R-S correspondence for $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_n$ and Klein-4 diagram algebras

M. Parvathi and B. Sivakumar*

Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai-600 005, India
sparvathi.riasm@gmail.com,b.sivakumar@hotmail.com

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Abstract

In [PS] a new family of subalgebras of the extended $\mathbb{Z}_2$-vertex colored algebras, called Klein-4 diagram algebras, are studied. These algebras are the centralizer algebras of $G_n := (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_n$ when it acts on $V^{\otimes k}$, where V is the signed permutation module for G_n. In this paper we give the Robinson-Schensted correspondence for G_n on 4-partitions of n, which gives a bijective proof of the identity $\sum_{[\lambda]\vdash n}(f^{[\lambda]})^2 = 4^n n!$, where $f^{[\lambda]}$ is the degree of the corresponding representation indexed by $[\lambda]$ for G_n. We give proof of the identity $2^k n^k = \sum_{[\lambda]\in\Gamma^G_{n,k}} f^{[\lambda]} m_k^{[\lambda]}$ where the sum is over 4-partitions which index the irreducible G_n-modules appearing in the decomposition of $V^{\otimes k}$ and $m_k^{[\lambda]}$ is the multiplicity of the irreducible G_n-module indexed by $[\lambda]$. Also, we develop an R-S correspondence for the Klein-4 diagram algebras by giving a bijection between the diagrams in the basis and pairs of vacillating tableau of same shape.

1 Introduction

The Bratteli diagrams of the group algebras arising out of the sequence of wreath product groups $\mathbb{Z}_4 \wr S_n$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_n$ are the same. The structures associated with the wreath product $\mathbb{Z}_4 \wr S_n$ are well studied. This motivated us to study the centralizer of wreath product of the Klein-4 group with S_n in [PS] and we obtained a new family of subalgebras of the extended $\mathbb{Z}_2$-vertex colored algebras, called Klein-4 diagram algebras. These algebras are the centralizer algebras of $G_n := (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_n$ when it acts on $V^{\otimes k}$ where V is the signed permutation module for G_n and are denoted by $R_k(n)$.

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The partition algebras were independently studied by Jones and Martin. In [J], Jones has given a description of the centralizer algebra $\mathrm{End}_{S_n}(V^{\otimes k})$, where S_n acts by permutations on V and acts diagonally on $V^{\otimes k}$. This algebra was independently introduced by Martin [M] and named the Partition algebra. The main motivation for studying the partition algebra is in generalizing the Temperly-Lieb algebras and the Potts model in statistical mechanics. In [PK1] Parvathi and Kennedy obtained a new class of algebras $P_k(x, G)$, the G-vertex colored partition algebras, where G is a finite group. These algebras, when $x = n \geq 2k$, were shown to be the centralizer algebra of the direct product group $S_n \times G$ acting on the tensor product space $V^{\otimes k}$ by the restricted action as a subgroup of the wreath product $G \wr S_n$ as in [B]. The extended G-vertex colored algebras obtained in [PK2] were shown to be the centralizer algebras of the the symmetric group S_n acting on the tensor product space $V^{\otimes k}$ by the restricted action as a subgroup of the direct product $S_n \times G, n \geq 2k$. These algebras have a basis consisting of G-vertex colored diagrams with a corresponding multiplication defined on the diagrams. In [PS] a new family of subalgebras of the extended $\mathbb{Z}_2$-vertex colored algebras, called Klein-4 diagram algebras are studied, which are the centralizer algebras of G_n when it acts on $V^{\otimes k}$ where V is the signed permutation module for G_n when $n \geq 2k$.

Let $G = \{e,\ g \mid g^2 = e\} \cong \mathbb{Z}_2$. Let Π_k denote the set of all $\mathbb{Z}_2$-vertex colored partition diagrams which have even number of e's and even number of g's as labeling of vertices appearing in each part. Let $\widehat{EP}_k(x)$ denote the subalgebra of $\widehat{P}_k(x, \mathbb{Z}_2)$ with a basis consisting of diagrams in Π_k. These algebras are known as the Klein-4 diagram algebras. For $n \geq 2k, \widehat{EP}_k(n) \cong R_k(n)$.

The number of standard Young 4-tableau, denoted by $f^{[\lambda]}$, is the degree of the corresponding representation for G_n. The Robinson-Schensted correspondence gives a bijective proof of the identity $\sum_{[\lambda]\vdash n}(f^{[\lambda]})^2 = 4^n n!$. In this paper we develop a Robinson-Schensted correspondence for the group $G_n := (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_n$ on 4-partitions of n. We give proof of the identity $2^k n^k = \sum_{[\lambda]\in\Gamma^G_{n,k}} f^{[\lambda]} m_k^{[\lambda]}$ where the sum is over 4-partitions appearing in the decomposition of $V^{\otimes k}$ as a G_n module and $m_k^{[\lambda]}$ is the multiplicity of the irreducible G_n module indexed by $[\lambda]$, by constructing a bijection between the k-tuples $((i_1, h_1), \ldots, (i_k, h_k))$ of pairs where $1 \leq i_j \leq n, h_j \in \{e, g\}$ and pairs $(T^{[\lambda]}, P^{[\lambda]})$ where $T^{[\lambda]}$ is a standard 4-tableau and $P^{[\lambda]}$ is a vacillating tableau of shape $[\lambda]$. Also, we develop an R-S correspondence for the Klein-4 diagram algebras by giving a bijection between the diagrams in the basis Π_k and pairs of vacillating tableau of same shape. As an application of the Robinson-Schensted correspondence we also define a Knuth relation for the elements in the basis Π_k.

2 Preliminaries

2.1 The group $G_n = (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_n$

We recall the definition of wreath product group from [JK].

$$(\mathbb{Z}_2 \times \mathbb{Z}_2)^n = \{f \mid f : \{1, 2, \ldots, n\} \to (\mathbb{Z}_2 \times \mathbb{Z}_2)\}$$

Let S_n denote the symmetric group on n symbols $\{1, 2, \ldots, n\}$. Let

$$(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_n := (\mathbb{Z}_2 \times \mathbb{Z}_2) \times S_n = \{(f; \pi) \mid f : \{1, 2, \ldots, n\} \to (\mathbb{Z}_2 \times \mathbb{Z}_2)^n, \pi \in S_n\}$$

For $f \in (\mathbb{Z}_2 \times \mathbb{Z}_2)^n$ and $\pi \in S_n,\ f_\pi \in (\mathbb{Z}_2 \times \mathbb{Z}_2)^n$ is defined by $f_\pi = f \circ \pi^{-1}$. Multiplication on $(\mathbb{Z}_2 \times \mathbb{Z}_2)^n$ is given by

$$(ff')(i) = f(i)f'(i), i \in \{1, 2, \ldots, n.\}$$

Using this and with a composition given by

$$(f; \pi)(f'; \pi') = (ff'_\pi; \pi\pi')$$

$(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_n$ is a group, the wreath product of $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ by S_n. Its order is $4^n n!$.
The group G_n is generated by $a, b, g_1, \ldots, g_{n-1}$ with the complete set of relations given by:

1. $a^2 = 1$
2. $b^2 = 1$
3. $g_i^2 = 1$, $1 \leq i \leq n - 1$
4. $ab = ba$
5. $ag_1ag_1 = g_1ag_1a$
6. $bg_1bg_1 = g_1bg_1b$
7. $bg_1ag_1 = g_1ag_1b$
8. $g_ig_j = g_jg_i, |i - j| \geq 2$
9. $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}, 1 \leq i \leq n - 2$
10. $g_ia = ag_i,\ i \geq 2$
11. $g_ib = bg_i,\ i \geq 2$

Remark 2.1.

1. The group generated by $g_1, \ldots, g_{n-1}$ and the relations 3 , 8 , 9 is isomorphic to the symmetric group S_n.
2. The group generated by $a, g_1, \ldots, g_{n-1}$ and the relations 1, 3 ,5, 8 , 9 ,10 is isomorphic to the hyperoctahedral group, denoted by H_n^a.

3. The group generated by $b, g_1, \ldots, g_{n-1}$ and the relations 2, 3, 6 , 8 , 9, 11 is isomorphic to the hyperoctahedral group, denoted by H_n^b.

4. The group generated by a, b and the relations 1, 2, 4 is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, the Klein-4 group.

Definition 2.2. [S] A *partition* of a non-negative integer n is a sequence of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_l)$ such that $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_l \geq 0$ and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_l = n$. The non-zero α_i's are called the *parts* of α and the number of non zero parts is called the length of α. It is denoted by $\alpha \vdash n$.

A *Young diagram* is a pictorial representation of a partition α as an array of n boxes with α_1 boxes in the first row, α_2 boxes in the second row and so on.

Definition 2.3. A 4-*partition* of size n, $[\lambda] = (\alpha, \beta, \gamma, \delta)$ is an ordered 4-tuple of partitions α, β, γ and δ such that $|(\alpha, \beta, \gamma, \delta)| = |\alpha| + |\beta| + |\gamma| + |\delta| = n$. A 4-partition corresponds to a 4-tuple of Young diagrams as follows:

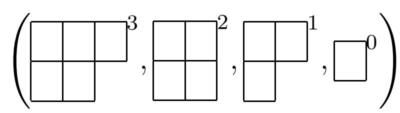


Figure 1. $(\alpha, \beta, \gamma, \delta) = ([3,2][2,2][2,1][1^2])$ and $|\alpha + \beta + \gamma + \delta| = 14$

Notation 2.1. The 4-partition $[\lambda] = [\alpha]^3[\beta]^2[\gamma]^1[\delta]^0$ is denoted by $[\lambda] \vdash n$, where the super scripts denote the residue of the partitions. Note that $\lambda \vdash n$ denotes a single partition of n, while $[\lambda] \vdash n$ denotes a 4-partition of n.

Definition 2.4. [PS] 4-*Tableau.* Let $[\lambda] = [\alpha]^3[\beta]^2[\gamma]^1[\delta]^0$ be a 4-partition of n, ie., $\alpha \vdash n_3, \beta \vdash n_2, \gamma \vdash n_1, \delta \vdash n_0$ such that $n_3 + n_2 + n_1 + n_0 = n$. A tableau of shape $[\lambda]$ is an array obtained by filling boxes in the Young diagram in each partition bijectively with $1, 2, \ldots, n$.

Definition 2.5. [PS] A 4-tableau $[t]$ of shape $[\lambda]$ is standard if in each of the residues, the corresponding tableau are standard i.e., the entries increase along the rows and columns i.e., t_3, t_2, t_1, t_0 are standard tableau of shape $\alpha, \beta, \gamma, \delta$ respectively.

Notation 2.2. [PS] Let $ST^4([\lambda]) = \{[t] \mid [t]$ is a standard tableau of shape $[\lambda]\}$.

From Corollary 4.4.4., [JK], we have the following,

Theorem 2.6. *A complete set of inequivalent irreducible representations of the wreath product* $G_n = (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_n$ *is indexed by a collection of* 4*-partitions* $(\alpha, \beta, \gamma, \delta)$ *such that* $\alpha \vdash k_1, \beta \vdash k_2, \gamma \vdash k_3, \delta \vdash k_4$ *with* $\sum_{i=1}^{4} k_i = n$.

The dimension of the irreducible G_n-module indexed by the 4-partition $[\lambda]$ is given by the number of 4-standard tableau of shape $[\lambda]$.

Theorem 2.7. *[Rb][Theorem 5.18.] The set of* 4*-partitions of* n *is in 1-1 correspondence with the set of partitions of* $4n$*-whose* 4*-core is empty.*

2.2 Double Centralizer Theory

A finite-dimensional associative algebra $\mathcal{A}$ with unit over $\mathbb{C}$, the field of complex numbers, is said to be *semisimple* if $\mathcal{A}$ is isomorphic to a direct sum of full matrix algebras:

$$\mathcal{A} \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} \mathcal{M}_{d_\lambda}(\mathbb{C}),$$

for $\widehat{\mathcal{A}}$ a finite index set, and d_λ positive integers. Corresponding to each $\lambda \in \widehat{\mathcal{A}}$, there is a singe irreducible $\mathcal{A}$-module, call it V^λ, which has dimension d_λ. If $\widehat{\mathcal{A}}$ is singleton set then $\mathcal{A}$ is said to be *simple.* Maschke's Theorem [GW] says that for G finite, $\mathbb{C}[G]$ is semisimple.

A finite dimensional $\mathcal{A}$-module M is *completely reducible* if it is the direct sum of irreducible $\mathcal{A}$ modules, i.e.,

$$M \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} m_\lambda V^\lambda$$

where the non-negative integer m_λ is the multiplicity of the irreducible $\mathcal{A}$-module V^λ in M (some of the m_λ may be zero). Wedderburn's Theorem [GW] tells us that for $\mathcal{A}$ semisimple, every $\mathcal{A}$-module is completely reducible.

The algebra $\mathrm{End}(M)$ comprises of all $\mathbb{C}$-linear transformations on M, where the composition of transformations is the algebra multiplication. If the representation $\rho : \mathcal{A} \to \mathrm{End}(M)$ is injective, we say that M is a *faithful* $\mathcal{A}$-module. The *centralizer algebra* of $\mathcal{A}$ on M, denoted $\mathrm{End}_{\mathcal{A}}(M)$, is the subalgebra of $\mathrm{End}(M)$ comprising of all operators that commute with the $\mathcal{A}$-action:

$$\mathrm{End}_{\mathcal{A}}(M) = \{T \in \mathrm{End}(M) \mid T\rho(a).m = \rho(a)T.m, \quad \text{for all} \quad a \in \mathcal{A},\ m \in M\}.$$

If M is irreducible, then Schur's Lemma says that $\mathrm{End}_{\mathcal{A}}(M) \cong \mathbb{C}$. If G is a finite group and M is a G-module, then we often write $\mathrm{End}_G(M)$ in place of $\mathrm{End}_{\mathbb{C}[G]}(M)$.

Theorem 2.8. ***Double Centralizer Theorem*** *[GW]*
Suppose that $\mathcal{A}$ and M decompose as above. Then

1. $$\mathrm{End}_{\mathcal{A}}(M) \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} \mathcal{M}_{m_\lambda}(\mathbb{C}).$$

2. *As an* $\mathrm{End}_{\mathcal{A}}(M)$*-module*
$$M \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} d_\lambda U^\lambda$$
where dim $U^\lambda = m_\lambda$, *and* U^λ *is an irreducible module for* $\mathrm{End}_{\mathcal{A}}(M)$ *when* $m_\lambda > 0$.

3. *As an* $\mathcal{A} \otimes \mathrm{End}_{\mathcal{A}}(M)$*-bi-module,*
$$M \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}, m_\lambda \neq 0} V^\lambda \otimes U^\lambda.$$

4. $\mathcal{A}$ generates $\mathrm{End}_{\mathrm{End}_{\mathcal{A}}(M)}(M)$.

This theorem tells us that if $\mathcal{A}$ is semisimple, then so is $\mathrm{End}_{\mathcal{A}}(M)$. It also says that the set $\widehat{\mathcal{A}}_M = \{m_\lambda \in \widehat{\mathcal{A}} \mid m_\lambda > 0\}$ indexes all the irreducible representations of $\mathrm{End}_{\mathcal{A}}(M)$. Finally, we see from this theorem that the roles of multiplicity and dimension are interchanged when we view M as an $\mathrm{End}_{\mathcal{A}}(M)$ module as against an $\mathcal{A}$-module. When the hypothesis of the above theorem are satisfied, we say that $\mathcal{A}$ and $\mathrm{End}_{\mathcal{A}}(M)$ generate full centralizers of each other in M. This is often called *Schur-Weyl Duality* between $\mathcal{A}$ and $\mathrm{End}_{\mathcal{A}}(M)$.

2.3 Extended G-vertex colored partition algebra [PK2]

In this paper we consider extended $\mathbb{Z}_2$-vertex colored partition algebra. Let $G = \{e,\ g \mid g^2 = e\} \cong \mathbb{Z}_2$, the group operation being multiplication.

We recall the definition of the partition algebra from [HL]. A k-partition diagram is a graph on two rows of k vertices, one row above the other, where each edge is incident to two distinct vertices and there is at most one edge between any two vertices. The connected components of a diagram partition the $2k$ vertices into l subsets, $1 \leq l \leq 2k$. An equivalence relation is defined on k-partition diagrams by saying that two diagrams are equivalent if they determine the same partition of the $2k$ vertices i.e., when we speak of the diagrams we are really talking about the associated equivalence classes.

Define the composition of two diagrams $d_1 \circ d_2$ of partition diagrams $d_1, d_2 \in P_k(x)$ to be the set partition $d_1 \circ d_2 \in P_k(x)$ obtained by placing d_1 above d_2, identifying the bottom dots of d_1 with top dots of d_2 and removing any connected components that are entirely in the middle row. Multiplication in $P_k(x)$ is defined by $d_1 d_2 = x^l (d_1 \circ d_2)$ where l is the number of blocks removed form the middle row when constructing the composition $d_1 \circ d_2$. The $\mathbb{C}(x)$ span of the partition diagrams with the above defined multiplication of diagrams is called the partition algebra.

Let $[k] = \{1, 2, \ldots, k\}$. Let $f \in G^{2k}$. We can write $f = (f_1, f_2)$ where $f_1, f_2 \in G^k$ are defined on $[k]$ by $f_1(p) = f(p), f_2(p) = f(k+p)$ for all $p \in [k]$. We say that f_1 and f_2 are the first and the second component of f respectively.

Let (d, f) and (d', f') be two (G, k)-diagrams, where d, d' are any two partitions and $f = (f_1, f_2), f' = (f'_1, f'_2) \in G^{2k}$.

$$(d', f') * (d, f) = \begin{cases} x^l(d", (f_1, f'_2)), & \text{if } f_2 = f'_1; \\ 0, & \text{otherwise.} \end{cases}$$

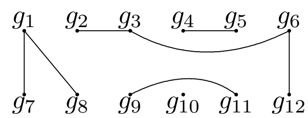
where $d'd = x^l d"$. The multiplication $*$ of two G-diagrams (d, f) and (d', f') defined above can be equivalently stated in other words as follows:

- Multiply the underlying partition diagrams d and d'. This will give the underlying partition diagram of the G diagram $(d', f') * (d, f)$.
- If the bottom label sequence of (d, f) is equal to the top label sequence of (d', f') then the top label sequence and the bottom label sequence of $(d', f') * (d, f)$ are the top label sequence of (d, f) and the bottom label sequence of (d', f') respectively.

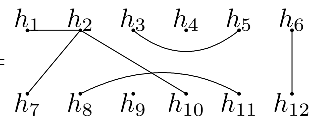
- If the bottom label sequence of (d, f) is not equal to the top label sequence of (d', f') then $(d', f') * (d, f) = 0$.
- For each connected component entirely in the middle row , a factor of x appears in the product.

For example, let $g_r, h_s \in G (1 \leq r, s \leq 12)$.

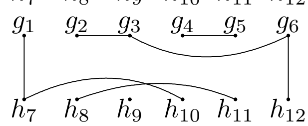
$(d, f) =$



$(d', f') =$



$(d', f') * (d, f) = \quad x^2 \delta^{(g_7, g_8, \ldots, g_{12})}_{(h_1, h_2, \ldots, h_6)}$

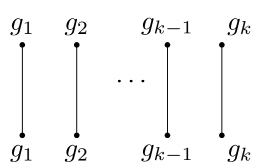


Note that $\delta^{g_7, g_8, \ldots, g_{12}}_{h_1, h_2, \ldots, h_6}$ is the Kronecker delta that is

$$\delta^{g_7, g_8, \ldots, g_{12}}_{h_1, h_2, \ldots, h_6} = \begin{cases} 1, & \text{if } (g_7, g_8, \ldots g_{12}) = (h_1, h_2, \ldots, h_6); \\ 0, & \text{if } (g_7, g_8, \ldots g_{12}) \neq (h_1, h_2, \ldots, h_6) \,. \end{cases}$$

The linear span of G-vertex colored partition diagrams with the multiplication above forms an associative algebra which is denoted by $\widehat{P}_k(x, G)$ called *Extended G-vertex colored partition algebra*. The identity in $\widehat{P}_k(x, G)$ is $\sum_{f \in G^{2k}, f_1 = f_2} (d, f)$ where d is the identity partition diagram

$\sum_{g_1, \ldots, g_k \in G}$



The dimension of the algebra $\widehat{P}_k(x, G)$ is the number of (G, k) diagrams, so that if G is finite, dim $\widehat{P}_k(x, G) = |G|^{2k} B(2k)$ where $B(2k)$ is the bell number of $2k$ i.e., the number of equivalence relations of $2k$-vertices.

2.4 Klein-4 diagram algebras [PS]

Let $G = \{e, g \mid g^2 = e\} \cong \mathbb{Z}_2$. Let $V = \mathbb{C}^n \otimes \mathbb{C}[G]$. Let $v_{i,h} = v_i \otimes h$ where $h \in \{e, g\}$ and $1 \leq i \leq n$. $\{v_{1,e}, \ldots v_{n,e}, v_{1,g}, \ldots, v_{n,g}\}$ is a basis of V.

G_n acts on V as follows:

$$\begin{aligned} av_{1,e} &= -v_{1,e} \\ av_{i,e} &= v_{i,e}, \quad i \neq 1 \\ av_{i,g} &= v_{i,g}, \quad \forall\ i \\ bv_{1,g} &= -v_{1,g} \\ bv_{i,g} &= v_{i,g}, \quad i \neq 1 \\ bv_{i,e} &= v_{i,e}, \quad \forall\ i \end{aligned}$$

For $\pi \in S_n \subset G_n$

$$\begin{aligned} \pi v_{i,g} &= v_{\pi(i),g}, \quad \forall\ i \\ \pi v_{i,e} &= v_{\pi(i),e}, \quad \forall\ i \end{aligned}$$

since the group G_n is generated by a, b and the group of permutations S_n. V is the signed permutation module for G_n.

This action of G_n on V is extended to $V^{\otimes k}$ diagonally. The authors studied the centralizer of the group G_n on $V^{\otimes k}$ in [PS]. The centralizer algebra $\mathrm{End}_{G_n} V^{\otimes k} = R_k(n)$. Since $S_n \subset G_n$, $\mathrm{End}_{G_n} V^{\otimes k} \subset \mathrm{End}_{S_n} V^{\otimes k} \cong \widehat{P}_k(n, G)$ for $n \geq 2k$ and $G \cong \mathbb{Z}_2$, where $\widehat{P}_k(n, G)$ is the extended G-vertex colored algebra studied in [PK2].

Notation 2.3. Let Π_k denote the set of all $\mathbb{Z}_2$-vertex colored partition diagrams which have even number of e's and even number of g's labeling of vertices appearing in each part.

Definition 2.9. [PS]*Klein-4 diagram algebras.* Let $\widehat{EP}_k(x)$ denote the subalgebra of $\widehat{P}_k(x, \mathbb{Z}_2)$ with a basis consisting of diagrams in Π_k. These algebras are known as the Klein-4 diagram algebras.

Proposition 2.10. [PS] $\widehat{EP}_k(n) \cong R_k(n), n \geq 2k$.

Lemma 2.11. [PS]*Let* $\Lambda_{k,n} = \{\lambda \mid \lambda$ *is obtained from a* $[\mu] \in \Lambda_{k-1}$ *by removal of a rim node in* 1 *or* 2 *residue and placing in* 0 *or* 3 *residue and vice versa* $\}$, *where* $\Lambda_{1,n} = \{[n-1]^3[1]^2, [n-1]^3[1]^1\}$. *Let* $\Gamma^G_{k,n} = \{([n-j, \alpha]^3[\beta]^2[\gamma]^1[\delta]^0)$ *such that* $\beta \vdash x_1,\ \gamma \vdash x_2,\ \alpha \vdash y_1,\ \delta \vdash y_2,\ x_1 + x_2 + y_1 + y_2 = j,\ x_1 + x_2 = k - 2i,\ y_1 + y_2 = r,\ 0 \leq i \leq \lfloor \frac{k}{2} \rfloor,\ 0 \leq r \leq i, 0 \leq j \leq k.\}$ *Then* $\Lambda_{k,n} = \Gamma^G_{k,n}$.

The main theorem in [PS] gives the decomposition of the tensor product of the G_n module $V^{\otimes k}$. This rule is used to recursively construct the Bratteli diagram for the Klein-4 diagram algebras $\mathrm{End}_{G_n}(V^{\otimes k})$.

Theorem 2.12. *[PS]As* G_n *modules,* $V^{\otimes k} = \bigoplus_{[\lambda] \in \Gamma^G_{k,n}} m_k^{[\lambda]} V_{[\lambda]}$, *where* $V_{[\lambda]}$ *is the irreducible* G_n *module indexed by* $[\lambda]$.

It follows from the double centralizer theorem

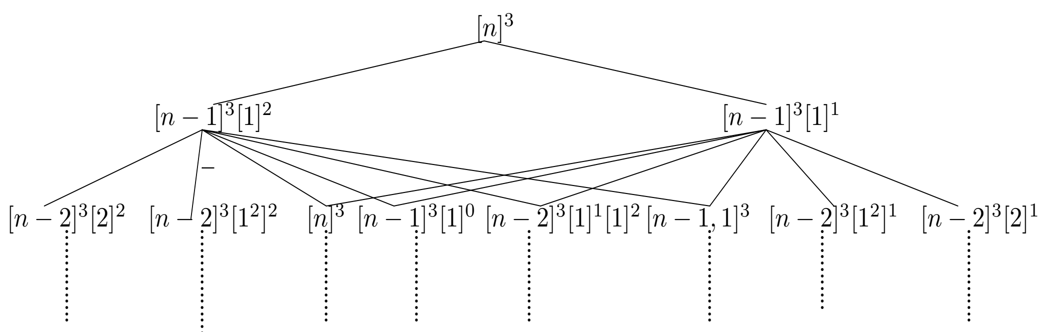
Theorem 2.13. *[PS]As $R_k(n)$ modules, $V^{\otimes k} = \bigoplus_{[\lambda]\in\Gamma^G_{k,n}} f^{[\lambda]}U_{[\lambda]}$, where $U_{[\lambda]}$ is the irreducible $R_k(n)$ module indexed by $[\lambda]$.*

Proposition 2.14. [PS] The Bratteli diagram of the chain

$$R_0(n) \subset R_1(n) \subset R_2(n) \subset R_3(n) \dots$$

is the graph where the vertices in the k^{th} level are labeled by the elements in the set $\Gamma^G_{k,n}, k \geq 0$ and the edges are defined as follows , a vertex $[n-j,\alpha]^3[\beta]^2[\gamma]^1[\delta]^0$ in the i^{th} level is joined to a vertex $[n-j,\lambda]^3[\mu]^2[\nu]^1[\rho]^0$ in the $(i+1)^{st}$ level if $[n-j,\lambda]^3[\mu]^2[\nu]^1[\rho]^0$ can be obtained from the four tuple $[n-j,\alpha]^3[\beta]^2[\gamma]^1[\delta]^0$ by removing a box in the Young diagram in 0 or 3 residue and adding it to the Young diagram in the 1 or 2 residue or removing a box from the young diagram in 1 or 2 residue and adding it to the Young diagram in 0 or 3 residue.

See the diagram below of the first three rows of the Bratteli diagram.



2.5 R-S correspondence for S_n [S]

In this section, we recall the R-S correspondence for S_n from [S]. Let $\pi \in S_n$. Suppose that π is given in two line notation as

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$$

We construct sequence of tableaux pairs (P,Q) where $\pi(1),\pi(2),\dots,\pi(n)$ where $\pi(i)'$s are inserted in the P's and $1,2,\dots,n$ are placed in the Q's. For inserting a positive integer x not in the partial tableau P we proceed as follows

1. Let R be the first row of P.
2. While x is less than some element in R, do
   (a) Let y be the smallest element of R greater than x,
   (b) Replace $y \in R$ with x;
   (c) Let $x := y$ and let R be the next row.

3. Place x at the end of row R and stop.

Definition 2.15. [S] A generalized permutation is a two line array of positive integers

$$\pi = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

whose columns are in lexicographic order, with the top entry taking precedence.

Theorem 2.16. *[S] A pair of generalized permutations are Knuth equivalent if and only if they have the same P-tableau.*

3 R-S correspondence for G_n

In this section, we establish the R-S correspondence for the wreath product $G_n = (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_n$.

Let $\sigma \in G_n$ such that $\sigma = (f;\pi)$. We shall denote this by
$\begin{pmatrix} 1 & 2 & \cdots & n \\ (f(1),\pi(1)) & (f(2),\pi(2)) & \cdots & (f(n),\pi(n)) \end{pmatrix}$

For every $\sigma \in G_n$ we define a pair of standard 4-tableau:

$$(P(\sigma), Q(\sigma)) \text{ where } \begin{cases} P(\sigma) = [P^e(\sigma), P^a(\sigma), P^b(\sigma), P^{ab}(\sigma)] \\ Q(\sigma) = [Q^e(\sigma), Q^a(\sigma), Q^b(\sigma), Q^{ab}(\sigma)], \end{cases}$$

such that $P(\sigma)$ and $Q(\sigma)$ have the same shape.

We construct a sequence of 4-tableaux pairs

$$(P_0, Q_0) = (\emptyset, \emptyset), (P_1, Q_1), (P_2, Q_2), \ldots, (P_n, Q_n) = (P, Q)$$

where$(f(1),\pi(1)), (f(2),\pi(2)), \cdots, (f(n),\pi(n))$ are inserted into the P's and $1, 2, \ldots, n$ are placed in the Q's so that $\mathrm{sh}P_k = \mathrm{sh}Q_k$ for all k. The operations and placement will now be described. Let P be a partial 4-tableau i.e., an array with distinct entries whose rows and columns increase in each of the residues. Also let $x = \pi(i)$ be an element to be inserted in P. Let the associated sign be $f(i) \in \{e, a, b, ab\}$. To each of the elements $\{e, a, b, ab\}$ we associate $3, 2, 1, 0$ residues of the 4-partition respectively. To row insert $(f(i), \pi(i))$ into P we insert $\pi(i)$ into the residue of the 4-partition associated with $f(i)$ in the 4-tableau using the usual insertion procedure for the symmetric group as in [S].

The following theorem establishes the identity $\sum_{[\lambda]\vdash n}(f^{[\lambda]})^2 = 4^n n!$.

Theorem 3.1. *The map $\sigma \xrightarrow{RS} (P, Q)$ is a bijection between the elements of G_n and the pairs of standard 4-tableaux of same shape $[\lambda] \vdash n$.*

Proof. To show that we have a bijection we create an inverse. This is done by reversing the algorithm above. In the above process we keep track of the sign $f(i)$ of the element $\pi(i)$ according to the part from which it is removed. Thus the element recovered from the process at kth step is $(f(i), \pi(i))$. Continuing in this way, we eventually recover all the elements of σ in reverse order. $\square$

4 Vacillating Tableau

We give the vacillating tableau in case of 4-partitions of n following the procedure outlined in [HL] for partitions of n. The dimensions of the irreducible G_n module $V_{[\lambda]}$ equals the number of standard 4-tableaux of shape $[\lambda]$. A standard 4-tableau of shape $[\lambda] = [n-j,\alpha]^3[\beta]^2[\gamma]^1[\delta]^0$ is a filling of the diagram with numbers $1, 2, \ldots, n$ in such a way that each number appears exactly once, the rows increase from left to right and the columns increase from top to bottom. We can identify a standard 4-tableau $T_{[\lambda]}$ of shape $[\lambda]$ with a sequence $(\emptyset, [\lambda]^{(0)}, [\lambda]^{(1)}, \ldots, [\lambda]^{(n)} = [\lambda])$ of 4-tableaux such that $|[\lambda]^{(i)}| = i$, $[\lambda]^{(i)} \subseteq [\lambda]^{(i+1)}$ and such that $[\lambda]^{(i)}/[\lambda]^{(i-1)}$ is the box containing i in $T_{[\lambda]}$. For example, the set of all standard 4-tableaux of shape $[2]^3[1]^2[1]^1[1]^0$ is 60.

The path associated with the standard tableau $\boxed{2 \mid 3}^{\,3}\boxed{4}^{\,2}\boxed{5}^{\,1}\boxed{1}^{\,0}$ is

$$\left(\emptyset, \square^{\,0}, \square^{\,3}\square^{\,0}, \square\square^{\,3}\square^{\,0}, \square\square^{\,3}\square^{\,0}\square^{\,2}, \square\square^{\,3}\square^{\,0}\square^{\,2}\square^{\,1}\right)$$

The number of standard 4-tableaux can be computed by using the hook formula for the 4-partition.

Let $[\lambda] \in \Gamma^G_{k,n}$. A vacillating tableau of shape $[\lambda]$ and length $2k$ is a sequence of 4-partitions

$$\left([n]^3 = [\lambda]^{(0)}, [\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, [\lambda]^{(1\frac{1}{2})}, \ldots, [\lambda]^{(k-\frac{1}{2})}[\lambda]^{(k)} = [\lambda]\right),$$

satisfying for each i,

1. $[\lambda]^{(i)} \in \Gamma^G_{i,n}$ and $[\lambda]^{(i+\frac{1}{2})} \in \Gamma^G_{i,n-1}$,
2. $[\lambda]^{(i)} \supseteq [\lambda]^{(i+\frac{1}{2})}$ and $|[\lambda]^{(i)}/[\lambda]^{(i+\frac{1}{2})}| = 1$,
3. $[\lambda]^{(i+\frac{1}{2})} \subseteq [\lambda]^{(i+1)}$ and $|[\lambda]^{(i+1)}/[\lambda]^{(i+\frac{1}{2})}| = 1$.

The vacillating tableau of shape $[\lambda]$ correspond exactly with paths from bottom of the Bratteli diagram starting from $[n]^3$ to $[\lambda]$. By the double centralizer theorem we have $m_k^{[\lambda]} = \dim(U_k^{[\lambda]})$. Thus if we let $VT^4_k([\lambda])$ denote the set of vacillating tableau of shape $[\lambda]$ and length k, then

$$m_k^{[\lambda]} = \dim(U_k^{[\lambda]}) = |VT^4_k([\lambda])|$$

where $m_k^{[\lambda]}$ is the multiplicity of $V_{[\lambda]}$ in the decomposition of $V^{\otimes k}$ as a G_n module.

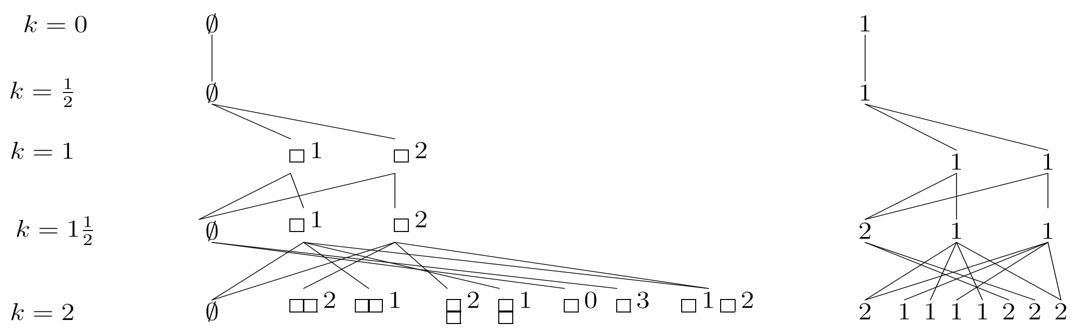
Let $n \geq 2k$ and for a partition $[\lambda] = [n-j,\alpha]^3[\beta]^2[\gamma]^1[\delta]^0$ associate the partition $[\alpha]^3[\beta^2][\gamma]^1[\delta]^0$. Let $\widehat{\Gamma}^G_{k,n} = \{\widehat{[\lambda]} \mid \widehat{[\lambda]} = [\alpha]^3[\beta^2][\gamma]^1[\delta]^0\}$. Then the sets $\widehat{\Gamma}^G_{k,n}$ is in bijection with the set $\Gamma^G_{k,n}$. Hence we can use either of the sets to index the irreducible representations of $R_k(n)$.

For example, the following sequences represent the same vacillating tableau $P_{[\lambda]}$, the first using diagrams in $\Gamma^G_{k,n}$ and the second from $\widehat{\Gamma}^G_{k,n}$

$$
\begin{aligned}
P_{[\lambda]} &= \left(\square\square\square\square^{\,3}, \square\square\square^{\,3}, \square\square\square^{\,3}\square^{\,2}, \square\square^{\,3}\square^{\,2}, \square\square^{\,3}\square\square^{\,2}, \square\square^{\,3}\square^{\,2},\right. \\
&\quad \left.\square\square^{\,3}\square^{\,2}\square^{\,0}\right) \\
&= \left(\emptyset, \emptyset, \square^{\,2}, \square^{\,2}, \square\square^{\,2}, \square^{\,2}, \square^{\,2}\square^{\,0}\right).
\end{aligned}
$$

For our bijection in section 5 we will use $\Gamma^G_{k,n}$ and for the bijection in section 6 we will use $\widehat{\Gamma}^G_{k,n}$.

The following diagram gives the Bratteli diagram for $R_k(n)$ upto $k = 2$ using the identification of $\Gamma^G_{k,n}$ with $\widehat{\Gamma}^G_{k,n}$ given above and the dimensions of the corresponding modules indexed by $[\lambda]'$s.



5 A bijective proof of $2^k n^k = \sum_{[\lambda]\in\Gamma^G_{k,n}} f^{[\lambda]} m_k^{[\lambda]}$

Using the double centralizer theorem and comparing dimensions on both sides for $V^{\otimes k} \cong \bigoplus_{[\lambda]\in\Gamma^G_{k,n}} V_{[\lambda]} \otimes U_{[\lambda]}$ where $V_{[\lambda]}$ are the irreducible G_n modules and $U_{[\lambda]}$ are the irreducible $R_k(n)$ modules, we establish the following identity:

$$
2^k n^k = \sum_{[\lambda]\in\ \Gamma^G_{k,n}} f^{[\lambda]} m_k^{[\lambda]} \tag{5.1}
$$

We follow the notations as given below:

1. $\underline{n} = \{1, 2, \ldots, n\}$
2. $\underline{k} = \{1, 2, \ldots, k\}$
3. $G = \{e, g\}$

To give a combinatorial proof of identity 5.1 we need to find a bijection of the form

$$
\{(i_1, h_1), \ldots, (i_k, h_k) \mid i_j \in \underline{n}, j \in \underline{k}, h_j \in G\} \longleftrightarrow \bigsqcup_{[\lambda]\in\Gamma^G_{k,n}} ST^4([\lambda]) \times VT^4_k([\lambda])
$$

To do so we construct an invertible function that turns the k-tuple of pairs $((i_1,h_1),\ldots,(i_k,h_k))$ into a pair $(T_{[\lambda]},P_{[\lambda]})$ consisting of a standard 4-tableau $T_{[\lambda]}$ of shape $[\lambda]$ and a vacillating tableau $P_{[\lambda]}$ of shape $[\lambda]$ and length $2k$ for some $[\lambda]\in\Gamma^G_{k,n}$.
Note: We use the Robinson-Schensted (RS) insertion and inverse RS algorithm as in [HL]. Also we use the same version of jeu de taquin in each of the residue of the 4-partition. The color h_j associated with i_j is used to give the choice of the residue into which the insertion of i_j will take place. Similarly for the reverse process h_j is determined by the path along which i_j is removed using reverse RS algorithm and placed in another residue. Let S be the standard 4 tableau with parts (S^3,S^2,S^1,S^0) in the corresponding residues. Let $S^l_{i,j}$ denote the entry of S in row i and column j in the l^{th} residue. A box whose removal leaves the diagram of a 4 partition is called a corner. Thus the corners of S are the corners appearing in each of the residues $3,2,1,0$. Thus the corners of S are boxes which appear both at the end of row and column in some residue. The following algorithm will delete x from T leaving a standard 4-tableau S with x removed. Let $T=T^3T^2T^1T^0$ where T^i denotes the tableau of numbers in the i^{th} residue. Let x appear in l^{th} residue (for some $l\in\{0,1,2,3\}$). Let $x\overset{jdt}{\leftarrow}T$ denote the usual jeu de taquin of removal of x from T^l when x appears in the l^{th} residue.

1. Let $c=S^l_{i,j}$ be the box containing x.
2. While c is not a corner, do
   (a) Let c' be the box containing $\min\{S^l_{i+1,j},S^l_{i,j+1}\}$;
   (b) Exchange the position of c and c'.
3. Delete c.

If only one of $S^l_{i+1,j},S^l_{i,j+1}$ exists at step $2(a)$ then the minimum is taken to be the single value.

Example 5.1. Let $T=\begin{array}{|c|c|}\hline 2&3\\\hline 6&7\\\hline\end{array}^3\ \begin{array}{|c|}\hline 4\\\hline\end{array}^2\ \begin{array}{|c|}\hline 5\\\hline 8\\\hline\end{array}^1\ \begin{array}{|c|}\hline 1\\\hline\end{array}^0$

$2\overset{jdt}{\leftarrow}T$ denotes in this case $2\overset{jdt}{\leftarrow}T^3$ since 2 appears in the 3 residue. The process jeu de taquin on the 4 tableau is as follows $T=\begin{array}{|c|c|}\hline 2&3\\\hline 6&7\\\hline\end{array}^3\ \begin{array}{|c|}\hline 4\\\hline\end{array}^2\ \begin{array}{|c|}\hline 5\\\hline 8\\\hline\end{array}^1\ \begin{array}{|c|}\hline 1\\\hline\end{array}^0 \Rightarrow \begin{array}{|c|c|}\hline 3&2\\\hline 6&7\\\hline\end{array}^3\ \begin{array}{|c|}\hline 4\\\hline\end{array}^2\ \begin{array}{|c|}\hline 5\\\hline 8\\\hline\end{array}^1\ \begin{array}{|c|}\hline 1\\\hline\end{array}^0 \Rightarrow$
$\begin{array}{|c|c|}\hline 3&7\\\hline 6&2\\\hline\end{array}^3\ \begin{array}{|c|}\hline 4\\\hline\end{array}^2\ \begin{array}{|c|}\hline 5\\\hline 8\\\hline\end{array}^1\ \begin{array}{|c|}\hline 1\\\hline\end{array}^0 \Rightarrow \begin{array}{|c|c|}\hline 3&7\\\hline 6\\\cline{1-1}\end{array}^3\ \begin{array}{|c|}\hline 4\\\hline\end{array}^2\ \begin{array}{|c|}\hline 5\\\hline 8\\\hline\end{array}^1\ \begin{array}{|c|}\hline 1\\\hline\end{array}^0$

For inserting a positive integer x not in the 4 tableau into the l^{th} residue we follow the usual RS algorithm $x\overset{RS}{\rightarrow}S$ which is given by $x\overset{RS}{\rightarrow}S^l$

1. Let R be the first row of S^l.
2. While x is less than some element in R, do

(a) Let y be the smallest element of R greater than x,

(b) Replace $y \in R$ with x;

(c) Let $x := y$ and let R be the next row.

3. Place x at the end of row R.

For example, the insertion of 2 into the 2-residue in the following 4-tableau yields $2 \overset{RS}{\rightarrow} S$.

Let $S = \begin{array}{|c|c|}\hline 1&3\\\hline\end{array}^{3}\begin{array}{|c|c|c|}\hline 4&5&6\\\hline 7&8\\\cline{1-2} 9\\\cline{1-1}\end{array}^{2}\begin{array}{|c|c|}\hline 10&11\\\hline\end{array}^{1}\begin{array}{|c|}\hline 12\\\hline\end{array}^{0}$

Now $2 \overset{RS}{\rightarrow} S^2$

$$S = \begin{array}{|c|c|}\hline 1&3\\\hline\end{array}^{3}\begin{array}{|c|c|c|}\hline 4&5&6\\\hline 7&8\\\cline{1-2} 9\\\cline{1-1}\end{array}^{2}\begin{array}{|c|c|}\hline 10&11\\\hline\end{array}^{1}\begin{array}{|c|}\hline 12\\\hline\end{array}^{0} \Rightarrow \begin{array}{|c|c|}\hline 1&3\\\hline\end{array}^{3}\begin{array}{|c|c|c|}\hline 2&5&6\\\hline 7&8\\\cline{1-2} 9\\\cline{1-1}\end{array}^{2}\begin{array}{|c|c|}\hline 10&11\\\hline\end{array}^{1}\begin{array}{|c|}\hline 12\\\hline\end{array}^{0}$$

$$\Rightarrow \begin{array}{|c|c|}\hline 1&3\\\hline\end{array}^{3}\begin{array}{|c|c|c|}\hline 2&5&6\\\hline 4&8\\\cline{1-2} 9\\\cline{1-1}\end{array}^{2}\begin{array}{|c|c|}\hline 10&11\\\hline\end{array}^{1}\begin{array}{|c|}\hline 12\\\hline\end{array}^{0} \Rightarrow \begin{array}{|c|c|}\hline 1&3\\\hline\end{array}^{3}\begin{array}{|c|c|c|}\hline 2&5&6\\\hline 4&8\\\cline{1-2} 7\\\cline{1-1}\end{array}^{2}\begin{array}{|c|c|}\hline 10&11\\\hline\end{array}^{1}\begin{array}{|c|}\hline 12\\\hline\end{array}^{0}$$

$$\Rightarrow \begin{array}{|c|c|}\hline 1&3\\\hline\end{array}^{3}\begin{array}{|c|c|c|}\hline 2&5&6\\\hline 4&8\\\cline{1-2} 7\\\cline{1-1} 9\\\cline{1-1}\end{array}^{2}\begin{array}{|c|c|}\hline 10&11\\\hline\end{array}^{1}\begin{array}{|c|}\hline 12\\\hline\end{array}^{0} = T$$

We construct the mapping Fw, i.e., given $((i_1, h_1), \ldots, (i_k, h_k))$ with $1 \leq i_j \leq n$ and $h_j \in G$ we will produce a pair $(T_{[\lambda]}, P_{[\lambda]})$, $[\lambda] \in \Gamma^G_{k,n}$, consisting of a standard 4-tableau $T_{[\lambda]}$ and a vacillating tableau $P_{[\lambda]}$. We first initialize the 0^{th} tableau to be standard tableau of shape $[n]^3$ namely

$$T^{(0)} = \begin{array}{|c|c|c|c|}\hline 1&2&\quad\cdots\quad&n\\\hline\end{array}$$

Then we recursively define standard 4-tableau for , $0 \leq j \leq k-1, T^{(j+\frac{1}{2})}$ and $T^{(j+1)}$ by

$$T^{(j+\frac{1}{2})} = i_{j+1} \overset{jdt}{\longleftarrow} T^{(j)}$$

$$T^{(j+1)} = i_{j+1} \overset{RS}{\longrightarrow} T^{(j+\frac{1}{2})}.$$

The removal of i_{j+1} using jdt is done from the residue where i_{j+1} appears. The insertion of i_{j+1} using RS into $T^{(j+\frac{1}{2})}$ is done according to the rules given below:

1. If i_{j+1} is removed from 3-residue and

   (a) $h_{j+1} = e$ then insert i_{j+1} in the 2 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

   (b) $h_{j+1} = g$ then insert i_{j+1} in the 1 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

2. If i_{j+1} is removed from 2-residue and

(a) $h_{j+1} = e$ then insert i_{j+1} in the 3 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

(b) $h_{j+1} = g$ then insert i_{j+1} in the 0 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

3. If i_{j+1} is removed from 1-residue and

(a) $h_{j+1} = e$ then insert i_{j+1} in the 0 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

(b) $h_{j+1} = g$ then insert i_{j+1} in the 3 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

4. If i_{j+1} is removed from 0-residue and

(a) $h_{j+1} = e$ then insert i_{j+1} in the 2 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

(b) $h_{j+1} = g$ then insert i_{j+1} in the 1 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

Let $[\lambda]^j \in \Gamma^G_{j,n}$ be the shape of $T^{(j)}$ and let $[\lambda]^{j+\frac{1}{2}} \in \Gamma^G_{j,n-1}$ be the shape of $T^{j+\frac{1}{2}}$. Let $P_{[\lambda]} = \left([\lambda]^{(0)}, [\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, [\lambda]^{(1\frac{1}{2})}, \ldots, [\lambda]^{(k)}\right)$ and $T_{[\lambda]} = T^k$ so that $P_{[\lambda]}$ is a vacillating tableau of shape $[\lambda] = [\lambda]^{(k)} \in \Gamma^G_{k,n}$ and $T_{[\lambda]}$ is a standard 4-tableau of same shape $[\lambda]$. We denote this iterative process of deletion and insertion that associates the pair $(T_{[\lambda]}, P_{[\lambda]})$ to the k-tuple of pairs $((i_1, h_1), \ldots, (i_k, h_k))$ by

$$((i_1, h_1), \ldots, (i_k, h_k)) \xrightarrow{Fw} (T_{[\lambda]}, P_{[\lambda]}).$$

Example 5.2.

For $((2, e), (3, g), (4, e), (3, e), (5, g))$ we have

$$\begin{array}{lll}
T^{(0)} & = \begin{array}{|c|c|c|c|}\hline 1&2&\cdots&n\\\hline\end{array}^{\,3} & \\
T^{(\frac{1}{2})} & = \begin{array}{|c|c|c|c|}\hline 1&3&\cdots&n\\\hline\end{array}^{\,3} & T^{(0)} \xrightarrow{jdt} 2 \\
T^{(1)} & = \begin{array}{|c|c|c|c|}\hline 1&3&\cdots&n\\\hline\end{array}^{\,3} \; \begin{array}{|c|}\hline 2\\\hline\end{array}^{\,2} & T^{(\frac{1}{2})} \xleftarrow{RS} 2 \\
T^{(1\frac{1}{2})} & = \begin{array}{|c|c|c|c|}\hline 1&4&\cdots&n\\\hline\end{array}^{\,3} \; \begin{array}{|c|}\hline 2\\\hline\end{array}^{\,2} & T^{(1)} \xrightarrow{jdt} 3 \\
T^{(2)} & = \begin{array}{|c|c|c|c|}\hline 1&4&\cdots&n\\\hline\end{array}^{\,3} \; \begin{array}{|c|}\hline 2\\\hline\end{array}^{\,2} \; \begin{array}{|c|}\hline 3\\\hline\end{array}^{\,1} & T^{(1\frac{1}{2})} \xleftarrow{RS} 3 \\
T^{(2\frac{1}{2})} & = \begin{array}{|c|c|c|c|}\hline 1&5&\cdots&n\\\hline\end{array}^{\,3} \; \begin{array}{|c|}\hline 2\\\hline\end{array}^{\,2} \; \begin{array}{|c|}\hline 3\\\hline\end{array}^{\,1} & T^{(2)} \xrightarrow{jdt} 4 \\
T^{(3)} & = \begin{array}{|c|c|c|c|}\hline 1&5&\cdots&n\\\hline\end{array}^{\,3} \; \begin{array}{|c|c|}\hline 2&4\\\hline\end{array}^{\,2} \; \begin{array}{|c|}\hline 3\\\hline\end{array}^{\,1} & T^{(2\frac{1}{2})} \xleftarrow{RS} 4 \\
T^{(3\frac{1}{2})} & = \begin{array}{|c|c|c|c|}\hline 1&5&\cdots&n\\\hline\end{array}^{\,3} \; \begin{array}{|c|c|}\hline 2&4\\\hline\end{array}^{\,2} & T^{(3)} \xrightarrow{jdt} 3 \\
T^{(4)} & = \begin{array}{|c|c|c|c|}\hline 1&5&\cdots&n\\\hline\end{array}^{\,3} \; \begin{array}{|c|c|}\hline 2&4\\\hline\end{array}^{\,2} \; \begin{array}{|c|}\hline 3\\\hline\end{array}^{\,0} & T^{(3\frac{1}{2})} \xleftarrow{RS} 3 \\
T^{(4\frac{1}{2})} & = \begin{array}{|c|c|c|c|}\hline 1&6&\cdots&n\\\hline\end{array}^{\,3} \; \begin{array}{|c|c|}\hline 2&4\\\hline\end{array}^{\,2} \; \begin{array}{|c|}\hline 3\\\hline\end{array}^{\,0} & T^{(4)} \xrightarrow{jdt} 5
\end{array}$$

$T^{(5)} \quad = \boxed{1}\boxed{6}\boxed{\quad\cdots\quad}\boxed{n}\ ^3\ \boxed{2}\boxed{4}\ ^2\ \boxed{5}\ ^1\boxed{3}\ ^0 \qquad T^{(4\frac{1}{2})} \xleftarrow{RS} 5$

$T^{(5)} \quad = T_{[\lambda]}$.

The associated vacillating tableau is $P_{[\lambda]} = ([n]^3, [n-1]^3, [n-1]^3[1]^2, [n-2]^3[1]^2, [n-2]^3[1]^2[1]^1, [n-3]^3[1]^2[1]^1, [n-3]^3[2]^2[1]^1, [n-3]^3[2]^2, [n-3]^3[2]^2[1]^0, [n-4]^3[2]^2[1]^0, [n-4]^3[2]^2[1]^1[1]^0)$.

Hence $((2,e),(3,g),(4,e),(3,e),(5,g)) \longrightarrow (T_{[\lambda]}, P_{[\lambda]})$ where $T_{[\lambda]}$ and $P_{[\lambda]}$ are as above.

Theorem 5.3. *The function* $((i_1,h_1),\ldots,(i_k,h_k)) \xrightarrow{Fw} (T_{[\lambda]}, P_{[\lambda]})$ *provides a bijection between k-tuples of pairs* $\{((i_1,h_1),\ldots,(i_k,h_k)) \xrightarrow{Fw} (T_{[\lambda]}, P_{[\lambda]}) \mid 1 \le i_j \le n, 1 \le j \le k, h_j \in G\}$ *and* $\bigsqcup_{[\lambda]\in\Gamma^G_{k,n}} ST^4([\lambda]) \times VT^4_k([\lambda])$ *and thus gives a combinatorial proof of the identity 5.1.*

Proof. We prove the theorem by constructing the inverse of $\xrightarrow{Fw}$. Let $[\lambda]^{(j+\frac{1}{2})} \subseteq [\lambda]^{(j+1)}$ with $[\lambda]^{(j+1)} \in \Gamma^G_{j+1,n}$ and $[\lambda]^{j+\frac{1}{2}} \in \Gamma^G_{j,n-1}$ and let $T^{(j+1)}$ be a standard tableau of shape $[\lambda]^{(j+1)}$. we can uniquely determine i_{j+1} together with a color h_{j+1} associated with i_{j+1} and a tableau $T^{(j+\frac{1}{2})}$ of shape $[\lambda]^{(j+\frac{1}{2})}$ such that $T^{(j+1)} = (i_{j+1}) \xrightarrow{RS} T^{(j+\frac{1}{2})}$. To do this let b be the box in $[\lambda]^{(j+1)}/[\lambda]^{(j+\frac{1}{2})}$. Let i_{j+1} and $T^{(j+\frac{1}{2})}$ be the result of applying reverse RS algorithm on the number in box b of $T^{(j+1)}$. Now let $T^{(j+\frac{1}{2})}$ be a tableau of shape $[\lambda]^{(j+\frac{1}{2})} \in \Gamma^G_{j,n-1}$ with increasing rows and columns and entries $\{1,2,\ldots,n\}\backslash\{i_{j+1}\}$ and let $[\lambda]^{(j)} \subseteq [\lambda]^{(j+\frac{1}{2})}$ with $[\lambda]^{(j)} \in \Gamma^G_{j,n}$. We can uniquely produce a standard tableau $T^{(j)}$ such that $T^{(j+\frac{1}{2})} = (i_{j+1} \xrightarrow{jdt} T^{(j)})$. To do this, let b be the box in $[\lambda]^{(j+1)}/[\lambda]^{(j+\frac{1}{2})}$ put i_{j+1} in position b of $T^{(j+\frac{1}{2})}$ and perform the reverse jeu de taquin to produce $T^{(j)}$, i.e., move i_{j+1} into a standard position by iteratively swapping it with larger of the numbers just above it or just to its left. We associate a color h_{j+1} to i_{j+1} as follows:

1. If i_{j+1} is removed from 3 residue in $T^{(j+1)}$ and
   (a) placed in 2 residue in $T^{(j)}$ then $h_{j+1} = e$
   (b) placed in 1 residue in $T^{(j)}$ then $h_{j+1} = g$
2. If i_{j+1} is removed from 2 residue in $T^{(j+1)}$ and
   (a) placed in 3 residue in $T^{(j)}$ then $h_{j+1} = e$
   (b) placed in 0 residue in $T^{(j)}$ then $h_{j+1} = g$
3. If i_{j+1} is removed from 1 residue in $T^{(j+1)}$ and
   (a) placed in 3 residue in $T^{(j)}$ then $h_{j+1} = g$
   (b) placed in 0 residue in $T^{(j)}$ then $h_{j+1} = e$
4. If i_{j+1} is removed from 0 residue in $T^{(j+1)}$ and
   (a) placed in 2 residue in $T^{(j)}$ then $h_{j+1} = g$

(b) placed in 1 residue in $T^{(j)}$ then $h_{j+1} = e$

Given $[\lambda] \in \Gamma^G_{k,n}$ and $(T_{[\lambda]}, P_{[\lambda]}) \in ST^4([\lambda]) \times VT^4_k([\lambda])$ we apply the above process to $[\lambda]^{(k-\frac{1}{2})} \subseteq [\lambda]^{(k)}$ and $T^{(k)} = T_{[\lambda]}$ producing (i_k, h_k) and $T^{(k-1)}$ continuing this way we can produce $((i_1, h_1), \ldots, (i_k, h_k))$ and $T^{(k)}, T^{(k-1)}, \ldots, T^{(1)}$ such that $((i_1, h_1), \ldots, (i_k, h_k))$ $\xrightarrow{Fw} (T_{[\lambda]}, P_{[\lambda]})$

$\square$

We illustrate the reverse algorithm below.

Let $T_{[\lambda]} = \begin{ytableau} 1 & 9 & \cdots & \text{n-1} \\ 2 \end{ytableau}^3 \begin{ytableau} 3 & 8 \\ 7 \end{ytableau}^2 \begin{ytableau} 4 & \text{n} \end{ytableau}^1 \begin{ytableau} 5 \\ 6 \end{ytableau}^0$ and

$P_{[\lambda]} = ([n]^3, [n-1]^3, [n-1]^3[1]^2, [n-1]^3, [n-1,1]^3, [n-2,1]^3, [n-2,1]^3[1]^1, [n-3,1]^3[1]^1,$ $[n-3,1]^3[2]^1, [n-4,1]^3[2]^1, [n-4,1]^3[3]^1, [n-4,1]^3[2]^1, [n-4,1]^3[2]^1[1]^0,$ $[n-4,1]^3[1]^1[1]^0, [n-4,1]^3[1]^1[1^2]^0, [n-5,1]^3[1]^1[1^2]^0, [n-5,1]^3[2]^1[1^2]^0,$ $[n-6,1]^3[2]^1[1^2]^0, [n-6,1]^3[1]^2[2]^1[1^2]^0, [n-7,1]^3[1]^2[2]^1[1^2]^0, [n-7,1]^3[2]^2[2]^1[1^2]^0,$ $[n-8,1]^3[2]^2[2]^1[1^2]^0, [n-8,1]^3[2,1]^2[2]^1[1^2]^0)$

$$T^{(11)} = \begin{ytableau} 1 & 9 & \cdots & \text{n-1} \\ 2 \end{ytableau}^3 \quad \begin{ytableau} 3 & 8 \\ 7 \end{ytableau}^2 \quad \begin{ytableau} 4 & \text{n} \end{ytableau}^1 \quad \begin{ytableau} 5 \\ 6 \end{ytableau}^0$$

$$T^{(10\frac{1}{2})} = \begin{ytableau} 1 & 9 & \cdots & \text{n-1} \\ 2 \end{ytableau}^3 \quad \begin{ytableau} 7 & 8 \end{ytableau}^2 \quad \begin{ytableau} 4 & \text{n} \end{ytableau}^1 \quad \begin{ytableau} 5 \\ 6 \end{ytableau}^0 \quad \xrightarrow{RS^{-1}} 3$$

$$T^{(10)} = \begin{ytableau} 1 & 3 & 9 & \cdots & \text{n-1} \\ 2 \end{ytableau}^3 \quad \begin{ytableau} 7 & 8 \end{ytableau}^2 \quad \begin{ytableau} 4 & \text{n} \end{ytableau}^1 \quad \begin{ytableau} 5 \\ 6 \end{ytableau}^0 \quad \xleftarrow{jdt^{-1}} 3 \quad (3, e)$$

$$T^{(9\frac{1}{2})} = \begin{ytableau} 1 & 3 & 9 & \cdots & \text{n-1} \\ 2 \end{ytableau}^3 \quad \begin{ytableau} 7 \end{ytableau}^2 \quad \begin{ytableau} 4 & \text{n} \end{ytableau}^1 \quad \begin{ytableau} 5 \\ 6 \end{ytableau}^0 \quad \xrightarrow{RS^{-1}} 8$$

$$T^{(9)} = \begin{ytableau} 1 & 3 & 8 & 9 & \cdots & \text{n-1} \\ 2 \end{ytableau}^3 \quad \begin{ytableau} 7 \end{ytableau}^2 \quad \begin{ytableau} 4 & \text{n} \end{ytableau}^1 \quad \begin{ytableau} 5 \\ 6 \end{ytableau}^0 \quad \xleftarrow{jdt^{-1}} 8 \quad (8, e)$$

$$T^{(8\frac{1}{2})} = \begin{ytableau} 1 & 3 & 8 & 9 & \cdots & \text{n-1} \\ 2 \end{ytableau}^3 \quad \begin{ytableau} 4 & \text{n} \end{ytableau}^1 \quad \begin{ytableau} 5 \\ 6 \end{ytableau}^0 \quad \xrightarrow{RS^{-1}} 7$$

$$T^{(8)} = \begin{ytableau} 1 & 3 & 7 & 8 & \cdots & \text{n-1} \\ 2 \end{ytableau}^3 \quad \begin{ytableau} 4 & \text{n} \end{ytableau}^1 \quad \begin{ytableau} 5 \\ 6 \end{ytableau}^0 \quad \xleftarrow{jdt^{-1}} 7 \quad (7, e)$$

$$T^{(7\frac{1}{2})} = \begin{ytableau} 1 & 3 & 7 & 8 & \cdots & \text{n-1} \\ 2 \end{ytableau}^3 \quad \begin{ytableau} 4 \end{ytableau}^1 \quad \begin{ytableau} 5 \\ 6 \end{ytableau}^0 \quad \xrightarrow{RS^{-1}} n$$

$$T^{(7)} = \begin{ytableau} 1 & 3 & 7 & \cdots & \text{n-1} & \text{n} \\ 2 \end{ytableau}^3 \quad \begin{ytableau} 4 \end{ytableau}^1 \quad \begin{ytableau} 5 \\ 6 \end{ytableau}^0 \quad \xleftarrow{jdt^{-1}} n \quad (n, g)$$

$$T^{(6\frac{1}{2})} = \begin{ytableau} 1 & 3 & 7 & \cdots & \text{n-1} & \text{n} \\ 2 \end{ytableau}^3 \quad \begin{ytableau} 4 \end{ytableau}^1 \quad \begin{ytableau} 6 \end{ytableau}^0 \quad \xrightarrow{RS^{-1}} 5$$

$$T^{(6)} = \begin{ytableau} 1 & 3 & 7 & \cdots & \text{n-1} & \text{n} \\ 2 \end{ytableau}^3 \quad \begin{ytableau} 4 & 5 \end{ytableau}^1 \quad \begin{ytableau} 6 \end{ytableau}^0 \quad \xleftarrow{jdt^{-1}} 5 \quad (5, e)$$

$$T^{(5\frac{1}{2})} = \begin{array}{|c|c|c|c|c|c|}\hline 1&3&7&\cdots&\text{n-1}&\text{n}\\\hline 2\\\cline{1-1}\end{array}^{3} \quad \begin{array}{|c|c|}\hline 4&5\\\hline\end{array}^{1} \quad \xrightarrow{RS^{-1}} 6$$

$$T^{(5)} = \begin{array}{|c|c|c|c|c|c|}\hline 1&3&7&\cdots&\text{n-1}&\text{n}\\\hline 2\\\cline{1-1}\end{array}^{3} \quad \begin{array}{|c|c|c|}\hline 4&5&6\\\hline\end{array}^{1} \quad \xleftarrow{jdt^{-1}} 6 \quad (6,e)$$

$$T^{(4\frac{1}{2})} = \begin{array}{|c|c|c|c|c|c|}\hline 1&3&7&\cdots&\text{n-1}&\text{n}\\\hline 2\\\cline{1-1}\end{array}^{3} \quad \begin{array}{|c|c|}\hline 4&5\\\hline\end{array}^{1} \quad \xrightarrow{RS^{-1}} 6$$

$$T^{(4)} = \begin{array}{|c|c|c|c|c|c|}\hline 1&3&6&\cdots&\text{n-1}&\text{n}\\\hline 2\\\cline{1-1}\end{array}^{3} \quad \begin{array}{|c|c|}\hline 4&5\\\hline\end{array}^{1} \quad \xleftarrow{jdt^{-1}} 6 \quad (6,g)$$

$$T^{(3\frac{1}{2})} = \begin{array}{|c|c|c|c|c|c|}\hline 1&3&6&\cdots&\text{n-1}&\text{n}\\\hline 2\\\cline{1-1}\end{array}^{3} \quad \begin{array}{|c|}\hline 4\\\hline\end{array}^{1} \quad \xrightarrow{RS^{-1}} 5$$

$$T^{(3)} = \begin{array}{|c|c|c|c|c|c|}\hline 1&3&5&\cdots&\text{n-1}&\text{n}\\\hline 2\\\cline{1-1}\end{array}^{3} \quad \begin{array}{|c|}\hline 4\\\hline\end{array}^{1} \quad \xleftarrow{jdt^{-1}} 5 \quad (5,g)$$

$$T^{(2\frac{1}{2})} = \begin{array}{|c|c|c|c|c|c|}\hline 1&3&5&\cdots&\text{n-1}&\text{n}\\\hline 2\\\cline{1-1}\end{array}^{3} \quad \xrightarrow{RS^{-1}} 4$$

$$T^{(2)} = \begin{array}{|c|c|c|c|c|c|}\hline 1&3&4&\cdots&\text{n-1}&\text{n}\\\hline 2\\\cline{1-1}\end{array}^{3} \quad \xleftarrow{jdt^{-1}} 4 \quad (4,g)$$

$$T^{(1\frac{1}{2})} = \begin{array}{|c|c|c|c|c|c|}\hline 2&3&4&\cdots&\text{n-1}&\text{n}\\\hline\end{array}^{3} \quad \xrightarrow{RS^{-1}} 1$$

$$T^{(1)} = \begin{array}{|c|c|c|c|c|c|}\hline 2&3&4&\cdots&\text{n-1}&\text{n}\\\hline\end{array}^{3} \quad \begin{array}{|c|}\hline 1\\\hline\end{array}^{2} \quad \xleftarrow{jdt^{-1}} 1 \quad (1,e)$$

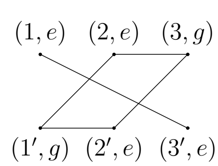
$$T^{(\frac{1}{2})} = \begin{array}{|c|c|c|c|c|c|}\hline 2&3&4&\cdots&\text{n-1}&\text{n}\\\hline\end{array}^{3} \quad \xrightarrow{RS^{-1}} 1$$

$$T^{(0)} = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&\cdots&\text{n-1}&\text{n}\\\hline\end{array}^{3} \quad \xleftarrow{jdt^{-1}} 1 \quad (1,e)$$

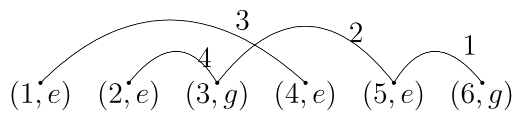
$$(T_{[\lambda]}, P_{[\lambda]}) \xrightarrow{Bw} ((1,e),(1,)e,(4,g),(5,g),(5,g),(6,e),(5,e),(n,g),(7,e),(8,e),(3,e)$$

6 RS correspondence for Klein-4 diagram algebras

For $n \geq 2k, \widehat{EP}_k(n) = R_k(n)$. Following notation 2.3, we will draw diagrams $d \in \Pi_k$ using a standard representation as single row with vertices in order $((1,h_1),\ldots,$ $(2k,h_{2k}))$ where we relabel the vertex $(j',h_{j'})$ with the label $(2k-j+1,h_{j'})$. We draw the edges of the standard representation of $d \in \Pi_k$ in a specific way. Connect the vertices (i,h_i) and (j,h_j) with $i \leq j$ if and only if (i,h_i) and (j,h_j) are related in d and there does not exists (l,h_l) related to (i,h_i) and (j,h_j) with $i < l < j$. In this way each vertex is connected only to its nearest neighbors in its block. For example, the following diagram $d \in \Pi_3$



has a standard one line representation given as follows:



We label each edge e of the diagram d with $2k+1-l$, where l is the right vertex of e. We define the insertion sequence of a diagram to be the sequence $E = (E_j, h_j)$ indexed by j in the sequence $\frac{1}{2}, 1, 1\frac{1}{2}, 2, \ldots, 2k-1, 2k-\frac{1}{2}, 2k$.

$$(E_j, h_j) = \begin{cases} (a, h_j), & \text{if } (j, h_j) \text{ is the left end point of edge } a; \\ \emptyset, & \text{if } (j, h_j) \text{ is not a left end point of edge } a. \end{cases}$$

$$(E_{j-\frac{1}{2}}, h_{j-\frac{1}{2}}) = \begin{cases} (a, h_{j-\frac{1}{2}}), & \text{if } (j, h_j) \text{ is the right end point of edge } a; \\ \emptyset, & \text{if } (j, h_j) \text{ is not a right end point of edge } a. \end{cases}$$

In the running example the insertion sequence is

j	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6
(E_j, h_j)	$\emptyset$	$(3, e)$	$\emptyset$	$(4, e)$	$(4, g)$	$(2, g)$	$(3, e)$	$\emptyset$	$(2, e)$	$(1, e)$	$(1, g)$	$\emptyset$

The insertion sequence of standard representation of a diagram completely determines the edges and thus the confected components of the diagram. For $d \in \Pi_k$ with insertion sequence $E = (E_j, h_j)$ we will produce a pair $(P_{[\lambda]}, Q_{[\lambda]})$ of vacillating tableaux. Begin with empty tableaux

$$T^{(0)} = \emptyset.$$

Then recursively define standard tableaux $T^{(j+\frac{1}{2})}$ and $T^{(j+1)}$ as follows

Then recursively define standard tableaux $T^{(j+\frac{1}{2})}$ and $T^{(j+1)}$ as follows: The number $E_{j+\frac{1}{2}}$ is removed from the tableau $T^{(j)}$ by the process of applying *jeu de taquin* on the residue in which it appears

$$T^{(j+\frac{1}{2})} = \begin{cases} E_{j+\frac{1}{2}} \xleftarrow{jdt} T^{(j)}, & \text{if } E_{j+\frac{1}{2}} \neq \emptyset; \\ T^{(j)}, & \text{if } E_{j+\frac{1}{2}} = \emptyset. \end{cases}$$

The process of insertion is as follows:

$$T^{(j+1)} = \begin{cases} E_{j+1} \xrightarrow{RS} T^{(j+\frac{1}{2})}, & \text{if } E_{j+1} \neq \emptyset \text{ (as given below)}; \\ T^{(j+\frac{1}{2})}, & \text{if } E_{j+1} = \emptyset. \end{cases}$$

Let

$$(E_{j+1}, h_{j+1}) \xrightarrow{RS} [T^{(j+\frac{1}{2})}]^m$$

denote the insertion of E_{j+1} into the m^{th} residue of $T^{(j+\frac{1}{2})}$ when the other parts of the tableau (i.e., in the other residues) remain unchanged.

If $E_{j+1} \neq \emptyset$, then

$$T^{(j+1)} = \begin{cases} E_{j+1} \xrightarrow{RS} [T^{(j+1)}]^3, & \text{if } h_{j+1} = e \text{ and } E_{j+\frac{1}{2}} \text{ was removed from 2-residue;} \\ E_{j+1} \xrightarrow{RS} [T^{(j+1)}]^3, & \text{if } h_{j+1} = g \text{ and } E_{j+\frac{1}{2}} \text{ was removed from 1-residue;} \\ E_{j+1} \xrightarrow{RS} [T^{(j+1)}]^1, & \text{if } h_{j+1} = g \text{ and } E_{j+\frac{1}{2}} \text{ was removed from 3-residue;} \\ E_{j+1} \xrightarrow{RS} [T^{(j+1)}]^1, & \text{if } h_{j+1} = e \text{ and } E_{j+\frac{1}{2}} \text{ was removed from 0-residue;} \\ E_{j+1} \xrightarrow{RS} [T^{(j+1)}]^2, & \text{if } h_{j+1} = e \text{ and } E_{j+\frac{1}{2}} \text{ was removed from 3-residue;} \\ E_{j+1} \xrightarrow{RS} [T^{(j+1)}]^2, & \text{if } h_{j+1} = g \text{ and } E_{j+\frac{1}{2}} \text{ was removed from 0-residue;} \\ E_{j+1} \xrightarrow{RS} [T^{(j+1)}]^0, & \text{if } h_{j+1} = g \text{ and } E_{j+\frac{1}{2}} \text{ was removed from 2-residue;} \\ E_{j+1} \xrightarrow{RS} [T^{(j+1)}]^0, & \text{if } h_{j+1} = e \text{ and } E_{j+\frac{1}{2}} \text{ was removed from 1-residue;} \end{cases}$$

Let $[\lambda]^{(i)}$ be the shape of $T^{(i)}$ and let $[\lambda]^{(i+\frac{1}{2})}$ be the shape of $T^{(i+\frac{1}{2})}$ and let $[\lambda] = [\lambda]^{(k)}$. Define

$$Q_{[\lambda]} = (\emptyset, [\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, \ldots, [\lambda]^{(k-\frac{1}{2})}, [\lambda]^{(k)}) \in VT_k^4([\lambda])$$

$$P_{[\lambda]} = (\emptyset, [\lambda]^{(2k)}, [\lambda]^{(2k-\frac{1}{2})}, \ldots, [\lambda]^{(k+\frac{1}{2})}, [\lambda]^{(k)}) \in VT_k^4([\lambda])$$

In this way we associate a pair of vacillating tableaux $(P_{[\lambda]}, Q_{[\lambda]})$ to a diagram $d \in \Pi_k$ which we denote by $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$. We explain the construction with our running example. For the insertion sequence

j	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6
(E_j, h_j)	$\emptyset$	$(3,e)$	$\emptyset$	$(4,e)$	$(4,g)$	$(2,g)$	$(3,e)$	$\emptyset$	$(2,e)$	$(1,e)$	$(1,g)$	$\emptyset$

the pair of vacillating tableau is given by

j	(E_j, h_j)		$T^{(j)}$	j	(E_j, h_j)		$T^{(j)}$
0			$\emptyset$	6	$\emptyset$	$\xrightarrow{RS}$	$\emptyset$
$\frac{1}{2}$	$\emptyset$	$\xleftarrow{jdt}$	$\emptyset$	$5\frac{1}{2}$	$(1,g)$	$\xleftarrow{jdt}$	$\emptyset$
1	$(3,e)$	$\xrightarrow{RS}$	$\boxed{3}^{\,2}$	5	$(1,e)$	$\xrightarrow{RS}$	$\boxed{1}^{\,1}$
$1\frac{1}{2}$	$\emptyset$	$\xleftarrow{jdt}$	$\boxed{3}^{\,2}$	$4\frac{1}{2}$	$(2,e)$	$\xleftarrow{jdt}$	$\emptyset$
2	$(4,e)$	$\xrightarrow{RS}$	$\boxed{3}\boxed{4}^{\,2}$	4	$\emptyset$	$\xrightarrow{RS}$	$\boxed{2}^{\,0}$
$2\frac{1}{2}$	$(4,g)$	$\xleftarrow{jdt}$	$\boxed{3}^{\,2}$	$3\frac{1}{2}$	$(3,e)$	$\xleftarrow{jdt}$	$\boxed{2}^{\,0}$
3	$(2,g)$	$\xrightarrow{RS}$	$\boxed{3}^{\,2}\ \boxed{2}^{\,0}$	3			$\boxed{3}^{\,2}\ \boxed{2}^{\,0}$

$$P_{[\lambda]} = \left(\emptyset, \emptyset, \square^{\,1}, \emptyset, \square^{\,0}, \square^{\,0}, \square^{\,0}\square^{\,2}\right)$$

$$Q_{[\lambda]} = \left(\emptyset, \emptyset, \square^{\,2}, \square^{\,2}, \square\square^{\,2}, \square^{\,2}, \square^{\,2}\square^{\,0}\right).$$

We have numbered the edges of the standard diagram of d in increasing order from right to left so if $E_{j+\frac{1}{2}} \neq \emptyset$ then $E_{j+\frac{1}{2}}$ will be the largest element of $T^{(j)}$, then in $T^{(j+\frac{1}{2})} = (E_{j+\frac{1}{2}} \xleftarrow{jdt} T^{(j)})$ we know that $E_{j+\frac{1}{2}}$ is in a corner box and jeu de taquin simple deletes the box.

Theorem 6.1. *The function $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$ provides a bijection between the set of diagrams in Π_k and pairs of vacillating tableaux in $\bigsqcup_{[\lambda]\in\Gamma^G_{k,n}} VT^4_k([\lambda]) \times VT^4_k([\lambda])$ and this gives a combinatorial proof of identity $d_k = \sum_{[\lambda]\in\ \Gamma^G_{k,n}} (m_k^{[\lambda]})^2$.*

Proof. We prove the theorem by constructing the inverse of $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$. First we use $Q_{[\lambda]}$ followed by $P_{[\lambda]}$ in the reverse order to construct the sequence

$$[\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, \ldots, [\lambda]^{(2k-\frac{1}{2})}, [\lambda]^{(2k)}.$$

We initialize $T^{(2k)} = \emptyset$. We now show how to construct $T^{(i+\frac{1}{2})}$ and E_{i+1} so that $T^{(i+1)} = (E_{i+1} \xrightarrow{RS} T^{(i+\frac{1}{2})})$. If $[\lambda]^{i+\frac{1}{2}} = [\lambda]^{i+1}$ then let $T^{(i+\frac{1}{2})} = T^{(i+1)}$ and $E_{i+1} = \emptyset$, otherwise $[\lambda]^{i+1}/[\lambda]^{i+\frac{1}{2}}$ is a box b and we use the reverse RS algorithm on the value in b to produce $T^{(i+\frac{1}{2})}$ and E_{i+1} such that $T^{(i+1)} = (E_{i+1} \xrightarrow{RS} T^{(i+\frac{1}{2})})$. Since we un-inserted the value in position b, we know that $T^{(i+\frac{1}{2})}$ has shape $[\lambda]^{i+\frac{1}{2}}$. For fixing h_{i+1} we follow the rules given below:

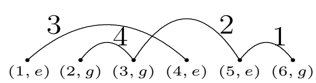
1. If box b lies in 2 residue then, $h_{i+1} = e$
2. If box b lies in 1 residue then, $h_{i+1} = g$
3. If box b lies in 0 residue then,
   (a) If $h_{i-1} = e$ then $h_{i+1} = g$,
   (b) If $h_{i-1} = g$ then $h_{i+1} = e$,
4. If box b lies in 3 residue then,
   (a) If $h_{i-1} = e$ then $h_{i+1} = e$,
   (b) If $h_{i-1} = g$ then $h_{i+1} = g$,

We then show how to construct $T^{(i)}$ and $E_{i+\frac{1}{2}}$ so that $T^{(i+\frac{1}{2})} = (E_{i+\frac{1}{2}} \xleftarrow{jdt} T^{(i)})$. If $[\lambda]^{(i)} = [\lambda]^{(i+\frac{1}{2})}$ then let $T^{(i)} = T^{(i+\frac{1}{2})}$ and $E_{i+\frac{1}{2}}$. Otherwise $[\lambda]^{(i)}/[\lambda]^{(i+\frac{1}{2})}$ is a box b let $T^{(i)}$ be the tableau of shape $[\lambda]^{(i)}$ with the same entries as $T^{(i+1)}$ and having the entry $2k-i$. At any step $i, 2k-i$ is the largest value added to the tableau this far. So $T^{(i)}$ is standard. Furthermore $T^{(i+\frac{1}{2})} = (E_{i+\frac{1}{2}} \xleftarrow{jdt} T^{(i)})$, since $E_{i+\frac{1}{2}} = 2k-i$ is already a corner and thus jeu de taquin simply deletes it. Thus the iterative procedure will produce $E_{2k}, E_{2k-\frac{1}{2}}, \ldots, E_{\frac{1}{2}}$ which completely determines the diagram d. By the way we have constructed d we have $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$.

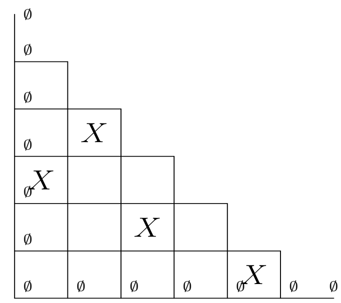
□

Notice that in the standard one line representation of the diagram d a flip corresponds to a reflection over the vertical line between the vertices k and $k+1$. We will show that if $d \longrightarrow (P, Q)$ then $flip(d) \longrightarrow (Q, P)$. Given a diagram $d \in R_k(n)$ we construct a triangular grid (as in the case of partition diagrams) whose vertices are $\{((i, h_i), (j, h_j))$

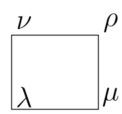
where $i, j \in \underline{k}$ and h_i, h_j are the corresponding colors associated with the vertices. Number the columns $1, \ldots, 2k$ from left to right and the rows from bottom to top. Place an X in the box in column i and row j if and only if , in the one row diagram of d the vertex (i, h_i) is the left end point of edge j. We can then label the vertices of the diagram on the bottom row and the left column with the empty partition $\emptyset$. For example, in the running example



$\longleftrightarrow$



Now we inductively label the remaining vertices using local rules of Fomin (as in the case of partition diagrams.) If a box is labeled with $[\mu], [\nu], [\lambda]$ then we add the label $[\rho]$ according to the rules given below:



LR1 If $[\mu] \neq [\nu]$ let $[\rho] = [\mu] \cup [\nu]$ i.e., for each residue j we have $\rho^j = \mu^j \cup \nu^j$ i.e., $\rho^j = max(\mu_i^j, \nu_i^j)$

LR2 If $[\mu] \neq [\nu], [\lambda] \subset [\mu]$ and $[\lambda] \neq [\mu]$ then this will automatically imply that $[\mu]$ can be obtained from $[\lambda]$ by adding a box to the i^{th} in the j^{th} residue λ_i^j of $[\lambda]$. Let $[\rho]$ be obtained from $[\mu]$ by adding a box to μ_{i+1}^j

LR3 If $[\mu] = [\nu] = [\lambda]$ then if the square does not contain X let $[\rho] = [\lambda]$ and if the square does contain X let $[\rho]$ be obtained from $[\lambda]$ by adding 1 to λ_1^i i.e., $[\lambda]_1$ of the i^{th} residue by following the rules given below:

1. If i_{j+1} is removed from 3-residue and
   (a) $h_{j+1} = e$ then insert i_{j+1} in the 2 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.
   (b) $h_{j+1} = g$ then insert i_{j+1} in the 1 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.
2. If i_{j+1} is removed from 2-residue and

(a) $h_{j+1} = e$ then insert i_{j+1} in the 3 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

(b) $h_{j+1} = g$ then insert i_{j+1} in the 0 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

3. If i_{j+1} is removed from 1-residue and

   (a) $h_{j+1} = e$ then insert i_{j+1} in the 0 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

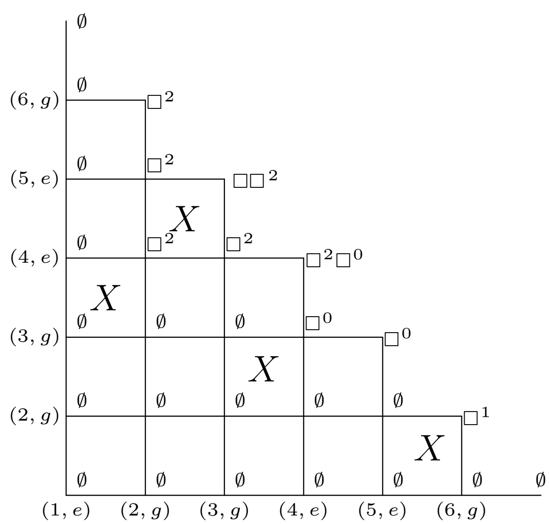
   (b) $h_{j+1} = g$ then insert i_{j+1} in the 3 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

4. If i_{j+1} is removed from 0-residue and

   (a) $h_{j+1} = e$ then insert i_{j+1} in the 2 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

   (b) $h_{j+1} = g$ then insert i_{j+1} in the 1 residue of $T^{(j+\frac{1}{2})}$ to obtain $T^{(j+1)}$ using RS algorithm.

Using these rules we can uniquely label every corner one step at a time and the resulting diagram is called the growth diagram Gr_d for the diagram d. The growth diagram Gr_d for the running example diagram d is



Let P_d denote the chain of partitions that follow the staircase paths on the diagonal of Gr_d from $(0, 2k)$ to (k, k) and let Q_d denote the chain of partitions that follows the staircase path on the diagonal of Gr_d from $(2k, 0)$ to (k, k) the pair (P_d, Q_d) represent a pair of vacillating tableau whose shape is the partition at (k, k).

Theorem 6.2. *Let $d \in \Pi_k$ with $d \to (P, Q)$ then $P_d = P$ and $Q_d = Q$.*

Proof. Proof follows as in the case of the partition diagrams as given in [HL], we give the proof for the sake of completion. The proof follows from the standard representation of d. A key advantage of the use of growth diagram is that the symmetry of the algorithm is nearly obvious. We have that in the left end point of edge labeled j in d if and only if j is the left endpoint of edge labeled in $flip(d)$. Then this growth diagram of Gr_d is the reflection over the line y=x of the growth diagram of $Gr_{flip(d)}$ and so

$$P_d = Q_{flip(d)} \text{ and } Q_d = P_{flip(d)}.$$

□

Corollary 6.3. If $d \to (P, Q)$ then $flip(d) \to (Q, P)$.

Corollary 6.4. A diagram $d \in \Pi_k$ is symmetric if and only if $d \to (P, P)$.

Proof. Proof follows as in the case of the partition diagrams as given in [HL], we give the proof for the sake of completion. If d is symmetric then by corollary we must have $P = Q$. To prove the converse we place the vacillating tableau on the stair case border of a growth diagram. The local rules are invertible. Given μ, ν and ρ one can trace the rules backwards uniquely to find $[\lambda]$ and determine whether there is an X in the box. Thus the interior of the growth diagram is uniquely determined. By the symmetry of having $P = Q$ along the staircase, the growth diagram must have a symmetric interior and a symmetric placement of the $X's$. This forces d to be symmetric. □

The corollary tells us that the number of symmetric diagrams in $R_k(n)$ is equal to the number of vacillating tableau of length $2k$ or the number of paths to level k in the Bratteli diagram $R_k(n)$.

6.1 Applications of R-S Correspondence

We define the Knuth relation for the diagrams in Π_k.

We can write each $d \in \Pi_k$ as a 6-tuple $d = [U(d), L(d), w_0, w_1, w_2, w_3]$. The G-vertex colored partition of k in the first row is $U(d)$ and the G-vertex colored partition of k in the second row is $L(d)$. Given a diagram $d \in \Pi_k$ it is made up of classes completely lying in $U(d)$ or completely lying in $L(d)$ or through classes. We distinguish the following types of through classes.

1. Through classes having even number of vertices with labels e in the first row and even number of vertices with labels e in the second row together with odd number of vertices with label g in the first row and odd number of vertices with label g in the second row.

2. Through classes having even number of vertices with labels e in the first row and even number of vertices with labels e in the second row together with even number of vertices with label g in the first row and even number of vertices with label g in the second row.

3. Through classes having odd number of vertices with labels e in the first row and odd number of vertices with labels e in the second row together with odd number of vertices with label g in the first row and odd number of vertices with label g in the second row.

4. Through classes having odd number of vertices with labels e in the first row and odd number of vertices with labels e in the second row together with even number of vertices with label g in the first row and even number of vertices with label g in the second row.

We write the diagrams in the standard form as follows, for any through class, an edge is drawn form the right most vertex in first row to the right most vertex in the second row. We now look at the generalized permutations arising from classes of each type.

Lemma 6.5. *Let $d \in \Pi_k$ have a through class of*

1. *type 1 then a node is added in the 1-residue in the final shape $[\lambda]$.*

2. *type 2 then a node is added in the 3-residue in the final shape $[\lambda]$.*

3. *type 3 then a node is added in the 0-residue in the final shape $[\lambda]$.*

4. *type 4 then a node is added in the 2-residue in the final shape $[\lambda]$.*

Proof. We prove the lemma by induction on k. For $k = 1$, the diagrams are
CASE 1.

The insertion sequence is

j	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
(E_j, h_j)	$\emptyset$	$\emptyset$	$(1, e)$	$(1, e)$	$\emptyset$

The pair of vacillating tableaux is

j	(E_j, h_j)		$T^{(j)}$	j	(E_j, h_j)		$T^{(j)}$
0			$\emptyset$	2	$\emptyset$	$\xrightarrow{RS}$	$\emptyset$
$\frac{1}{2}$	$\emptyset$	$\xleftarrow{jdt}$	$\emptyset$	$1\frac{1}{2}$	$(1, e)$	$\xleftarrow{jdt}$	$\emptyset$
1	$(1, e)$	$\xrightarrow{RS}$	$\boxed{1}\ ^{2}$	1	$(1, e)$	$\xrightarrow{RS}$	$\boxed{1}\ ^{2}$

CASE 2.

The insertion sequence is

j	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
(E_j, h_j)	$\emptyset$	$\emptyset$	$(1, g)$	$(1, g)$	$\emptyset$

The pair of vacillating tableaux is

j	(E_j, h_j)		$T^{(j)}$	j	(E_j, h_j)		$T^{(j)}$
0			$\emptyset$	2	$\emptyset$	$\xrightarrow{RS}$	$\emptyset$
$\frac{1}{2}$	$\emptyset$	$\xleftarrow{jdt}$	$\emptyset$	$1\frac{1}{2}$	$(1, g)$	$\xleftarrow{jdt}$	$\emptyset$
1	$(1, g)$	$\xrightarrow{RS}$	$\boxed{1}\ ^{1}$	1	$(1, g)$	$\xrightarrow{RS}$	$\boxed{1}\ ^{1}$

Let $d' \in \Pi_k$. Let d' be obtained from d by addition of the vertices $k+1$ in the first row and $k+2$ in the second row. Now the lower row vertices of d' are renamed as $j := j+2$. We show the labeling of the vertices here.

$$d = \begin{array}{cccc} 1 & 2 & & k \\ \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \cdots & \bullet \\ 2k & 2k-1 & & k+1 \end{array}$$

$$d' = \begin{array}{ccccc} 1 & 2 & & k & k+1 \\ \bullet & \bullet & \cdots & \bullet & \bullet \\ \bullet & \bullet & \cdots & \bullet & \bullet \\ 2k+2 & 2k+1 & & k+3 & k+2 \end{array}$$

The effect of adding the above edge to the diagram d is as follows: In the insertion sequence for

1. $1 \leq j \leq k$ we have $j' = j$ and $E_{j'} = E_j + 2$
2. $k + 2\frac{1}{2} \leq j \leq 2k+2$ we have $j' = j+2$ and $E_{j'} = E_j$
3.

j'	$k + \frac{1}{2}$	$k+1$	$k + 1\frac{1}{2}$	$k+2$
$(E_{j'}, h_{j'})$	$(k+2, h_{k+1})$	$(k+1, h_{k+1})$	$(k+1, h_{k+2})$	(k, h_{k+2})

We have the following cases for h_{k+1} and h_{k+2}

1. If the class to which the new edge is added is of type 1 and
   (a) if $h_{k+1} = e$ and $h_{k+2} = e$, then a node is added in the 0-residue .
   (b) if $h_{k+1} = g$ and $h_{k+2} = g$, then a node is added in the 3-residue.
2. If the class to which the new edge is added is of type 2 and
   (a) if $h_{k+1} = e$ and $h_{k+2} = e$, then a node is added in the 2-residue.
   (b) if $h_{k+1} = g$ and $h_{k+2} = g$, then a node is added in the 1-residue.
3. If the class to which the new edge is added is of type 2 and

(a) if $h_{k+1} = e$ and $h_{k+2} = e$, then a node is added in the 1-residue.

(b) if $h_{k+1} = g$ and $h_{k+2} = g$, then a node is added in the 2-residue.

4. If the class to which the new edge is added is of type 3 and

   (a) if $h_{k+1} = e$ and $h_{k+2} = e$, then a node is added in the 3-residue.

   (b) if $h_{k+1} = g$ and $h_{k+2} = g$, then a node is added in the 0-residue.

using the rules for the R-S correspondence for the Klein-4 diagram algebras given in section 6. □

Notation 6.1. By the above lemma, the generalized permutation arising out of through classes of type 1 correspond to insertion in 1-residue and we denote the generalized permutation by w_1. The generalized permutation arising out of through classes of type 2 correspond to insertion in 3-residue and we denote the generalized permutation by w_3. The generalized permutation arising out of through classes of type 3 correspond to insertion in 0-residue and we denote the generalized permutation by w_0. The generalized permutation arising out of through classes of type 4 correspond to insertion in 2-residue and we denote the generalized permutation by w_2.

Definition 6.6. Knuth relation

Let $d, d' \in \Pi_k$. The two vertex colored diagrams are said to be Knuth related if

1. $L(d) = L(d')$
2. As generalized permutations $w_0 \underset{K}{\sim} w'_0, w_1 \underset{K}{\sim} w'_1, w_2 \underset{K}{\sim} w'_2, w_3 \underset{K}{\sim} w'_3$

We denote this relation by $d \underset{K}{\sim} d'$

Notation 6.2.

1. $K_{[\lambda]}(d) = \{d' \in \Pi_k | d' \underset{K}{\sim} d\}$.
2. $P_{[\lambda]}(d) = \{d' \in \Pi_k | P(d') = P(d)\}$.

Proposition 6.7. $K_{[\lambda]}(d) = P_{[\lambda]}(d)$

Proof. We draw the standard one line representations for the diagrams d and d'. Draw a line between k and $k + \frac{1}{2}$ in the diagram. The part of the diagram from 0 to k corresponds to $U(d)$ and the part of the diagram from $k + \frac{1}{2}$ to $2k$ corresponds to $L(d)$. Let $d, d' \in K_{[\lambda]}$. By the above lemma we have that the corresponding string which form the through classes are Knuth related i.e., by 2.16 [S] the insertion gives rise to the same shape in each residue. Since $L(d) = L(d')$, the second parts of the standard representations for both the diagrams d and d' are same which give rise to the same insertion sequence (E_j, h_j). Now from the same 4-tableau insertion and removal of the same sequence of (E_j, h_j) leads to the same P-vacillating tableau. Hence $K_{[\lambda]} \subseteq P_{[\lambda]}$.

Conversely, if the two diagrams have the same insertion sequence then $L(d) = L(d')$ i.e., we have the same edges emanating from the vertices in both the diagrams so that the numbers $E_i(d)$ and $E_i(d')$ are equal. By the above lemma we have the insertion corresponding to each type of through class is done in different residues. By theorem 2.16 [S], $w_0 \underset{K}{\sim} w'_0, w_1 \underset{K}{\sim} w'_1, w_2 \underset{K}{\sim} w'_2, w_3 \underset{K}{\sim} w'_3$ if and only if $P(w_0) = P(w'_0), P(w_1) = P(w'_1), P(w_2) = P(w'_2), P(w_3) = P(w'_3)$, respectively and hence we have $d_1 \underset{K}{\sim} d_2$. Therefore $P_{[\lambda]}(d) \subseteq K_{[\lambda]}(d)$, which completes the proof.

$\square$

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