vDash n$. If I is a composition, there is a partition $\pi(I)$ given by forgetting the ordering: the monomial symmetric function m_λ corresponding to a partition λ is given by

$$m_\lambda = \sum_{\pi(I)=\lambda} M_I,$$

and the monomial symmetric functions generate Sym as a vector space. For a partition λ, we use the notations $\ell(\lambda)$ and $|\lambda|$ in the same way as for compositions; if $|\lambda| = n$ we say λ is a partition of n and write $\lambda \vdash n$.

For a composition $(i_1, i_2, \dots, i_k)$ with $i_1 > 1$, the corresponding multiple zeta value is the k-fold infinite series

$$\zeta(i_1, i_2, \dots, i_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}. \tag{3}$$

Multiple zeta values were introduced in [12] and [25], but the case $k = 2$ actually goes back to Euler [7]. They have been studied extensively in recent decades, and have appeared in a surprising number of contexts, including knot theory and particle physics. Surveys include [4, 5, 15].

The multiple zeta value (3) can be obtained from the monomial quasi-symmetric function $M_{(i_k, i_{k-1}, \dots, i_1)}$ by sending x_n to $\frac{1}{n}$, but the series won't converge unless $i_1 > 1$. If we let QSym^0 be the subspace of QSym generated by the monomial quasi-symmetric functions M_I with the last part of I greater than 1, then it turns out that QSym^0 is a subalgebra of QSym, and we have a homomorphism $\zeta : \mathrm{QSym}^0 \to \mathbf{R}$ whose images are the multiple zeta values.

In fact (as we explain in the next section) $\mathrm{QSym} = \mathrm{QSym}^0[M_{(1)}]$, so to extend ζ to a homomorphism defined on all of QSym it suffices to define $\zeta(M_{(1)})$. As the author noted in [13], setting $\zeta(M_{(1)}) = \gamma$ (the Euler-Mascheroni constant) is a fruitful choice. If

$$H(t) = 1 + h_1 t + h_2 t^2 + \cdots \in \mathrm{Sym}[[t]]$$

is the generating function for the complete symmetric functions

$$h_n = \sum_{I \vDash n} M_I = \sum_{\lambda \vdash n} m_\lambda,$$

then setting $\zeta(M_{(1)}) = \gamma$ implies [13, Theorem 5.1]

$$\zeta(H(t)) = \Gamma(1-t), \tag{4}$$

where Γ is the usual gamma function. This is equivalent to

$$\zeta(E(t)) = \frac{1}{\Gamma(1+t)}, \tag{5}$$

where

$$E(t) = 1 + e_1 t + e_2 t^2 + \cdots$$

is the generating function of the elementary symmetric functions $e_j = M_{(1^j)}$ (we write 1^j for a string of j 1's). The identity (4) was a key step in the proof in [13] that

$$\sum_{n,m\geq 1} \zeta(n+1, 1^{m-1}) s^n t^m = 1 - \exp\left(\sum_{i\geq 2} \zeta(i) \frac{t^i + s^i - (t+s)^i}{i}\right).$$

This latter identity (proved by a different method in [3]) is interesting since it shows that any multiple zeta value of the form $\zeta(n+1, 1, \dots, 1)$ can be expressed as a polynomial with rational coefficients in the ordinary zeta values $\zeta(i)$.

Libgober [17] showed that the Γ-genus appears in formulas that relate Chern classes of certain manifolds to the periods of their mirrors. The Γ-genus is the Hirzebruch [11] genus associated with the power series $Q(x) = \Gamma(1+x)^{-1}$, i.e., the genus coming from the multiplicative sequence of polynomials $\{Q_i(c_1, \dots, c_i)\}$ in Chern classes, where

$$\sum_{i=0}^{\infty} Q_i(e_1, \dots, e_i) = \prod_{i=1}^{\infty} \frac{1}{\Gamma(1+x_i)}.$$

As shown in [14], the coefficient of the monomial $c_\lambda = c_{\lambda_1} c_{\lambda_2} \cdots$ in $Q_i(c_1, \dots, c_i)$ is $\zeta(m_\lambda)$, for any partition λ. For example, using the tables in the Appendix, we have

$$Q_3(c_1, c_2, c_3) = \zeta(3) c_3 + (\gamma\zeta(2) - \zeta(3)) c_1 c_2 + \frac{1}{6}(\gamma^3 - 3\gamma\zeta(2) + 2\zeta(3)) c_1^3.$$

More recently Lu [19] defined a similar $\hat{\Gamma}$-genus $\{P_i\}$ by using the generating function $P(x) = e^{-\gamma x}\Gamma(1+x)^{-1}$ in place of $Q(x) = \Gamma(1+x)^{-1}$, and related this new genus to an S^1-equivariant Euler class. The coefficient of c_λ in $P_i(c_1, \dots, c_i)$ can be obtained by setting $\gamma = 0$ in $\zeta(m_\lambda)$. Thus

$$P_3(c_1, c_2, c_3) = \zeta(3) c_3 - \zeta(3) c_1 c_2 + \frac{1}{3}\zeta(3) c_1^3$$

(cf. Table 1 of [19]). If we write $\hat{\zeta}$ for the function on QSym that sends $M_{(1)}$ to zero and agrees with ζ on QSym^0, then

$$\hat{\zeta}(E(t)) = \frac{1}{e^{\gamma t}\Gamma(1+t)}.$$

Following the proof of the result of [14], we then have

$$\sum_{i=0}^{\infty} P_i(e_1,\dots,e_i)t^i = \prod_{i=1}^{\infty} \frac{1}{e^{\gamma x_i t}\Gamma(1+x_i t)} = \sum_{\lambda} \hat{\zeta}(e_\lambda) m_\lambda t^{|\lambda|} = \sum_{\lambda} \hat{\zeta}(m_\lambda) e_\lambda t^{|\lambda|}. \qquad (6)$$

While equation (6) appears in [19] (see Prop. 4.3), it has a nice corollary that doesn't. Recall [11, Theorem 4.10.2] that the Chern classes of the tangent bundle of projective space $\mathbf{C}\mathrm{P}^n$ are given by

$$c_i = \binom{n+1}{i} a^i$$

with $a \in H^2(\mathbf{C}\mathrm{P}^n;\mathbf{Z})$ such that $\langle a^n, [\mathbf{C}\mathrm{P}^n]\rangle = 1$, where $[\mathbf{C}\mathrm{P}^n] \in H_{2n}(\mathbf{C}\mathrm{P}^n;\mathbf{Z})$ is the fundamental class. Now by [20, p. 26] the specialization

$$x_i = \begin{cases} 1, & i = 1,2,\dots,n+1 \\ 0, & i > n+1 \end{cases}$$

sends e_i to $\binom{n+1}{i}$. It then follows from equation (6) that

$$\hat{\Gamma}(\mathbf{C}\mathrm{P}^n) = \langle P_n(c_1,\dots,c_n), [\mathbf{C}\mathrm{P}^n]\rangle = \text{coefficient of } t^n \text{ in } \frac{1}{(e^{\gamma t}\Gamma(1+t))^{n+1}}$$

(cf. Table 2 of [19]).

As another occurrence of ζ, we cite the following result about values of the derivatives of the gamma function at positive integers from [22]: if n and k are positive integers, then

$$\frac{\Gamma^{(k)}(n)}{k!} = \sum_{j=0}^{k} \begin{bmatrix} n \\ k+1-j \end{bmatrix} (-1)^j \zeta(h_j),$$

where $\begin{bmatrix} n \\ j \end{bmatrix}$ is the number of permutations of degree n with exactly j cycles (Stirling number of the first kind). Cf. [23, pp. 40-44].

These examples suggest that the homomorphism $\zeta : \mathrm{QSym} \to \mathbf{R}$ may be useful to calculate. Now QSym is actually a Hopf algebra, as we discuss in the next section. Aguiar, Bergeron and Sottille [1] develop a theory of graded connected Hopf algebras endowed with characters (scalar-valued homomorphisms), in which "even" and "odd" characters are defined. A key result is that any such character χ is uniquely expressible as the convolution product $\chi_+\chi_-$ of an even character χ_+ times an odd one χ_-. In this paper we discuss some results on the character $\zeta : \mathrm{QSym} \to \mathbf{R}$ and its factors ζ_+ and ζ_-, and particularly on the restrictions of these characters to $\mathrm{Sym} \subset \mathrm{QSym}$. (Note that for the computation of the Γ- and $\hat{\Gamma}$-genera, the restriction of ζ to Sym suffices.)

After developing some properties of the Hopf algebras QSym and Sym in §2, we discuss the factorization $\zeta = \zeta_+\zeta_-$ on the full algebra QSym in §3. In §4 we consider the restriction of ζ, ζ_+ and ζ_- to Sym. First we show how to use the character table of the symmetric group to compute ζ on Schur functions. Then we consider the effect of ζ on the elementary and complete symmetric functions. We show equation (4) splits as

$$\zeta_+(H(t)) = \sqrt{\frac{\pi t}{\sin \pi t}} \quad \text{and} \quad \zeta_-(H(t)) = \Gamma(1-t)\sqrt{\frac{\sin \pi t}{\pi t}},$$

which makes it easier to compute ζ on elementary and complete symmetric functions by computing ζ_+ and ζ_- separately. Next we consider the values of the three characters on the monomial symmetric functions m_λ. While there is an explicit formula for m_λ in terms of the p_λ (Theorem 7 below), it is somewhat ineffective computationally since it involves a sum over set partitions. We develop some further methods by which the values of ζ, ζ_+, and ζ_- can computed on m_λ, including an efficient algorithm for the case where λ is a hook partition, i.e., λ has at most one part greater than 1 (see equations (33) and (34) below). Finally, we discuss a family of symmetric functions in the kernel of ζ_-. Values of ζ, ζ_+, and ζ_- on m_λ for $|\lambda| \leq 7$ are listed in the Appendix.

2 The Hopf Algebras QSym and Sym

As noted above, the monomial quasi-symmetric functions M_I generate QSym as a vector space. The multiplication of the M_I is given by a "quasi-shuffle" product, which involves combining parts of the associated compositions as well as shuffling them. For example,

$$M_{(1)}M_{(i_1,i_2,\dots,i_l)} = M_{(1,i_1,\dots,i_l)} + M_{(i_1+1,i_2,\dots,i_l)} + M_{(i_1,1,i_2,\dots,i_l)} + \cdots + M_{(i_1,i_2,\dots,i_{l-1},i_l+1)} + M_{(i_1,i_2,\dots,i_l,1)}. \quad (7)$$

In fact, QSym is a polynomial algebra, as shown by by by Malvenuto and Reutenauer [21]. To state their result, we first define what it means for a composition I to be Lyndon. If we order the compositions lexicographically, i.e.,

$$(1) < (1,1) < (1,1,1) < \cdots < (1,2) < \cdots < (2) < (2,1) < \cdots < (3) < \cdots,$$

then a composition I is called Lyndon if $I < K$ for any nontrivial decomposition $I = JK$ of I as a juxtaposition of shorter compositions. For example, (1) and $(1,2,2)$ are Lyndon, but $(2,1)$ is not. Then the result of [21] as follows.

Theorem 1. QSym *is the polynomial algebra on the set* $\{M_I : I \text{ Lyndon}\}$.

The only Lyndon composition ending in 1 is (1) itself, so QSym^0 is the subalgebra of QSym generated by the set $\{M_I : I \text{ Lyndon}, I \neq (1)\}$. Thus $\mathrm{QSym} = \mathrm{QSym}^0[M_{(1)}]$, and we can be more specific as follows.

Theorem 2. *Each monomial quasi-symmetric function M_I can be expressed as a polynomial in $M_{(1)}$ with coefficients in* QSym^0*, of degree equal to the number of trailing 1's in I.*

Proof. Let $t(I)$ be the number of trailing 1's in I. Suppose the result holds for M_J with $t(J) \le n$, and consider M_I with $t(I) = n+1$. Writing I as the juxtaposition $I'(1)$, it follows from equation (7) with $(i_1, \dots, i_l) = I'$ that

$$M_{(1)} M_{I'} = \sum_{k=1}^{2\ell(I')-n} M_{J_k} + (n+1) M_I$$

where each J_k has $t(J_k) \le n$, so the result follows. $\square$

Now QSym is a graded connected Hopf algebra. If we adopt the convention that $M_\emptyset = 1$, then the grade-n part of QSym is generated by $\{M_I : \ |I| = n\}$. The counit ϵ is given by

$$\epsilon(M_I) = \begin{cases} 1, & \text{if } I = \emptyset; \\ 0, & \text{otherwise}; \end{cases}$$

and coproduct Δ by

$$\Delta(M_I) = \sum_{JK=I} M_J \otimes M_K. \tag{8}$$

It follows immediately from equation (8) that the only M_I which are primitives are the power sums $p_n = M_{(n)}$.

The antipode $S : \mathrm{QSym} \to \mathrm{QSym}$ is given by (see [6, Prop. 3.4])

$$S(M_I) = (-1)^{\ell(I)} \sum_{\bar{I} \succeq J} M_J, \tag{9}$$

where $\bar{I}$ is the reverse of I and $I \succeq J$ means I is a refinement of J, i.e., J is obtainable by combining some parts of I. Since QSym is commutative, S is an automorphism of QSym with $S^2 = \mathrm{id}$.

The algebra Sym is generated by the elementary symmetric functions e_n, and also by the complete symmetric functions h_n. The generating functions $E(t)$ and $H(t)$ for these symmetric functions are related by $E(t) = H(-t)^{-1}$. The power-sums p_n also generate Sym as an algebra, and have generating function

$$P(t) = p_1 + p_2 t + p_3 t^2 + \cdots = \frac{H'(t)}{H(t)}.$$

Now Sym is a sub-Hopf-algebra of QSym, and its structure is described succinctly by Geissinger [8]. As follows from equation (9), $S(e_n) = (-1)^n h_n$. The power-sums p_n are primitive, and both the e_n and h_n are divided powers, i.e.,

$$\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j$$

and similarly for h_n. Stated in terms of generating functions, we have

$$\Delta(E(t)) = E(t) \otimes E(t) \quad \text{and} \quad \Delta(H(t)) = H(t) \otimes H(t). \tag{10}$$

as well as

$$\Delta(P(t)) = P(t) \otimes 1 + 1 \otimes P(t).$$

As a vector space, Sym has the basis $\{m_\lambda : \lambda \in \Pi\}$, where Π is the set of partitions. We also have bases

$$\{e_\lambda : \lambda \in \Pi\}, \quad \{h_\lambda : \lambda \in \Pi\}, \quad \text{and} \quad \{p_\lambda : \lambda \in \Pi\},$$

where $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}$ for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, and similarly for h_λ and p_λ. Another important basis for Sym is the Schur functions $\{s_\lambda : \lambda \in \Pi\}$ (see [20, I,§3]). For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ with $\lambda_1 \geq \lambda_2 \geq \cdots$, the corresponding Schur function s_λ is the determinant

$$\begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+l-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+l-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_l-l+1} & h_{\lambda_l-l+2} & \cdots & h_{\lambda_l} \end{vmatrix},$$

where h_i is interpreted as 1 if $i = 0$ and 0 if $i < 0$. Then $s_{(n)} = h_n$ and $s_{(1^n)} = e_n$.

There is an inner product on Sym defined by

$$\langle h_\mu, m_\lambda \rangle = \delta_{\mu,\lambda} \tag{11}$$

for all $\mu, \lambda \in \Pi$. As shown in [20, I,§4], this inner product is symmetric and positive definite. The Schur functions are an orthonormal basis with respect to it, i.e.,

$$\langle s_\mu, s_\lambda \rangle = \delta_{\mu,\lambda}.$$

For any symmetric function f we can define its adjoint $f^\perp$ by

$$\langle f^\perp u, v \rangle = \langle u, fv \rangle.$$

For later use we recall from [20, p. 76] that

$$p_r^\perp = r \frac{\partial}{\partial p_r}. \tag{12}$$

3 The character ζ and its factors ζ_+ and ζ_-

Define $\zeta : \mathrm{QSym} \to \mathbf{R}$ by $\zeta(1) = 1$,

$$\zeta(M_{(i_1,\dots,i_k)}) = \zeta(i_k, i_{k-1}, \dots, i_1)$$

for $i_k > 1$, and $\zeta(M_{(1)}) = \gamma$. It follows from Theorem 2 that $\zeta(M_I)$ can be expressed as a polynomial in γ with coefficients in the multiple zeta values, of degree equal to the number of trailing 1's of I.

Following [1], we say a character of QSym (i.e., an algebra homomorphism $\chi : \mathrm{QSym} \to \mathbf{R}$) is even if $\chi(u) = (-1)^{|u|}\chi(u)$ for homogeneous elements u, and odd if $\chi(u) = (-1)^{|u|}\chi(S(u))$ for all homogeneous u. From [1] we have the following result.

Theorem 3. *For any character χ of* QSym*, there is a unique even character χ_+ and a unique odd character χ_- so that χ is the convolution product $\chi_+\chi_-$.*

From the preceding theorem, there are unique characters ζ_- and ζ_+ of QSym so that ζ_+ is even, ζ_- is odd, and $\zeta = \zeta_+\zeta_-$, i.e.,

$$\zeta(u) = \sum_u \zeta_+(u')\zeta_-(u'') \tag{13}$$

for all elements u of QSym, where

$$\Delta(u) = \sum_u u' \otimes u''.$$

Since $M_{(n)} = p_n$ is primitive, we have from equation (13)

$$\zeta(n) = \zeta_+(p_n) + \zeta_-(p_n). \tag{14}$$

This gives us the following result.

Theorem 4. *If n is even, $\zeta_+(p_n) = \zeta(n)$ and $\zeta_-(p_n) = 0$. If n is odd, $\zeta_+(p_n) = 0$ and $\zeta_-(p_n) = \zeta(n)$ (or γ if $n = 1$).*

Proof. For even n, the oddness of ζ_- implies

$$\zeta_-(p_n) = \zeta_-(S(p_n)) = -\zeta_-(p_n),$$

and the first statement follows from equation (14). If n is odd, then $\zeta_+(p_n) = 0$ and the second statement follows from equation (14). □

The result that $\zeta_-(p_n) = 0$ for n even can be generalized as follows. Call a composition I even if all its parts are even.

Theorem 5. *If I is even, then $\zeta_-(M_I) = 0$.*

Proof. We make use of the universal character $\zeta_Q : \mathrm{QSym} \to \mathbf{R}$ given by

$$\zeta_Q(M_I) = \begin{cases} 1, & \text{if } \ell(I) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

By [1, Theorem 4.1], there is a unique homomorphism $\Psi : \mathrm{QSym} \to \mathrm{QSym}$ such that $\zeta_Q \circ \Psi = \zeta$. Further, Ψ is given by

$$\Psi(M_I) = \sum_{I = I_1 I_2 \cdots I_h} \zeta(M_{I_1})\zeta(M_{I_2}) \cdots \zeta(M_{I_h}) M_{(|I_1|,\dots,|I_h|)},$$

where the sum is over all decompositions of I into a juxtaposition $I_1 I_2 \cdots I_h$ of compositions. Lemma 2.2 of [2] implies that $\zeta_{Q-} \circ \Psi = \zeta_-$, so

$$\zeta_-(M_I) = \sum_{I = I_1 I_2 \cdots I_h} \zeta(M_{I_1})\zeta(M_{I_2}) \cdots \zeta(M_{I_h}) \zeta_{Q-}(M_{(|I_1|,\dots,|I_h|)}).$$

Now an explicit formula for $\zeta_{Q-}(M_J)$ is given by [2, Theorem 3.2], which implies that $\zeta_{Q-}(M_J) = 0$ whenever the last part of J is even. Since $(|I_1|, |I_2|, \dots, |I_h|)$ is even whenever I is, the conclusion follows. □

It follows from the preceding result and equation (13) that $\zeta_+(M_I) = \zeta(M_I)$ for I even. Nevertheless, for most compositions I with $|I|$ even it is no easier to compute $\zeta_+(M_I)$ or $\zeta_-(M_I)$ than $\zeta(M_I)$. In fact, the bound on the degree of γ in $\zeta(M_I)$ given by Theorem 2 need not hold for $\zeta_+(M_I)$ and $\zeta_-(M_I)$. For example,

$$\zeta(M_{(1,2,3)}) = \zeta(3,2,1) = 3\zeta(3)^2 - \frac{203}{48}\zeta(6),$$

while

$$\zeta_+(M_{(1,2,3)}) = -\gamma\zeta(2)\zeta(3) + \frac{11}{4}\gamma\zeta(5) + \frac{5}{2}\zeta(3)^2 - \frac{203}{48}\zeta(6)$$

and

$$\zeta_-(M_{(1,2,3)}) = \gamma\zeta(2)\zeta(3) - \frac{11}{4}\gamma\zeta(5) + \frac{1}{2}\zeta(3)^2$$

(note equation (13) gives $\zeta(M_{(1,2,3)}) = \zeta_+(M_{(1,2,3)}) + \zeta_-(M_{(1,2,3)})$ here). As we see in the next section, the situation is dramatically different when these characters are restricted to $\mathrm{Sym} \subset \mathrm{QSym}$.

4 The restriction of ζ to Sym

The vector space Sym has the various bases m_λ, e_λ, h_λ, p_λ and s_λ discussed in §2. We shall consider the last two bases first. We know the values of ζ in the basis elements p_λ immediately from the definition, since

$$\zeta(p_i) = \begin{cases} \gamma, & \text{if } i = 1, \\ \zeta(i), & \text{otherwise.} \end{cases}$$

From Theorem 4 and Euler's identity for $\zeta(i)$, i even,

$$\zeta_+(p_i) = \begin{cases} \frac{2^{i-1}|B_i|}{i!}\pi^i, & \text{if } i \text{ is even}, \\ 0, & \text{otherwise,} \end{cases}$$

so it follows that $\zeta_+(u)$ for an element $u \in \mathrm{Sym}$ of even degree d is a rational multiple of π^d (or alternatively of $\zeta(d)$). Of course $\zeta_+(u) = 0$ if u has odd degree. Also

$$\zeta_-(p_i) = \begin{cases} \gamma, & \text{if } i = 1, \\ \zeta(i), & \text{if } i > 1 \text{ is odd}, \\ 0, & \text{otherwise,} \end{cases}$$

so the value $\zeta_-(u)$ on any $u \in \mathrm{Sym}$ is a polynomial in $\gamma, \zeta(3), \zeta(5), \dots$.

Now the transition matrix from the p_λ to the Schur functions s_λ is provided by the character table of the symmetric group S_n (see [20, I,§7]). The irreducible characters of S_n are indexed by the partitions of n: let χ^λ be the character associated with λ. The value $\chi^\lambda(\sigma)$ of the character χ^λ on a permutation $\sigma \in S_n$ only depends on the conjugacy class of σ, i.e., its cycle-type: the cycle-type corresponding to the partition $\rho \vdash n$ is

$$\{\sigma \in S_n : \sigma \text{ has } m_i(\rho)\ i\text{-cycles for } 1 \le i \le n\},$$

where $m_i(\rho)$ is the number of parts of ρ equal to i. If we let $\chi^\lambda_\rho = \chi^\lambda(\sigma)$ for σ of cycle-type ρ, the numbers χ^λ_ρ completely determine the character χ^λ. From [20] we have the following result.

Proposition. *For any partition λ of n,*

$$s_\lambda = \sum_{\rho \vdash n} \frac{\chi^\lambda_\rho}{z_\rho} p_\rho, \tag{15}$$

where

$$z_\rho = m_1(\rho)! m_2(\rho)! 2^{m_2(\rho)} m_3(\rho)! 3^{m_3(\rho)} \cdots .$$

Two special cases are worth noting: $\lambda = (n)$ and $\lambda = (1^n)$. In the first case $\chi^{(n)}$ is the trivial character, and equation (15) is

$$h_n = \sum_{i_1+2i_2+\cdots=n} \frac{1}{i_1!1^{i_1} i_2! 2^{i_2} \cdots i_n! n^{i_n}} p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n}. \tag{16}$$

In the second, $\chi^{(1^n)}$ is the alternating character of S_n, i.e.,

$$\chi^{(1^n)}(\sigma) = \text{sign of } \sigma = (-1)^{m_2(\rho)+m_4(\rho)+\cdots}$$

where ρ is the cycle-type of σ. In this case equation (15) becomes

$$e_n = \sum_{i_1+2i_2+\cdots=n} \frac{(-1)^{i_2+i_4+\cdots}}{i_1!1^{i_1} i_2! 2^{i_2} \cdots i_n! n^{i_n}} p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n}. \tag{17}$$

At this point we could compute ζ on the bases h_λ and e_λ by applying ζ to equations (16) and (17) respectively (cf. [10, Prop. 2]). But as we shall see shortly, it is much more efficient to split ζ into even and odd parts.

Applying ζ to equation (15), we obtain

$$\zeta(s_\lambda) = \sum_{\rho \vdash n} \frac{\chi^\lambda_\rho}{z_\rho} \gamma^{m_1(\rho)} \zeta(2)^{m_2(\rho)} \zeta(3)^{m_3(\rho)} \cdots ,$$

which can be written in the alternative form

$$\zeta(s_\lambda) = \sum_{\rho \vdash n} \frac{\chi^\lambda_\rho N(\rho)}{n!} \gamma^{m_1(\rho)} \zeta(2)^{m_2(\rho)} \zeta(3)^{m_3(\rho)} \cdots , \tag{18}$$

where $N(\rho)$ is the number of permutations of cycle-type ρ. For example, using the tables of group characters in [18], equation (18) gives

$$\begin{aligned}\zeta(s_{(3,2,1)}) &= \frac{16}{6!}\gamma^6 - \frac{2\cdot 40}{6!}\gamma^3\zeta(3) + \frac{144}{6!}\gamma\zeta(5) - \frac{2\cdot 40}{6!}\zeta(3)^2\\ &= \frac{1}{45}\gamma^6 - \frac{1}{9}\gamma^3\zeta(3) + \frac{1}{5}\gamma\zeta(5) - \frac{1}{9}\zeta(3)^2.\end{aligned}$$

Now we turn to the values of ζ on the bases e_λ and h_λ. The two are closely related, because the automorphism ω of Sym defined by $\omega(u) = (-1)^{|u|}S(u)$ simply exchanges the two (see [20, I,§2]). The values of $\zeta_+(e_n)$ and $\zeta_+(h_n)$ are given by the following result.

Theorem 6.

$$\zeta_+(H(t)) = \sqrt{\frac{\pi t}{\sin \pi t}} \quad \textit{and} \quad \zeta_+(E(t)) = \sqrt{\frac{\sin \pi t}{\pi t}}.$$

Proof. The oddness of ζ_- means it is ω-invariant, so $\omega(E(t)) = H(t)$ implies

$$\zeta_-(H(t)) = \zeta_-(E(t)). \tag{19}$$

Since $\zeta_+(H(t))$ is an even function of t,

$$\zeta_+(H(t)) = \zeta_+(H(-t)) = \zeta_+(E(t)^{-1}) = \zeta_+(E(t))^{-1}. \tag{20}$$

Now using equations (4) and (5) together with (10), we have

$$\begin{aligned}\Gamma(1-t)\Gamma(1+t) &= \zeta(H(t))\zeta(E(t))^{-1}\\ &= \zeta_+(H(t))\zeta_-(H(t))\zeta_+(E(t))^{-1}\zeta_-(E(t))^{-1},\end{aligned}$$

and from equations (19) and (20) the right-hand side simplifies to $\zeta_+(H(t))^2$. Using the reflection formula for the gamma function and taking square roots, we have

$$\zeta_+(H(t)) = \sqrt{\frac{\pi t}{\sin \pi t}}$$

and thus, by equation (20), the conclusion. $\square$

From the preceding result, the $\zeta_-(e_n)$ are given by

$$\zeta_-(E(t)) = \zeta_-(H(t)) = \frac{\zeta(H(t))}{\zeta_+(H(t))} = \Gamma(1-t)\sqrt{\frac{\sin \pi t}{\pi t}}.$$

We can also apply ζ_- to both sides of equation (17) to obtain

$$\zeta_-(e_n) = \sum_{i_1+3i_3+5i_5\cdots=n} \frac{\gamma^{i_1}\zeta(3)^{i_3}\zeta(5)^{i_5}\cdots}{i_1!1^{i_1}i_3!3^{i_3}i_5!5^{i_5}\cdots} \tag{21}$$

for all positive integers n.

Since the e_n are divided powers,

$$\zeta(e_n) = \sum_{i+j=n} \zeta_+(e_i)\zeta_-(e_j), \tag{22}$$

and we need only consider those terms in the sum (22) with i even. So from

$$\sqrt{\frac{\sin \pi t}{\pi t}} = 1 - \frac{\pi^2 t}{12} + \frac{\pi^4 t^4}{1440} - \frac{\pi^6 t^6}{24192} + \cdots$$

we can compute $\zeta(e_6)$ using equations (21) and (22) as

$$\begin{aligned}
&\zeta_-(e_6) + \zeta_+(e_2)\zeta_-(e_4) + \zeta_+(e_4)\zeta_-(e_2) + \zeta_+(e_6) = \\
&\quad \frac{\gamma^6}{720} + \frac{\gamma^3\zeta(3)}{18} + \frac{\gamma\zeta(5)}{5} + \frac{\zeta(3)^2}{18} - \frac{\pi^2}{12}\left(\frac{\gamma^4}{24} + \frac{\gamma\zeta(3)}{3}\right) + \frac{\pi^4}{1440}\frac{\gamma^2}{2} - \frac{\pi^6}{24192} \\
&\qquad = \frac{\gamma^6}{720} - \frac{\gamma^4\pi^2}{288} + \frac{\gamma^3\zeta(3)}{18} + \frac{\gamma^2\pi^4}{2880} + \gamma\left(\frac{\zeta(5)}{5} - \frac{\pi^2\zeta(3)}{36}\right) + \frac{\zeta(3)^2}{18} - \frac{\pi^6}{24192}.
\end{aligned}$$

The h_n are also divided powers, so we can compute $\zeta(h_n)$ similarly, using Theorem 6 and equation (21) (since $\zeta_-(e_n) = \zeta_-(h_n)$).

Finally, we consider the basis m_λ. Since the power-sums p_i generate Sym over the rational numbers, there exists for each partition λ a polynomial P_λ (with rational coefficients) so that

$$m_\lambda = P_\lambda(p_1, p_2, \dots).$$

From Theorem 4 we then have

$$\zeta_+(m_\lambda) = P_\lambda(0, \zeta(2), 0, \zeta(4), 0, \dots) \tag{23}$$

and

$$\zeta_-(m_\lambda) = P_\lambda(\gamma, 0, \zeta(3), 0, \zeta(5), 0, \dots), \tag{24}$$

since ζ_+ and ζ_- are homomorphisms. (Of course, $\zeta_+(m_\lambda) = 0$ if $|\lambda|$ is odd.) Once ζ_+ and ζ_- are known on the monomial basis, the values of ζ can be computed using the fact [8] that

$$\Delta(m_\lambda) = \sum_{\alpha \cup \beta = \lambda} m_\alpha \otimes m_\beta,$$

where $\alpha \cup \beta$ means the union as multisets. Therefore

$$\zeta(m_\lambda) = \sum_{\alpha \cup \beta = \lambda} \zeta_+(m_\alpha)\zeta_-(m_\beta). \tag{25}$$

Note that we need only consider those terms in (25) with $|\alpha|$ even.

In fact the polynomials P_λ have an explicit formula, which follows from [16, Theorem 2.3] (see also [12, Theorem 2.2]). We need some notation. If $\mathcal{B} = \{B_1, \dots, B_l\}$ is a partition of the set $\{1, 2, \dots, k\}$, we write

$$c(\mathcal{B}) = (-1)^{k-l}(\operatorname{card} B_1 - 1)!(\operatorname{card} B_2 - 1)! \cdots (\operatorname{card} B_l - 1)!.$$

Then our formula is as follows.

Theorem 7. *For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Pi$,*

$$m_\lambda = \frac{1}{m_1(\lambda)!m_2(\lambda)!\cdots} \sum_{\textit{partitions } \mathcal{B} = \{B_1,\ldots,B_l\} \textit{ of } \{1,\ldots,k\}} c(\mathcal{B}) p_{b_1} p_{b_2} \cdots p_{b_l},$$

where $b_i = \sum_{j \in B_i} \lambda_j$.

For example, taking partitions into two parts we have

$$m_{(a,b)} = p_a p_b - p_{a+b} \tag{26}$$

$$m_{(a,a)} = \frac{1}{2}(p_a^2 - p_{2a}) \tag{27}$$

and taking those with three parts gives

$$m_{(a,b,c)} = p_a p_b p_c - p_a p_{b+c} - p_b p_{a+c} - p_c p_{a+b} + 2p_{a+b+c} \tag{28}$$

$$m_{(a,b,b)} = \frac{1}{2}(p_a p_b^2 - 2p_b p_{a+b} - p_a p_{2b} + 2p_{a+2b}) \tag{29}$$

$$m_{(a,a,a)} = \frac{1}{6}(p_a^3 - 3p_a p_{2a} + 2p_{3a}) \tag{30}$$

for a, b, c distinct.

The characters ζ, ζ_+, ζ_- can be computed on any m_λ by applying them to Theorem 7. But since it involves a sum over set partitions, the theorem is less effective computationally than it appears. Nevertheless, for some partitions λ the sum in Theorem 7 reduces to a sum over integer partitions. With a little work, equation (17) can be derived from Theorem 7 with $\lambda = (1^n)$. We also have the following result on hook partitions $\lambda = (n, 1^t)$.

Corollary. *If $n > 1$, then*

$$m_{(n,1^t)} = \sum_{j=0}^{t} (-1)^j p_{n+j} e_{t-j}.$$

Proof. Let $\lambda = (n, 1^t)$, and consider a partition

$$\mathcal{B} = \{B_1, \ldots, B_l\} \quad \text{of} \quad \{1, 2, \ldots, t+1\}.$$

We order the blocks B_i so that B_1 always includes 1. The b_i as in the conclusion of Theorem 7 are $b_1 = n + \operatorname{card} B_1 - 1$, and $b_i = \operatorname{card} B_i$ for $i > 1$. The sets $B_1 - \{1\}, B_2, \ldots, B_l$ form a partition of $\{2, \ldots, t+1\}$: let $c_1, \ldots, c_l$ be their respective cardinalities. Then

$$c(\mathcal{B}) = (-1)^{t+1-l} c_1!(c_2 - 1)! \cdots (c_l - 1)!. \tag{31}$$

The number of distinct partitions of $\{2, \ldots, t+1\}$ corresponding to given values of $c_1, c_2, \ldots, c_l$ is

$$\binom{t}{c_1 \; c_2 \; \cdots \; c_l} \frac{1}{i_1! i_2! \cdots}, \tag{32}$$

where $i_j = \mathrm{card}\{m \geq 2 : c_m = j\}$. The factors (31) and (32) have product

$$(-1)^{t-(l-1)} \frac{t!}{c_2 c_3 \cdots c_l} \frac{1}{i_1! i_2! \cdots} = (-1)^{t+i_1+i_2+\cdots} \frac{t!}{i_1! 1^{i_1} i_2! 2^{i_2} \cdots},$$

and from Theorem 7 it follows that

$$m_{(n,1^t)} = \sum_{i_1+2i_2+\cdots \leq t} \frac{(-1)^{t+i_1+i_2+\cdots}}{i_1! 1^{i_1} i_2! 2^{i_2} \cdots} p_{n+t-i_1-2i_2-\cdots} p_1^{i_1} {p_2}^{i_2} \cdots .$$

Now apply equation (17) to obtain the conclusion. □

Applying ζ_+ and ζ_- to both sides of the preceding result,

$$\zeta_+(m_{(n,1^t)}) = (-1)^t \sum_{i=0}^{t} \zeta_+(p_{n+i}) \zeta_+(e_{t-i}) \tag{33}$$

and

$$\zeta_-(m_{(n,1^t)}) = (-1)^{n+1} \sum_{i=0}^{t} \zeta_-(p_{n+i}) \zeta_-(e_{t-i}), \tag{34}$$

where in equation (33) we only include terms with $n+i$ even, and in (34) we only take terms with $n+i$ odd. To illustrate these formulas, we compute the values of ζ, ζ_+ and ζ_- on $m_{(5,1^3)}$. Equations (33) and (34) give respectively

$$\zeta_+(m_{(5,1^3)}) = (-1)^3(\zeta(6)\zeta_+(e_2) + \zeta(8)) = -\frac{1}{6}\zeta(8)$$

and

$$\zeta_-(m_{(5,1^3)}) = (-1)^6(\zeta(5)\zeta_-(e_3) + \zeta(7)\zeta_-(e_1)) = \frac{1}{6}\gamma^3\zeta(5) + \gamma\zeta(7) + \frac{1}{3}\zeta(3)\zeta(5)$$

Now apply equation (25) to get

$$\begin{aligned} \zeta(m_{(5,1^3)}) &= \zeta_-(m_{(5,1^3)}) + \zeta_+(e_2)\zeta_-(m_{(5,1)}) + \zeta_+(m_{(5,1)})\zeta_-(e_2) + \zeta_+(m_{(5,1^3)}) \\ &= \frac{1}{6}\gamma^3\zeta(5) - \frac{1}{2}\gamma^2\zeta(6) + \gamma(\zeta(7) - \frac{1}{2}\zeta(2)\zeta(5)) + \frac{1}{3}\zeta(3)\zeta(5) - \frac{1}{6}\zeta(8). \end{aligned}$$

A table of the polynomials P_λ can also be built up by formal antidifferentiation, as we now explain.

Theorem 8. *For any partitions λ and μ, define*

$$P_{\lambda-\mu} = \begin{cases} P_\pi, & \text{if } \lambda = \mu \cup \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\frac{\partial P_\lambda}{\partial p_r} = \frac{1}{r} \sum_{\mu \vdash r} c_\mu P_{\lambda-\mu},$$

where

$$p_r = \sum_{\mu \vdash r} c_\mu h_\mu.$$

Proof. We recall the inner product $\langle\cdot,\cdot\rangle$ defined by (11). For all partitions π we have

$$\langle\frac{\partial P_\lambda}{\partial p_r}(p_1,p_2,\dots),h_\pi\rangle=\frac{1}{r}\langle p_r^\perp m_\lambda,h_\pi\rangle=\frac{1}{r}\langle m_\lambda,p_rh_\pi\rangle=$$
$$\frac{1}{r}\sum_{\mu\vdash r}c_\mu\langle m_\lambda,h_{\mu\cup\pi}\rangle=\langle\frac{1}{r}\sum_{\mu\vdash r}c_\mu P_{\lambda-\mu}(p_1,p_2,\dots),h_\pi\rangle,$$

where we have used equation (12). ☐

Thus, from

$$\begin{aligned}
p_1&=h_1\\
p_3&=h_1^3-3h_1h_2+3h_3\\
p_5&=h_1^5-5h_1^3h_2+5h_1^2h_3+5h_1h_2^2-5h_1h_4-5h_2h_3+5h_5
\end{aligned}$$

it follows that

$$\begin{aligned}
\frac{\partial P_\lambda}{\partial p_1}&=P_{\lambda-(1)}\\
\frac{\partial P_\lambda}{\partial p_3}&=\frac{1}{3}P_{\lambda-(1^3)}-P_{\lambda-(2,1)}+P_{\lambda-(3)}\\
\frac{\partial P_\lambda}{\partial p_5}&=\frac{1}{5}P_{\lambda-(1^5)}-P_{\lambda-(2,1^3)}+P_{\lambda-(3,1,1)}+P_{\lambda-(2,2,1)}-P_{\lambda-(4,1)}-P_{\lambda-(2,3)}+P_{\lambda-(5)}.
\end{aligned}$$

Now to find, e.g., $\zeta_-(m_{(5,3,1,1)})$ we begin with

$$\zeta_-(m_{(5,3,1)})=\gamma\zeta(3)\zeta(5)+2\zeta(9),$$

obtainable by applying ζ_- to equation (28). From the preceding result with $r=1$,

$$\zeta_-(m_{(5,3,1,1)})=\frac{1}{2}\gamma^2\zeta(3)\zeta(5)+2\gamma\zeta(9)+\alpha\zeta(3)\zeta(7)+\beta\zeta(5)^2$$

for some rational numbers α,β. Since

$$\frac{\partial P_{(5,3,1,1)}}{\partial p_3}=P_{(5,1,1)}$$

we have

$$\frac{1}{2}\gamma^2\zeta(5)+\alpha\zeta(7)=\zeta_-(m_{(5,1,1)}),$$

and comparing with ζ_- applied to equation (29) (with $a=5$ and $b=1$) gives $\alpha=1$. But also

$$\frac{\partial P_{(5,3,1,1)}}{\partial p_5}=P_{(3,1,1)}+P_{(5)},$$

so

$$\frac{1}{2}\gamma^2\zeta(3)+2\beta\zeta(5)=\zeta_-(m_{(3,1,1)})+\zeta(5)$$

from which we see that $\beta = 1$. Thus

$$\zeta_-(m_{(5,3,1,1)}) = \frac{1}{2}\gamma^2\zeta(3)\zeta(5) + 2\gamma\zeta(9) + \zeta(3)\zeta(7) + \zeta(5)^2.$$

Another check on tables of $\zeta_-(m_\lambda)$ is provided by a series of identities that show certain symmetric functions are in the kernel of ζ_-. For $\pi \in \Pi$, let $L(\pi)$ be the number of parts of π of size greater than 1. Set

$$L_{n,k} = \sum_{|\pi|=n,\ L(\pi)=k} m_\pi.$$

Note that $L_{n,k} = 0$ unless $k \le \lfloor \frac{n}{2} \rfloor$. We define the "excess" of $L_{n,k}$ by $e(L_{n,k}) = n - 2k$, so the excess of a nonzero $L_{n,k}$ is always nonnegative.

Lemma. *For integers $e \ge 1$ and $k \ge 0$,*

$$p_1 L_{2k+e-1,k} + p_2 L_{2k+e-2,k} + \cdots + p_e L_{2k,k} = eL_{2k+e,k} + 2(k+1)L_{2k+e,k+1}.$$

Proof. First note that if we define $L(I)$ for a composition I to be the number of parts of I of size greater than 1, then

$$L_{n,k} = \sum_{|I|=n,\ L(I)=k} M_I.$$

Consider an individual monomial quasi-symmetric function in $L_{2k+e,k}$, say $M_{(2,1,4,2,1)}$ in the case $k = 3$ and $e = 4$. It can arise in the sum

$$M_{(1)}L_{2k+e-1,k} + M_{(2)}L_{2k+e-2,k} + \cdots + M_{(e)}L_{2k,k} \tag{35}$$

from $M_{(1)}M_{(2,4,2,1)}$, $M_{(1)}M_{(2,1,4,2)}$, $M_{(1)}M_{(2,1,3,2,1)}$, and $M_{(2)}M_{(2,1,2,2,1)}$. More generally, M_I in $L_{2k+e,k}$ arises in (35) in

$$P_1(I) - P_2(I) + P_3(I) + P_4(I) + \cdots = |I| - 2P_2(I) \tag{36}$$

ways, where $P_r(I)$ is the number of parts of I of size $\ge r$. But in fact (36) is just the excess e of $L_{2k+e,k}$, thus establishing the coefficient e in the lemma. Now consider a monomial quasi-symmetric function in $L_{2k+e,k+1}$, say $M_{(2,3,2,3)}$ in the case $k = 3, e = 4$. It can arise in (35) from any of the eight terms

$$M_{(1)}M_{(1,3,2,3)},\ M_{(1)}M_{(2,2,2,3)},\ M_{(1)}M_{(2,3,1,3)},\ M_{(1)}M_{(2,3,2,2)},$$
$$M_{(2)}M_{(3,2,3)},\ M_{(2)}M_{(2,3,3)},\ M_{(3)}M_{(2,2,3)},\ M_{(3)}M_{(2,3,2)}.$$

In general, M_I in $L_{2k+e,k+1}$ arises in (35) in $2(k+1)$ ways, giving the coefficient $2(k+1)$. □

Theorem 9. *If $k \ge 1$, then $\zeta_-(L_{n,k}) = 0$.*

Proof. We use induction on the excess of $L_{n,k}$. Since for $k \geq 1$

$$\zeta_-(L_{2k,k}) = \zeta_-(m_{(2,2,\ldots,2)}) = 0,$$

the theorem evidently holds for excess 0. Suppose it holds for excess $\leq n$. From the lemma we have

$$L_{n+2k+1,k} = \frac{1}{n+1}\left[p_1 L_{n+2k,k} + \cdots + p_{n+1} L_{2k,k} - 2(k+1)L_{n+2k+1,k+1}\right],$$

and every L on the right-hand side has excess n or less. Applying ζ_- to both sides, it follows from the induction hypothesis that the theorem also holds for $L_{n+2k+1,k}$. □

Remark. *If $k = 0$, then $\zeta_-(L_{n,k}) = \zeta_-(e_n)$ is given by equation (21).*

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Appendix: ζ, ζ_+ and ζ_- on Sym in monomial basis for weight ≤ 7

	ζ	ζ_+	ζ_-
$m_{(1)}$	γ	0	γ
$m_{(2)}$	$\zeta(2)$	$\zeta(2)$	0
$m_{(1^2)}$	$\frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2)$	$-\frac{1}{2}\zeta(2)$	$\frac{1}{2}\gamma^2$
$m_{(3)}$	$\zeta(3)$	0	$\zeta(3)$
$m_{(2,1)}$	$\gamma\zeta(2) - \zeta(3)$	0	$-\zeta(3)$
$m_{(1^3)}$	$\frac{1}{6}\gamma^3 - \frac{1}{2}\gamma\zeta(2) + \frac{1}{3}\zeta(3)$	0	$\frac{1}{6}\gamma^3 + \frac{1}{3}\zeta(3)$
$m_{(4)}$	$\zeta(4)$	$\zeta(4)$	0
$m_{(3,1)}$	$\gamma\zeta(3) - \zeta(4)$	$-\zeta(4)$	$\gamma\zeta(3)$
$m_{(2^2)}$	$\frac{3}{4}\zeta(4)$	$\frac{3}{4}\zeta(4)$	0
$m_{(2,1^2)}$	$\frac{1}{2}\gamma^2\zeta(2) - \gamma\zeta(3) - \frac{1}{4}\zeta(4)$	$-\frac{1}{4}\zeta(4)$	$-\gamma\zeta(3)$
$m_{(1^4)}$	$\frac{1}{24}\gamma^4 - \frac{1}{4}\gamma^2\zeta(2) + \frac{1}{3}\gamma\zeta(3) + \frac{1}{16}\zeta(4)$	$\frac{1}{16}\zeta(4)$	$\frac{1}{24}\gamma^4 + \frac{1}{3}\gamma\zeta(3)$
$m_{(5)}$	$\zeta(5)$	0	$\zeta(5)$
$m_{(4,1)}$	$\gamma\zeta(4) - \zeta(5)$	0	$-\zeta(5)$
$m_{(3,2)}$	$\zeta(2)\zeta(3) - \zeta(5)$	0	$-\zeta(5)$
$m_{(3,1^2)}$	$\frac{1}{2}\gamma^2\zeta(3) - \gamma\zeta(4) - \frac{1}{2}\zeta(2)\zeta(3) + \zeta(5)$	0	$\frac{1}{2}\gamma^2\zeta(3) + \zeta(5)$
$m_{(2^2,1)}$	$\frac{3}{4}\gamma\zeta(4) - \zeta(2)\zeta(3) + \zeta(5)$	0	$\zeta(5)$
$m_{(2,1^3)}$	$\frac{1}{6}\gamma^3\zeta(2) - \frac{1}{2}\gamma^2\zeta(3) - \frac{1}{4}\gamma\zeta(4)$ $+\frac{5}{6}\zeta(2)\zeta(3) - \zeta(5)$	0	$-\frac{1}{2}\gamma^2\zeta(3) - \zeta(5)$
$m_{(1^5)}$	$\frac{1}{120}\gamma^5 - \frac{1}{12}\gamma^3\zeta(2) + \frac{1}{6}\gamma^2\zeta(3) + \frac{1}{16}\gamma\zeta(4)$ $-\frac{1}{6}\zeta(2)\zeta(3) + \frac{1}{5}\zeta(5)$	0	$\frac{1}{120}\gamma^5 + \frac{1}{6}\gamma^2\zeta(3) + \frac{1}{5}\zeta(5)$

	ζ
$m_{(6)}$	$\zeta(6)$
$m_{(5,1)}$	$\gamma\zeta(5)-\zeta(6)$
$m_{(4,2)}$	$\frac{3}{4}\zeta(6)$
$m_{(3^2)}$	$\frac{1}{2}\zeta(3)^2-\frac{1}{2}\zeta(6)$
$m_{(4,1^2)}$	$\frac{1}{2}\gamma^2\zeta(4)-\gamma\zeta(5)+\frac{1}{8}\zeta(6)$
$m_{(3,2,1)}$	$\gamma\zeta(2)\zeta(3)-\gamma\zeta(5)-\zeta(3)^2+\frac{1}{4}\zeta(6)$
$m_{(2^3)}$	$\frac{3}{16}\zeta(6)$
$m_{(3,1^3)}$	$\frac{1}{6}\gamma^3\zeta(3)-\frac{1}{2}\gamma^2\zeta(4)-\frac{1}{2}\gamma\zeta(2)\zeta(3)+\gamma\zeta(5)+\frac{1}{3}\zeta(3)^2-\frac{1}{8}\zeta(6)$
$m_{(2^2,1^2)}$	$\frac{3}{8}\gamma^2\zeta(4)-\gamma\zeta(2)\zeta(3)+\gamma\zeta(5)+\frac{1}{2}\zeta(3)^2-\frac{13}{32}\zeta(6)$
$m_{(2,1^4)}$	$\frac{1}{24}\gamma^4\zeta(2)-\frac{1}{6}\gamma^3\zeta(3)-\frac{1}{8}\gamma^2\zeta(4)+\frac{5}{6}\gamma\zeta(2)\zeta(3)-\gamma\zeta(5)$ $-\frac{1}{3}\zeta(3)^2+\frac{15}{64}\zeta(6)$
$m_{(1^6)}$	$\frac{1}{720}\gamma^6-\frac{1}{48}\gamma^4\zeta(2)+\frac{1}{18}\gamma^3\zeta(3)+\frac{1}{32}\gamma^2\zeta(4)-\frac{1}{6}\gamma\zeta(2)\zeta(3)$ $+\frac{1}{5}\gamma\zeta(5)+\frac{1}{18}\zeta(3)^2-\frac{5}{128}\zeta(6)$

	ζ_+	ζ_-
$m_{(6)}$	$\zeta(6)$	0
$m_{(5,1)}$	$-\zeta(6)$	$\gamma\zeta(5)$
$m_{(4,2)}$	$\frac{3}{4}\zeta(6)$	0
$m_{(3^2)}$	$-\frac{1}{2}\zeta(6)$	$\frac{1}{2}\zeta(3)^2$
$m_{(4,1^2)}$	$\frac{1}{8}\zeta(6)$	$-\gamma\zeta(5)$
$m_{(3,2,1)}$	$\frac{1}{4}\zeta(6)$	$-\gamma\zeta(5)-\zeta(3)^2$
$m_{(2^3)}$	$\frac{3}{16}\zeta(6)$	0
$m_{(3,1^3)}$	$-\frac{1}{8}\zeta(6)$	$\frac{1}{6}\gamma^3\zeta(3)+\gamma\zeta(5)+\frac{1}{3}\zeta(3)^2$
$m_{(2^2,1^2)}$	$-\frac{13}{32}\zeta(6)$	$\gamma\zeta(5)+\frac{1}{2}\zeta(3)^2$
$m_{(2,1^4)}$	$\frac{15}{64}\zeta(6)$	$-\frac{1}{6}\gamma^3\zeta(3)-\gamma\zeta(5)-\frac{1}{3}\zeta(3)^2$
$m_{(1^6)}$	$-\frac{5}{128}\zeta(6)$	$\frac{1}{720}\gamma^6+\frac{1}{18}\gamma^3\zeta(3)+\frac{1}{5}\gamma\zeta(5)+\frac{1}{18}\zeta(3)^2$

	ζ
$m_{(7)}$	$\zeta(7)$
$m_{(6,1)}$	$\gamma\zeta(6)-\zeta(7)$
$m_{(5,2)}$	$\zeta(2)\zeta(5)-\zeta(7)$
$m_{(4,3)}$	$\zeta(3)\zeta(4)-\zeta(7)$
$m_{(5,1^2)}$	$\frac{1}{2}\gamma^2\zeta(5)-\gamma\zeta(6)-\frac{1}{2}\zeta(2)\zeta(5)+\zeta(7)$
$m_{(4,2,1)}$	$\frac{3}{4}\gamma\zeta(6)-\zeta(2)\zeta(5)-\zeta(3)\zeta(4)+2\zeta(7)$
$m_{(3^2,1)}$	$\frac{1}{2}\gamma\zeta(3)^2-\frac{1}{2}\gamma\zeta(6)-\zeta(3)\zeta(4)+\zeta(7)$
$m_{(3,2^2)}$	$\frac{3}{4}\zeta(3)\zeta(4)-\zeta(2)\zeta(5)+\zeta(7)$
$m_{(4,1^3)}$	$\frac{1}{6}\gamma^3\zeta(4)-\frac{1}{2}\gamma^2\zeta(5)+\frac{1}{8}\gamma\zeta(6)+\frac{1}{3}\zeta(3)\zeta(4)+\frac{1}{2}\zeta(2)\zeta(5)-\zeta(7)$
$m_{(3,2,1^2)}$	$\frac{1}{2}\gamma^2\zeta(2)\zeta(3)-\frac{1}{2}\gamma^2\zeta(5)-\gamma\zeta(3)^2+\frac{1}{4}\gamma\zeta(6)+\frac{3}{4}\zeta(3)\zeta(4)$ $+\frac{3}{2}\zeta(2)\zeta(5)-3\zeta(7)$
$m_{(2^3,1)}$	$\frac{3}{16}\gamma\zeta(6)-\frac{3}{4}\zeta(3)\zeta(4)+\zeta(2)\zeta(5)-\zeta(7)$
$m_{(3,1^4)}$	$\frac{1}{24}\gamma^4\zeta(3)-\frac{1}{6}\gamma^3\zeta(4)-\frac{1}{4}\gamma^2\zeta(2)\zeta(3)+\frac{1}{2}\gamma^2\zeta(5)+\frac{1}{3}\gamma\zeta(3)^2-\frac{1}{8}\gamma\zeta(6)$ $-\frac{13}{48}\zeta(3)\zeta(4)-\frac{1}{2}\zeta(2)\zeta(5)+\zeta(7)$
$m_{(2^2,1^3)}$	$\frac{1}{8}\gamma^3\zeta(4)-\frac{1}{2}\gamma^2\zeta(2)\zeta(3)+\frac{1}{2}\gamma^2\zeta(5)+\frac{1}{2}\gamma\zeta(3)^2-\frac{13}{32}\gamma\zeta(6)$ $+\frac{1}{2}\zeta(3)\zeta(4)-\frac{3}{2}\zeta(2)\zeta(5)+2\zeta(7)$
$m_{(2,1^5)}$	$\frac{1}{120}\gamma^5\zeta(2)-\frac{1}{24}\gamma^4\zeta(3)-\frac{1}{24}\gamma^3\zeta(4)+\frac{5}{12}\gamma^2\zeta(2)\zeta(3)-\frac{1}{2}\gamma^2\zeta(5)$ $-\frac{1}{3}\gamma\zeta(3)^2+\frac{15}{64}\gamma\zeta(6)-\frac{7}{48}\zeta(3)\zeta(4)+\frac{7}{10}\zeta(2)\zeta(5)-\zeta(7)$
$m_{(1^7)}$	$\frac{1}{5040}\gamma^7-\frac{1}{240}\gamma^5\zeta(2)+\frac{1}{72}\gamma^4\zeta(3)+\frac{1}{96}\gamma^3\zeta(4)-\frac{1}{12}\gamma^2\zeta(2)\zeta(3)+\frac{1}{10}\gamma^2\zeta(5)$ $+\frac{1}{18}\gamma\zeta(3)^2-\frac{5}{128}\gamma\zeta(6)+\frac{1}{48}\zeta(3)\zeta(4)-\frac{1}{10}\zeta(2)\zeta(5)+\frac{1}{7}\zeta(7)$

	ζ_+	ζ_-
$m_{(7)}$	0	$\zeta(7)$
$m_{(6,1)}$	0	$-\zeta(7)$
$m_{(5,2)}$	0	$-\zeta(7)$
$m_{(4,3)}$	0	$-\zeta(7)$
$m_{(5,1^2)}$	0	$\frac{1}{2}\gamma^2\zeta(5)+\zeta(7)$
$m_{(4,2,1)}$	0	$2\zeta(7)$
$m_{(3^2,1)}$	0	$\frac{1}{2}\gamma\zeta(3)^2+\zeta(7)$
$m_{(3,2^2)}$	0	$\zeta(7)$
$m_{(4,1^3)}$	0	$-\frac{1}{2}\gamma^2\zeta(5)-\zeta(7)$
$m_{(3,2,1^2)}$	0	$-\frac{1}{2}\gamma^2\zeta(5)-\gamma\zeta(3)^2-3\zeta(7)$
$m_{(2^3,1)}$	0	$-\zeta(7)$
$m_{(3,1^4)}$	0	$\frac{1}{24}\gamma^4\zeta(3)+\frac{1}{2}\gamma^2\zeta(5)+\frac{1}{3}\gamma\zeta(3)^2+\zeta(7)$
$m_{(2^2,1^3)}$	0	$\frac{1}{2}\gamma^2\zeta(5)+\frac{1}{2}\gamma\zeta(3)^2+2\zeta(7)$
$m_{(2,1^5)}$	0	$-\frac{1}{24}\gamma^4\zeta(3)-\frac{1}{2}\gamma^2\zeta(5)-\frac{1}{3}\gamma\zeta(3)^2-\zeta(7)$
$m_{(1^7)}$	0	$\frac{1}{5040}\gamma^7+\frac{1}{72}\gamma^4\zeta(3)+\frac{1}{10}\gamma^2\zeta(5)+\frac{1}{18}\gamma\zeta(3)^2+\frac{1}{7}\zeta(7)$