A note on the distance-balanced property of generalized Petersen graphs*

Rui Yang, Xinmin Hou,† Ning Li, Wei Zhong
Department of Mathematics
University of Science and Technology of China
Hefei, Anhui, 230026, P. R. China

Submitted: Aug 25, 2008; Accepted: Nov 13, 2009; Published: Nov 24, 2009
Mathematics Subject Classifications: 05C75, 05C12

Abstract

A graph G is said to be distance-balanced if for any edge uv of G, the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. Let $GP(n,k)$ be a generalized Petersen graph. Jerebic, Klavžar, and Rall [*Distance-balanced graphs, Ann. Comb.* 12 (2008) 71–79] conjectured that: For any integer $k \geqslant 2$, there exists a positive integer n_0 such that the $GP(n,k)$ is not distance-balanced for every integer $n \geqslant n_0$. In this note, we give a proof of this conjecture.

Keywords: generalized Petersen graph, distance-balanced graph

1 Introduction

Let G be a simple undirected graph and $V(G)$ $(E(G))$ be its vertex (edge) set. The distance $d(u,v)$ between vertices u and v of G is the length of a shortest path between u and v in G. For a pair of adjacent vertices $u, v \in V(G)$, let W_{uv} denote the set of all vertices of G closer to u than to v, that is

$$W_{uv} = \{x \in V(G) \mid d(u,x) < d(v,x)\}.$$

Similarly, let ${}_uW_v$ be the set of all vertices of G that are at the same distance to u and v, that is

$${}_uW_v = \{x \in V(G) \mid d(u,x) = d(v,x)\}.$$

A graph G is called *distance-balanced* if

$$|W_{uv}| = |W_{vu}|$$

*The work was supported by NNSF of China (No.10701068).
†Corresponding author: xmhou@ustc.edu.cn

holds for every pair of adjacent vertices $u, v \in V(G)$.

Let uv be an arbitrary edge of G. Then $d(u,x) - d(v,x) \in \{1, 0, -1\}$. Hence $W_{uv} = \{x \in V(G) \mid d(v,x) - d(u,x) = 1\}$, ${}_uW_v = \{x \in V(G) \mid d(v,x) - d(u,x) = 0\}$, and $W_{vu} = \{x \in V(G) \mid d(v,x) - d(u,x) = -1\}$ form a partition of $V(G)$. The following proposition follows immediately from the above comments.

Proposition 1 *If $|W_{uv}| > |V(G)|/2$ for an edge uv of G, then G is not distance-balanced.*

Let $n \geqslant 3$ be a positive integer, and let $k \in \{1, ..., n-1\} \setminus \{n/2\}$. The generalized Petersen graph $GP(n,k)$ is defined to have the following vertex set and edge set:

$$V(GP(n,k)) = \{u_i \mid i \in \mathbb{Z}_n\} \cup \{v_i \mid i \in \mathbb{Z}_n\},$$

$$E(GP(n,k)) = \{u_i u_{i+1} \mid i \in \mathbb{Z}_n\} \cup \{v_i v_{i+k} \mid i \in \mathbb{Z}_n\} \cup \{u_i v_i \mid i \in \mathbb{Z}_n\}.$$

Jerebic, Klavžar, Rall [1] posed the following conjecture.

Conjecture 1 *For any integer $k \geqslant 2$, there exists a positive integer n_0 such that the generalized Petersen graph $GP(n,k)$ is not distance-balanced for every integer $n \geqslant n_0$.*

Motivated by this conjecture, Kutnar et al. [3] studied the strongly distance-balanced property of the generalized Petersen graphs and gave a slightly weaker result that: For any integer $k \geqslant 2$ and $n \geqslant k^2 + 4k + 1$, the generalized Petersen graph $GP(n,k)$ is not strongly distance-balanced (strongly distance-balanced graph was introduced by Kutnar et al. in [2]).

In this note, we prove the following theorem.

Theorem 2 *For any integer $k \geqslant 2$ and $n > 6k^2$, $GP(n,k)$ is not distance-balanced.*

Theorem 2 gives a positive answer to Conjecture 1.

2 The Proof of Theorem 2

First we give a direct observation.

Proposition 3 *For any $i = 0, 1, 2, ..., n-1$, $d(u_0, u_i) - d(v_0, u_i) = 1$ if and only if there exists a shortest path from u_0 to u_i which passes through the edge $u_0 v_0$ first.*

We call the cycle induced by the vertices $\{u_0, u_1, \cdots, u_{n-1}\}$ the *outer cycle* of $GP(n,k)$, and the cycles induced by the vertices $\{v_0, v_1, \cdots, v_{n-1}\}$ the *inner cycles* of $GP(n,k)$. The edge $u_i v_i$ $(0 \leqslant i \leqslant n-1)$ is called a *spoke* of $GP(n,k)$.

Proposition 4 *Let $GP(n,k)$ be a generalized Petersen graph with $n \geqslant 6k$ and $k \geqslant 2$. If $3k \leqslant i \leqslant n - 3k$, then there exists a shortest path between u_0 and u_i which passes through the edge $u_0 v_0$ first.*

Proof. By symmetry, we only need consider the case $3k \leqslant i \leqslant n/2$. Let $P(u_0, u_i)$ be a shortest path between u_0 and u_i. Note that the path between u_0 and u_i contained in the outer cycle has length i. The path:

$$u_0 \rightarrow v_0 \rightarrow v_k \rightarrow v_{2k} \rightarrow v_{3k} \rightarrow u_{3k} \rightarrow u_{3k+1} \rightarrow \cdots \rightarrow u_i$$

between u_0 and u_i has length $5+i-3k$. Since $k \geqslant 2$, $i+5-3k < i$. Hence $P(u_0, u_i)$ contains spokes. Let $u_s v_s$ and $v_l u_l$ be the first spoke and the last one in $P(u_0, u_i)$, respectively. If $s = 0$, then the result follows. If $s > 0$, let $P(u_s, u_l)$ be the segment of $P(u_0, u_i)$ from u_s to u_l. Define a map $f : V(P(u_s, u_l)) \mapsto V(GP(n, k))$ such that $f(u_j) = u_{j-s}$ and $f(v_j) = v_{j-s}$ for $u_j \in V(P(u_s, u_l))$. Then the segment $f(P(u_s, u_l))$ is a segment from u_0 to u_{l-s} which first passes through the edge $u_0 v_0$. Hence the path which first passes through the segment $P(u_0, u_{l-s})$, then from u_{l-s} to u_i along the outer cycle is a shortest path between u_0 and u_i, as desired. $\square$

In what follows, we give the proof of the main theorem.

Proof of Theorem 2: By Proposition 4, there exists a shortest path from u_0 to u_i which passes through $u_0 v_0$ first for each $3k \leqslant i \leqslant n - 3k$. By Proposition 3, $d(u_0, u_i) - d(v_0, u_i) = 1$. Hence there are more than $n - 6k$ vertices in the outer cycle which satisfy $d(u_0, u_i) - d(v_0, u_i) = 1$.

Now we count the number of vertices in the inner cycle of $GP(n, k)$ satisfying $d(u_0, v_i) - d(v_0, v_i) = 1$. For $i = mk$ ($m = 0, 1, 2, \cdots, \lfloor n/2k \rfloor$), it is easy to check that $d(u_0, v_i) = m + 1$ and $d(v_0, v_i) = m$. Hence $d(u_0, v_i) - d(v_0, v_i) = 1$. By symmetry, $d(u_0, v_i) - d(v_0, v_i) = 1$ for $i = n - mk$ ($m = 1, 2, \cdots, \lfloor n/2k \rfloor$). Hence there are at least $2\lfloor n/2k \rfloor$ vertices in the inner cycle satisfying $d(u_0, v_i) - d(v_0, v_i) = 1$.

If $n \geqslant 6k^2$, then the number of the vertices x satisfying $d(u_0, x) - d(v_0, x) = 1$ is more than $n - 6k + 2\lfloor n/2k \rfloor \geqslant n - 6k + 2\lfloor 6k^2/2k \rfloor = n$. Hence $|W_{v_0 u_0}| > n = |V(GP(n, k))|/2$. By Proposition 1, $GP(n, k)$ is not distance-balanced for $n \geqslant 6k^2$ and $k \geqslant 2$. $\square$

References

[1] J. Jerebic, S. Klavžar, D. F. Rall, Distance-balanced graphs, Ann.Comb. 12 (2008) 71-79.

[2] K. Kutnar, A. Malnič, D. Marušič, Š. Miklavič, Distance-balanced graphs: symmetry conditions, Discrete Math. 306 (2006), 1881-1894.

[3] K. Kutnar, A. Malnič, D. Marušič, S. Miklavič, The strongly distance-balanced property of the generalized Petersen graphs, Ars math Contemp., 2 (2009), 41-47.