On the Energy of Unitary Cayley Graphs

H.N. Ramaswamy and C.R. Veena

Department of Studies in Mathematics, University of Mysore
Manasagangothri, Mysore 570 006, INDIA

hnrama@gmail.com, veena_maths@rediffmail.com

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Abstract

In this note we obtain the energy of unitary Cayley graph X_n which extends a result of R. Balakrishnan for power of a prime and also determine when they are hyperenergetic. We also prove that $\frac{E(X_n)}{2(n-1)} \geq \frac{2^k}{4k}$, where k is the number of distinct prime divisors of n. Thus the ratio $\frac{E(X_n)}{2(n-1)}$, measuring the degree of hyperenergeticity of X_n, grows exponentially with k.

Keywords: Spectrum of a graph; Energy of a graph; Unitary Cayley graphs; Hyperenergetic graphs.

1 Introduction

Let G be a simple finite undirected graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of graph G. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, assumed in non-increasing order, are the eigenvalues of the graph G called the **Spectrum of** G denoted by **Spec** G. If the distinct eigenvalues of G are $\mu_1 > \mu_2 > \cdots > \mu_s$, and their multiplicities are $m(\mu_1), m(\mu_2), \ldots, m(\mu_s)$, then we write

$$\text{Spec}\, G = \begin{pmatrix} \mu_1 & \mu_2 & \ldots & \mu_s \\ m(\mu_1) & m(\mu_2) & \ldots & m(\mu_s) \end{pmatrix}.$$

Spec G is independent of labelling of the vertices of G. As A is a real symmetric matrix with zero trace, these eigenvalues are real with sum equal to zero.

The **energy E(G)** of G was defined by I. Gutman [6] in 1978 as the sum of the absolute values of its eigenvalues.

Since the energy of a graph is not affected by isolated vertices, we assume throughout that graphs have no isolated vertices implying, in particular, that $m \geq \frac{n}{2}$. If a graph is not connected, its energy is the sum of the energies of its connected components. Thus there is no loss in generality in assuming that graphs are connected.

The complete graph K_n has simple eigenvalue $n-1$ and eigenvalue -1 of multiplicity $n-1$. Thus its energy is given by $E(K_n) = 2(n-1)$. The graph G of order n whose energy satisfies $E(G) > 2(n-1)$ is called **hyperenergetic** and graph with energy $E(G) \leq 2(n-1)$ is called **non-hyperenergetic**.

The **Line graph** $L(G)$ of a graph G is constructed by taking the edges of G as vertices of $L(G)$, and joining two vertices in $L(G)$ whenever the corresponding edges in G have a common vertex. It is proved in [11] that the line graph of all k-regular graphs, for $k \geq 4$, are hyperenegetic.

Let Γ be a finite multiplicative group with identity 1. For $S \subseteq \Gamma, 1 \notin S$ and $S^{-1} = \{s^{-1} : s \in S\} = S$, the **Cayley graph** $X = \text{Cay}\ (\Gamma, S)$ is the undirected graph having vertex set $V(X) = \Gamma$ and edge set $\{(a, b) : ab^{-1} \in S\}$. By the right multiplication Γ may be considered as a group of automorphisms of X acting transitively on $V(X)$. The Cayley graph X is a regular graph of degree $|S|$. Its connected components are the right cosets of the subgroup generated by S. So X is connected, if S generates Γ.

For a positive integer $n > 1$ the **unitary Cayley graph** $X_n = \text{Cay}\ (Z_n, U_n)$ is defined by the additive group of the ring Z_n of integers modulo n and the multiplicative group U_n of its units. If we represent the elements of Z_n by the integers $0, 1, \ldots, n-1$, then $U_n = \{a \in Z_n : \gcd\ (a, n) = 1\}$. So, X_n has the vertex set $V(X_n) = Z_n = \{0, 1, \ldots, n-1\}$ and the edge set $\{(a, b) : a, b \in Z_n,$
$\gcd\ (a - b, n) = 1\}$.

The concept of graph energy arose in theoretical chemistry. The total π-electron energy of some conjugated carbon molecule, computed using Hückel theory, coincides with the energy of its "molecular" graph. Recently there has been a tremendous research activity in the areas like hyperenergetic graphs, maximum energy graphs, equienergetic graphs. We refer to the survey papers by Gutman [7] and by Brualdi [3] for details. The study of the energy of circulant graphs is also of number theoretic interest as it is related to the Gauss sum (see for instance [2], [9] and [10]). Cayley graphs are important class of circulant graphs defined through finite groups. The unitary Cayley graphs have number theoretic aspects as illustrated by Klotz and Sander [8] and Fuchs [5], wherein, the basic invariants, the eigenvalues and the largest induced cycles were determined.

The energy of X_n when n is a power of a prime was determined by Balakrishnan [1] using the computations involving the cyclotonic polynomials $\phi_n(x)$. In this note we extend the result of Balakrishnan by obtaining the energy of all unitary Cayley graphs X_n

and determine when they are hyperenergetic. We also obtain a lower bound for the ratio of the energy of the unitary Cayley graph and the complete graph, thus measuring the degree of hyperenergeticity. This ratio grows exponentially with the number of distinct prime factors of n.

2. PRELIMINARIES

We give a brief account of some of the results of Klotz and Sander [8] on the eigenvalues of unitary Cayley graphs which will be used in this note.

It is well known that X_n is a connected $\phi(n)$ - regular graph. If $n = p$ is a prime number, then X_n is the complete graph on p vertices and if $n = p^\alpha$ is a prime power, then X_n is a complete p - partite graph. The unitary Cayley graph $X_n,\ n \geq 2$, is bipartite if and only if n is even. Klotz and Sander [8] have determined the chromatic number, the clique number, the independence number, the diameter and the vertex connectivity of X_n. They have also shown that all nonzero eigenvalues of X_n are integers dividing $\phi(n)$.

The eigenvalues of X_n are given by

$$\lambda_{r+1} = \sum_{\substack{1\leq j<n, \\ \gcd\ (j,n)=1}} \omega^{rj}, 0 \leq r \leq n-1, \tag{2.1}$$

where $\omega = \exp(\frac{2\pi i}{n})$. The sum in equation (2.1) is the well known Ramanujan sum $c(r,n)$. Thus, we have,

$$\lambda_{r+1} = c(r,n), \quad 0 \leq r \leq n-1. \tag{2.2}$$

The value of $c(r,n)$ is an integer and so all the eigenvalues of X_n are integers which are given by:

$$c(r,n) = \mu(t_r)\frac{\phi(n)}{\phi(t_r)},\ \text{ where }\ t_r = \frac{n}{\gcd\ (r,n)},\ \ 0 \leq r \leq n-1, \tag{2.3}$$

where μ denotes the Möbius function. Klotz and Sander [8] have obtained the following results:

Theorem 2.1 [8] For $n \geq 2$, the following statements hold:

1. Every nonzero eigenvalue of X_n is a divisor of $\phi(n)$.
2. Let m be the maximal squarefree divisor of n. Then

$$\lambda_{min} = \mu(m)\frac{\phi(n)}{\phi(m)}$$

is a nonzero eigenvalue of X_n of minimal absolute value and multiplicity $\phi(m)$. Every eigenvalue of X_n is a multiple of λ_{min}. If n is odd, then λ_{min} is the only nonzero eigenvalue of X_n with minimal absolute value. If n is even, then $-\lambda_{min}$ is also an eigenvalue of X_n with multiplicity $\phi(m)$.

Theorem 2.2 [8] Let m be the maximal squarefree divisor of n and let M be the set of positive divisors of m. Then the following statements for the unitary Cayley graph X_n, $n \geq 2$, hold:

1. Repeating $\phi(t)$-times every term of the sequence $S = \left(\mu(t)\frac{\phi(n)}{\phi(t)}\right)_{t \in M}$ results in a sequence $\tilde{\mathrm{S}}$ of length m which consists of all nonzero eigenvalues of X_n such that the number of appearances of an eigenvalue is its multiplicity.

2. The multiplicity of zero as an eigenvalue of X_n is $n - m$.

3. If $\alpha(\lambda)$ is the multiplicity of the eigenvalue λ of X_n, then $\lambda\alpha(\lambda)$ is a multiple of $\phi(n)$.

3. ENERGY OF UNITARY CAYLEY GRAPHS

We first give a direct proof of the result of Balakrishnan [1] when n is a power of a prime.

Theorem 3.1. If $n = p^\alpha$ is a prime power, then the energy of the unitary Cayley graph X_n is given by $E(X_n) = 2\phi(n)$.
Proof. When $\alpha = 1$, the graph X_n is the complete graph K_p. Clearly $E(K_p) = 2(p-1) = 2\phi(p)$. Hence we can assume $\alpha \geq 2$.
The eigenvalues of the unitary Cayley graph X_{p^α} are given by

$$\lambda_{r+1} = c(r, p^\alpha) = \mu(t_r)\frac{\phi(p^\alpha)}{\phi(t_r)}, \text{where } t_r = \frac{p^\alpha}{\gcd(r, p^\alpha)}, 0 \leq r \leq p^\alpha - 1.$$

We consider three cases:

Case(1): If $\gcd(r, p^\alpha) = p^\alpha$ then $r = 0$ and so $t_0 = 1$. Hence $\lambda_1 = \phi(p^\alpha) = p^\alpha - p^{\alpha-1}$.

Case(2): If gcd $(r, p^\alpha) = 1$ then $t_r = p^\alpha$ and hence we get $\lambda_{r+1} = 0$.

Case(3): If $1 < \gcd(r, p^\alpha) < p^\alpha$ then $\gcd(r, p^\alpha) = p^m$, where $1 \leq m \leq \alpha - 1$. When $\gcd(r, p^\alpha) = p^{\alpha-1}$, we get $\lambda_{r+1} = -p^{\alpha-1}$. For all other remaining values of m we get $\lambda_{r+1} = 0$.

Therefore the Spectrum of X_{p^α} is

$$\text{Spec } X_{p^\alpha} = \begin{pmatrix} p^\alpha - p^{\alpha-1} & -p^{\alpha-1} & 0 \\ 1 & p-1 & p^\alpha - p \end{pmatrix}.$$

Thus, $E(X_{p^\alpha}) = p^\alpha - p^{\alpha-1} + (p-1)p^{\alpha-1} = 2(p^\alpha - p^{\alpha-1}) = 2\phi(p^\alpha)$.
Hence the proof. □

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The **direct product** of G_1 and G_2 is the graph $G = (V, E)$ denoted by $G_1 \otimes G_2$ (also by $G_1 \wedge G_2$) where $V = V_1 \times V_2$, the cartesian product of V_1 and V_2, with (v_1, v_2) and (u_1, u_2) are adjacent in G if and only if v_1, u_1 are adjacent in G_1 and v_2, u_2 are adjacent in G_2.

Theorem 3.2. If $(m, n) = 1$, then the direct product of the unitary Cayley graphs X_m and X_n is isomorphic to X_{mn}.

Proof. Since $(m, n) = 1$, by the Chinese Remainder theorem, there is an isomorphism $\phi : Z_m \times Z_n \longrightarrow Z_{mn}$. This isomorphism induces an isomorphism between their groups of units $U_m \times U_n$ and U_{mn}. Let $k_{i,j}$ be the element in Z_{mn} corresponding to the element $(i, j) \in Z_m \times Z_n$. Then $(i, m) = 1 = (j, n)$ if and only if $(k_{i,j}, mn) = 1$. The vertex set of X_{mn} is Z_{mn} and the vertex set of $X_m \times X_n$ is $Z_m \times Z_n$. The isomorphism ϕ gives the bijective correspondence between their vertex sets. Let i_1 be adjacent to i_2 in X_m and let j_1 be adjacent to j_2 in X_n. Then $(i_1 - i_2, m) = 1 = (j_1 - j_2, n)$. Now consider $k_{i_1,j_1}, k_{i_2,j_2} \in Z_{mn}$. Since ϕ is an isomorphism, $k_{i_1-i_2,j_1-j_2} = k_{i_1,j_1} - k_{i_2,j_2}$. Now $(k_{i_1-i_2,j_1-j_2}, mn) = 1$ and so k_{i_1,j_1} and k_{i_2,j_2} are adjacent in X_{mn}.

Conversely, if k_{i_1,j_1} is adjacent to k_{i_2,j_2} in X_{mn}, then, $k_{i_1-i_2,j_1-j_2} = k_{i_1,j_1} - k_{i_2,j_2} \in U_{mn}$ and so $i_1 - i_2 \in U_m$ and $j_1 - j_2 \in U_n$. Thus i_1 is adjacent to i_2 in X_m and j_1 is adjacent to j_2 in X_n. Hence $X_m \otimes X_n$ and X_{mn} are isomorphic. This completes the proof. $\square$

Corollary 3.3. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, then the direct product of unitary Cayley graphs $X_{p_1^{\alpha_1}} \otimes X_{p_2^{\alpha_2}} \otimes \cdots \otimes X_{p_k^{\alpha_k}}$ is isomorphic to X_n.

Definition 3.4. The tensor product $A \otimes B$ of the $r \times s$ matrix $A = (a_{ij})$ and the $t \times u$ matrix $B = (b_{ij})$ is defined as the $rt \times su$ matrix got by replacing each entry a_{ij} of A by the double array $a_{ij}B$.

It is easy to check that for any two graphs G_1 and G_2 the adjacency matrix $A(G_1 \otimes G_2)$ of $G_1 \otimes G_2$ is given by

$$A(G_1 \otimes G_2) = A(G_1) \otimes A(G_2).$$

Lemma 3.5. [4] If A is a matrix of order r with Spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_r\}$, and B, a matrix of order s with Spectrum $\{\mu_1, \mu_2, \dots, \mu_s\}$, then the spectrum of $A \otimes B$ is $\{\lambda_i \mu_j : 1 \le i \le r; 1 \le j \le s\}$.

Corollary 3.6. If G_1 and G_2 are any two graphs, then,

$$E(G_1 \otimes G_2) = E(G_1)E(G_2).$$

Theorem 3.7. If $n > 1$ is of the form $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $p_1, p_2, \dots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_k$ are positive integers, then,

$$E(X_n) = 2^k \phi(n).$$

Proof. By Corollary 3.3, X_n is isomorphic to the product $X_{p_1^{\alpha_1}} \otimes \dots \otimes X_{p_k^{\alpha_k}}$. Now by Corollary 3.6, the energy of the direct product of graphs is the product of their energies. Hence, it follows that, $E(X_n) = E(X_{p_1^{\alpha_1}}) \dots E(X_{p_k^{\alpha_k}})$. Now by Theorem 3.1, $E(X_{p_i^{\alpha_i}}) = 2\phi(p_i^{\alpha_i})$ for $1 \le i \le k$ and so we have,

$$
\begin{aligned}
E(X_n) &= 2^k\phi(p_1^{\alpha_1})\dots\phi(p_k^{\alpha_k}) \\
&= 2^k\phi(p_1^{\alpha_1}\dots p_k^{\alpha_k}) \\
&= 2^k\phi(n), \text{ since } \phi \text{ is multiplicative.}
\end{aligned}
$$
□

Corollary 3.8. $\dfrac{E(X_n)}{2(n-1)} > 2^{k-1}\dfrac{\phi(n)}{n}$.

We note that if $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, then,

$$\frac{\phi(n)}{n} = \left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_k}\right).$$

This will be used in the characterization of hyperenergetic unitary Cayley graphs. First we state the following Lemma whose proof follows by induction and is elementary.

Lemma 3.9. For $k \geq 3$ and $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, we have,

$$\frac{\phi(n)}{n} > \frac{1}{2^{k-1}} \;.$$

By making use of the Theorem 3.7, we now characterise the hyperenergetic unitary Cayley graphs.

Theorem 3.10 Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$ where $p_1, p_2, \dots, p_k$ are distinct prime divisors of n. Then the unitary Cayley graph X_n is hyperenergetic if and only if $k \geq 3$ or $k = 2$ and n is odd.

Proof. We consider three cases:

Case 1: For $k = 1, n = p^\alpha$, if X_n is hyperenergetic then, we have,

$$2\phi(p^\alpha) > 2(p^\alpha - 1) \quad\Rightarrow\quad 2(p^\alpha - p^{\alpha-1}) > 2(p^\alpha - 1)$$

i.e., $\qquad 2 > 2p^{\alpha-1} \quad\Rightarrow\quad 1 > p^{\alpha-1}$

which is impossible.

Therefore X_n is not hyperenergetic.

Case 2: For $k = 2, n = p^\alpha q^\beta$ $(p < q)$.

Here we consider two subcases:

(i) $p = 2, n = 2^\alpha q^\beta,\ 2 < q$

Then, we have,

$E(X_n) = 4\phi(n) = 4 \cdot 2^{\alpha-1}q^{\beta-1}(q-1)$

$$= 2n\left(\frac{q-1}{q}\right) < 2n\left(\frac{n-1}{n}\right) = 2(n-1).$$

Therefore X_n is non-hyperenergetic.

(ii) $p \geq 3, \;\; q \geq 3, \;\; p < q$
Since $q \geq 5$, we have, $E(X_n) > 2n$.

Therefore X_n is hyperenergetic.

Case 3: Let $k \geq 3$. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, then, by Corollary 3.8 and Lemma 3.9, we have,

$$\frac{E(X_n)}{2(n-1)} > 1,$$

and so X_n is hyperenergetic.
This completes the proof of the theorem. □

In the next theorem we show that the degree of hyperenergeticity grows at least exponentially with the number of distinct prime divisors of n by making use of the sharper lower bound for $\frac{\phi(n)}{n}$, namely $\frac{\phi(n)}{n} > \frac{1}{2k}$.

Theorem 3.11. Let k denote the number of distinct prime divisors of n. Then

$$\frac{E(X_n)}{2(n-1)} > \frac{2^k}{4k}.$$

Proof. Let $n = q_1^{\alpha_1} \dots q_k^{\alpha_k}$ where $q_1, \dots, q_k$ are distinct primes such that $q_1 < q_2 < \dots < q_k$. When $k = 1$, we have, $n = q^\alpha$ and so

$$\frac{E(X_n)}{2(n-1)} = \frac{2\phi(n)}{2(n-1)} > \frac{\phi(n)}{n} = \left(1 - \frac{1}{q}\right) \geq \frac{1}{2}.$$

Suppose $k \geq 2$. Let p_j denote the j^{th} prime. Then clearly $p_j \geq 2j - 1$ for $j \geq 2$. Thus

$$1 - \frac{1}{q_j} \geq 1 - \frac{1}{p_j} \geq \frac{2j-2}{2j-1}.$$

Hence,

$$\frac{\phi(n)}{n} \geq \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2k-2}{2k-1} \geq \frac{1}{2k-1} > \frac{1}{2k}.$$

Now the result follows from Corollary 3.8. □

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