Note on generating all subsets of a finite set with disjoint unions

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Abstract

We call a family $\mathcal{G} \subset \mathbb{P}[n]$ a *k-generator* of $\mathbb{P}[n]$ if every $x \subset [n]$ can be expressed as a union of at most k disjoint sets in $\mathcal{G}$. Frein, Lévêque and Sebő [1] conjectured that for any $n \geq k$, such a family must be at least as large as the k-generator obtained by taking a partition of $[n]$ into classes of sizes as equal as possible, and taking the union of the power-sets of the classes. We generalize a theorem of Alon and Frankl [2] in order to show that for fixed k, any k-generator of $\mathbb{P}[n]$ must have size at least $k2^{n/k}(1 - o(1))$, thereby verifying the conjecture asymptotically for multiples of k.

1 Introduction

We call a family $\mathcal{G} \subset \mathbb{P}[n]$ a *k-generator* of $\mathbb{P}[n]$ if every $x \subset [n]$ can be expressed as a union of at most k disjoint sets in $\mathcal{G}$. Frein, Lévêque and Sebő [1] conjectured that for any $n \geq k$, such a family must be at least as large as the k-generator

$$\mathcal{F}_{n,k} := \bigcup_{i=1}^{k} \mathbb{P}V_i \setminus \{\emptyset\} \tag{1}$$

where (V_i) is a partition of $[n]$ into k classes of sizes as equal as possible. For $k = 2$, removing the disjointness condition yields the stronger conjecture of Erdős – namely, if $\mathcal{G} \subset \mathbb{P}[n]$ is a family such that any subset of $[n]$ is a union (not necessarily disjoint) of at most two sets in $\mathcal{G}$, then $\mathcal{G}$ is at least as large as

$$\mathcal{F}_{n,2} = \mathbb{P}V_1 \cup \mathbb{P}V_2 \setminus \{\emptyset\} \tag{2}$$

where (V_1, V_2) is a partition of $[n]$ into two classes of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. We refer the reader to for example Füredi and Katona [5] for some results around the Erdős conjecture. In fact, Frein, Lévêque and Sebő [1] made the analagous conjecture for all k. (We call a family $\mathcal{G} \subset \mathbb{P}[n]$ a *k-base* of $\mathbb{P}[n]$ if every $x \subset [n]$ can be expressed as a union of at most k sets in $\mathcal{G}$; they conjectured that for any $k \leq n$, any k-base of $\mathbb{P}[n]$ is at least as large as $\mathcal{F}_{n,k}$.)

In this paper, we show that for k fixed, a k-generator must have size at least $k2^{n/k}(1-o(1))$; when n is a multiple of k, this is asymptotic to $f(n,k) = |\mathcal{F}_{n,k}| = k(2^{n/k}-1)$. Our main tool is a generalization of a theorem of Alon and Frankl, proved via an Erdős-Stone type result.

As observed in [1], for a k-generator $\mathcal{G}$, we have the following trivial bound on $|\mathcal{G}| = m$. The number of ways of choosing at most k sets in $\mathcal{G}$ must be at least the number of subsets of $[n]$, i.e.:

$$\sum_{i=0}^{k} \binom{m}{i} \geq 2^n$$

For fixed k, the number of subsets of $[n]$ of size at most $k-1$ is $\sum_{i=0}^{k-1} \binom{m}{i} = \Theta(1/m)\binom{m}{k}$, so

$$\sum_{i=0}^{k} \binom{m}{i} = (1+\Theta(1/m))\binom{m}{k} = (1+\Theta(1/m))m^k/k!$$

Hence,

$$m \geq (k!)^{1/k} 2^{n/k}(1-o(1))$$

Notice that this ignores disjointness, and is therefore also a lower bound on the size of a k-base; it also ignores the fact that some unions may occur several times. We will improve the constant from $(k!)^{1/k} \approx k/e$ to k by taking into account disjointness. Namely, we will show that for any fixed $k \in \mathbb{N}$ and $\delta > 0$, if $m \geq 2^{(1/(k+1)+\delta)n}$, then any family $\mathcal{G} \subset \mathbb{P}[n]$ of size m contains at most

$$\left(\frac{k!}{k^k} + o(1)\right)\binom{m}{k}$$

unordered k-tuples $\{A_1, \ldots, A_k\}$ of pairwise disjoint sets, where the $o(1) = o_{k,\delta}(1)$ term tends to 0 as $m \to \infty$ for fixed k, δ. In other words, if we consider the 'Kneser graph' on $\mathbb{P}[n]$, with edge set consisting of the disjoint pairs of subsets, the density of K_k's in any sufficiently large $\mathcal{G} \subset \mathbb{P}[n]$ is at most $k!/k^k + o(1)$. The proof uses an Erdős-Stone type result (Theorem 1) together with a result of Alon and Frankl (Lemma 4, which is Lemma 4.3 in [2]).

The $k = 2$ case of this was proved by Alon and Frankl (Theorem 1.3 of [2]): for any fixed $\delta > 0$, if $m \geq 2^{(1/3+\delta)n}$, then any family $\mathcal{G} \subset \mathbb{P}[n]$ of size m contains at most

$$\left(\tfrac{1}{2} + o(1)\right)\binom{m}{2}$$

disjoint pairs, where the $o(1)$ term tends to 0 as $m \to \infty$ for fixed δ. In other words, the edge-density in any sufficiently large subset of the Kneser graph is at most $\frac{1}{2} + o(1)$.

Our result will follow quickly from this. From the trivial bound above, any k-generator $\mathcal{G} \subset \mathbb{P}[n]$ has size $m \geq 2^{n/k}$, so putting $\delta = 1/k(k+1)$, we will see that the number of unordered k-tuples of pairwise disjoint sets in $\mathcal{G}$ is at most

$$\left(\frac{k!}{k^k}+o(1)\right)\binom{m}{k}$$

so

$$2^n \leq \left(\frac{k!}{k^k}+o(1)+\Theta(1/m)\right)\binom{m}{k}=\left(\frac{m}{k}\right)^k(1+o(1))$$

and therefore

$$m \geq k2^{n/k}(1-o(1))$$

where the $o(1)$ term tends to 0 as $n \to \infty$ for fixed $k \in \mathbb{N}$.

For $k=2$, this improves the estimate $m \geq \sqrt{2}2^{n/2}-1$ in [1] (Theorem 5.3) by a factor of $\sqrt{2}$. For n even, it is asymptotically tight, but for n odd, the conjectured smallest 2-generator (2) has size $(3/\sqrt{2})2^{n/2}-1$, so our constant is 'out' by a factor of $3/(2\sqrt{2}) = 1.061$ (to 3 d.p.)

For general k and $n = qk + r$, the conjectured smallest k-generator (1) has size

$$(k-r)2^q + r2^{q+1} - k = (k+r)2^{-r/k}2^{n/k} - k$$

so our constant is out by a factor of $(1+r/k)2^{-r/k} \leq 2^{1-1/\ln 2}/\ln 2 = 1.061$ (to 3 d.p.).

It seems that different arguments will be required to improve the constant for $k \nmid n$, or to prove the exact result. Further, it seems likely that proving the same bounds for k-bases (i.e. without the assumption of disjoint unions) would be much harder, and require different techniques altogether.

2 A preliminary Erdős-Stone type result

We will need the following generalization of the Erdős-Stone theorem:

Theorem 1 *Given $r \leq s \in \mathbb{N}$ and $\epsilon > 0$, if n is sufficiently large depending on r, s and ϵ, then any graph G on n vertices with at least*

$$\left(\frac{s(s-1)(s-2)\dots(s-r+1)}{s^r}+\epsilon\right)\binom{n}{r}$$

K_r's contains a copy of $K_{s+1}(t)$, where $t \geq C_{r,s,\epsilon}\log n$ for some constant $C_{r,s,\epsilon}$ depending on r, s, ϵ.

Note that the density $\eta = \eta_{r,s} := \frac{s(s-1)(s-2)\dots(s-r+1)}{s^r}$ above is the density of K_r's in the s-partite Turán graph with classes of size T, $K_s(T)$, when T is large.

Proof:
Let G be a graph with K_r density at least $\eta+\epsilon$; let N be the number of l-subsets $U \subset V(G)$

such that $G[U]$ has K_r-density at least $\eta + \epsilon/2$. Then, double counting the number of times an l-subset contains a K_r,

$$N\binom{l}{r} + \left(\binom{n}{r} - N\right)(\eta + \epsilon/2)\binom{l}{r} \geq (\eta + \epsilon)\binom{n}{r}\binom{n-r}{l-r}$$

so rearranging,

$$N \geq \frac{\epsilon/2}{1 - \eta - \epsilon/2}\binom{n}{l} \geq \tfrac{\epsilon}{2}\binom{n}{l}$$

Hence, there are at least $\frac{\epsilon}{2}\binom{n}{l}$ l-sets U such that $G[U]$ has K_r-density at least $\eta + \epsilon/2$. But Erdős proved that the number of K_r's in a K_{s+1}-free graph on l vertices is maximized by the s-partite Turán graph on l vertices (Theorem 3 in [3]), so provided l is chosen sufficiently large, each such $G[U]$ contains a K_{s+1}. Each K_{s+1} in G is contained in $\binom{n-s-1}{l-s-1}$ l-sets, and therefore G contains at least

$$\tfrac{\epsilon}{2}\frac{\binom{n}{l}}{\binom{n-s-1}{l-s-1}} \geq \frac{\epsilon}{2}(n/l)^{s+1}$$

K_{s+1}'s, i.e. a positive density of K_{s+1}'s. Let $a = s+1,\ c = \frac{\epsilon}{2l^{s+1}}$ and apply the following 'blow up' theorem of Nikiforov (a slight weakening of Theorem 1 in [4]):

Theorem 2 *Let $a \geq 2$, $c^a \log n \geq 1$. Then any graph on n vertices with at least cn^a K_a's contains a $K_a(t)$ with $t = \lfloor c^a \log n \rfloor$.*

We see that provided n is sufficiently large depending on r, s and ϵ, G must contain a $K_{s+1}(t)$ for $t = \lfloor c^{s+1} \log n \rfloor = \lfloor (\frac{\epsilon}{2l^{s+1}})^{s+1} \log n \rfloor \geq C_{r,s,\epsilon} \log n$, proving Theorem 1. □

3 Density of K_k's in large subsets of the Kneser graph

We are now ready for our main result, a generalization of Theorem 1.3 in [2]:

Theorem 3 *For any fixed $k \in \mathbb{N}$ and $\delta > 0$, if $m \geq 2^{\left(\frac{1}{k+1}+\delta\right)n}$, then any family $\mathcal{G} \subset \mathbb{P}[n]$ of size $|\mathcal{G}| = m$ contains at most*

$$\left(\frac{k!}{k^k} + o(1)\right)\binom{m}{k}$$

unordered k-tuples $\{A_1, \ldots, A_k\}$ of pairwise disjoint sets, where the $o(1)$ term tends to 0 as $m \to \infty$ for fixed k, δ.

Proof:

By increasing δ if necessary, we may assume $m = 2^{\left(\frac{1}{k+1}+\delta\right)n}$. Consider the subgraph G of the 'Kneser graph' on $\mathbb{P}[n]$ induced on the set $\mathcal{G}$, i.e. the graph G with vertex set $\mathcal{G}$ and edge set $\{xy : x \cap y = \emptyset\}$. Let $\epsilon > 0$; we will show that if n is sufficiently large depending on k,δ and ϵ, the density of K_k's in G is less than $\frac{k!}{k^k}+\epsilon$. Suppose the density of K_k's in G is at least $\frac{k!}{k^k}+\epsilon$; we will obtain a contradiction for n sufficiently large. Let $l=m^f$ (we will choose $f<\frac{\delta}{2(1+(k+1)\delta)}$ maximal such that m^f is an integer). By the argument above, there are at least $\frac{\epsilon}{2}\binom{m}{l}$ l-sets U such that $G[U]$ has K_k-density at least $\frac{k!}{k^k}+\frac{\epsilon}{2}$. Provided m is sufficiently large depending on k,δ and ϵ, by Theorem 1, each such $G[U]$ contains a copy of $K:=K_{k+1}(t)$ where $t\geq C_{k,k,\epsilon/2}\log l = fC'_{k,\epsilon}\log m = C''_{k,\delta,\epsilon}\log m$. Any copy of K is contained in $\binom{m-(k+1)t}{l-(k+1)t}$ l-sets, so G must contain at least $\frac{\epsilon}{2}\frac{\binom{m}{l}}{\binom{m-(k+1)t}{l-(k+1)t}}\geq\frac{\epsilon}{2}(m/l)^{(k+1)t}$ copies of K.

But we also have the following lemma of Alon and Frankl (Lemma 4.3 in [2]), whose proof we include for completeness:

Lemma 4 *G contains at most $(k+1)2^{n(1-\delta t)}\binom{m}{t}^{k+1}\frac{1}{(k+1)!}$ copies of $K_{k+1}(t)$.*

Proof:
The probability that a t-subset $\{A_1,\ldots,A_t\}$ chosen uniformly at random from $\mathcal{G}$ has union of size at most $\frac{n}{k+1}$ is at most

$$\sum_{S\subset[n]:|S|\leq n/(k+1)}\binom{2^{|S|}}{t}\Big/\binom{m}{t}\leq 2^n(2^{n/(k+1)}/m)^t=2^{n(1-\delta t)}$$

Choose at random $k+1$ such t-sets; the probability that at least one has union of size at most $n/(k+1)$ is at most

$$(k+1)2^{n(1-\delta)t}$$

But this condition holds if our $k+1$ t-sets are the vertex classes of a $K_{k+1}(t)$ in G. Hence, the number of copies of $K_{k+1}(t)$ in G is at most

$$(k+1)2^{n(1-\delta t)}\binom{m}{t}^{k+1}\frac{1}{(k+1)!}$$

as required. $\square$

If m is sufficiently large depending on k,δ and ϵ, we may certainly choose $t\geq\lceil 4/\delta\rceil$, and comparing our two bounds gives

$$\tfrac{\epsilon}{2}(m/l)^{(k+1)t}\leq(k+1)2^{n(1-\delta t)}\binom{m}{t}^{k+1}\frac{1}{(k+1)!}\leq\tfrac{1}{2}2^{n(1-\delta t)}m^{(k+1)t}$$

Substituting in $l=m^f$, we get

$$\epsilon\leq 2^{n(1-\delta t)}m^{f(k+1)t}$$

Substituting in $m=2^{\left(\frac{1}{k+1}+\delta\right)n}$, we get

$$\epsilon\leq 2^{n(1-t(\delta-f(1+(k+1)\delta)))}\leq 2^{-n}$$

since we chose $f < \frac{\delta}{2(1+(k+1)\delta)}$ and $t \geq 4/\delta$. This is a contradiction if n is sufficiently large, proving Theorem 3. $\square$

As explained above, our result on k-generators quickly follows:

Theorem 5 *For fixed $k \in \mathbb{N}$, any k-generator $\mathcal{G}$ of $\mathbb{P}[n]$ must contain at least $k2^{n/k}(1 - o(1))$ sets.*

Proof:
Let $\mathcal{G}$ be a k-generator of $\mathbb{P}[n]$, with $|\mathcal{G}| = m$. As observed in the introduction, the trivial bound gives $m \geq 2^{n/k}$, so applying Theorem 3 with $\delta = 1/k(k+1)$, we see that the number of ways of choosing k pairwise disjoint sets in $\mathcal{G}$ is at most

$$\left(\frac{k!}{k^k} + o(1)\right)\binom{m}{k}$$

The number of ways of choosing less than k pairwise disjoint sets is, very crudely, at most $\sum_{i=0}^{k-1}\binom{m}{i} = \Theta(1/m)\binom{m}{k}$; since every subset of $[n]$ is a disjoint union of at most k sets in $\mathcal{G}$, we obtain

$$2^n \leq \left(\frac{k!}{k^k} + o(1) + \Theta(1/m)\right)\binom{m}{k} = \left(\frac{m}{k}\right)^k (1 + o(1))$$

(where the o(1) term tends to 0 as $m \to \infty$), and therefore

$$m \geq k2^{n/k}(1 - o(1))$$

(where the $o(1)$ term tends to 0 as $n \to \infty$). $\square$

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