# A short proof, based on mixed volumes, of Liggett's theorem on the convolution of ultra-logconcave sequences

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**Abstract**

R. Pemantle conjectured, and T. M. Liggett proved in 1997, that the convolution of two ultra-logconcave is ultra-logconcave. Liggett's proof is elementary but long. We present here a short proof, based on the mixed volume of convex sets.

## 1 Introduction

Let $\mathbf{a} = (a_0, ..., a_m)$ and $\mathbf{b} = (b_0, ..., b_n)$ be two real sequences. Their convolution $\mathbf{c} = \mathbf{a} \star \mathbf{b}$ is defined as $c_k = \sum_{i+j=k} a_i b_j, 0 \le k \le n+m$. A nonnegative sequence $\mathbf{a} = (a_0, ..., a_m)$ is said to be *logconcave* if

$$a_i^2 \ge a_{i-1}a_{i+1}, 1 \le i \le m-1. \tag{1}$$

Following Permantle and [5], we say that a nonnegative sequence $\mathbf{a} = (a_0, ..., a_m)$ is *ultra-logconcave of order* $d \ge m$ $(ULC(d))$ if the sequence $\frac{a_i}{\binom{d}{i}}, 0 \le i \le m$ is *logconcave*, i.e.

$$\left(\frac{a_i}{\binom{d}{i}}\right)^2 \ge \frac{a_{i-1}}{\binom{d}{i-1}} \frac{a_{i+1}}{\binom{d}{i+1}}, 1 \le i \le m-1. \tag{2}$$

The next result was conjectured by R. Pemantle and proved by T.M. Liggett in 1997 [5].

**Theorem 1.1:** *The convolution of a $ULC(l)$ sequence* $\mathbf{a}$ *and a $ULC(d)$ sequence* $\mathbf{b}$ *is $ULC(l+d)$.*

**Remark 1.2:** It is easy to see, by a standard perturbation argument, that it is sufficient to consider a positive case:

$$\mathbf{a} = (a_0, ..., a_l); a_i > 0, 0 \le i \le l \quad \text{and} \quad \mathbf{b} = (b_0, ..., b_d); b_i > 0, 0 \le i \le d.$$

The (relatively simple) fact that the convolution of *logconcave* sequences is also *logconcave* was proved in [3] in 1949. ∎

We present in this paper a short proof of Theorem(1.1).

# 2 The Minkowski sum and the mixed volume

## 2.1 The Minkowski sum

**Definition 2.1:**

1. Let $K_1, K_2 \subset \mathbf{R}^n$ be two subsets of the Euclidean space $\mathbf{R}^n$. Their *Minkowski* sum is defined as
$$K_1 + K_2 = \{X + Y : X \in K_1, Y \in K_2\}.$$
The *Minkowski* sum is obviously commutative, i.e $K_1 + K_2 = K_2 + K_1$, and associative, i.e
$$K_1 + K_2 + K_3 = K_1 + (K_2 + K_3).$$

2. Let $A \subset \mathbf{R}^l$, $B \subset \mathbf{R}^d$. Their cartesian product is defined as
$$A \times B := \{(X, Y) \in \mathbf{R}^{l+d} : X \in A, Y \in B\}.$$
Define the next two subsets of $\mathbf{R}^{l+d}$:
$$\mathrm{Lift}_1(A) = \{(X, 0) \in \mathbf{R}^{l+d} : X \in A\}, \mathrm{Lift}_2(B) = \{(0, Y) \in \mathbf{R}^{l+d} : Y \in B\}. \qquad (3)$$
Then the next set equalities holds:
$$A \times B = \mathrm{Lift}_1(A) + \mathrm{Lift}_2(B). \qquad (4)$$

∎

The next simple fact will be used below.

**Fact 2.2:** Let $K_1, K_2 \subset \mathbf{R}^l$ and $C_1, C_2 \subset \mathbf{R}^d$.
Define the next two subsets of $\mathbf{R}^{l+d}$:
$$P = K_1 \times C_1, Q = K_2 \times C_2.$$
Then the following set equality holds:
$$tP + Q = (tK_1 + K_2) \times (tC_1 + C_2), t \in \mathbf{R}. \qquad (5)$$

**Proof:** Using (4), we get that
$$(tK_1 + K_2) \times (tC_1 + C_2) = Lift_1(tK_1 + K_2) + Lift_2(tC_1 + C_2).$$
It follows from the definition (3) that
$$Lift_1(tK_1 + K_2) = tLift_1(K_1) + Lift_1(K_2); Lift_2(tC_1 + C_2) = tLift_2(C_1) + Lift_2(C_2).$$
Therefore, we get by the associativity and commutativity of the *Minkowski* sum that
$$(tK_1 + K_2) \times (tC_1 + C_2) = t(Lift_1(K_1) + Lift_2(C_1)) + (Lift_1(K_2) + Lift_2(C_2)) = tP + Q.$$
∎

## 2.2 The mixed volume

Let $\mathbf{K} = (K_1, ..., K_n)$ be a $n$-tuple of convex compact subsets in the Euclidean space $\mathbf{R}^n$, and let $V_n(\cdot)$ be the Euclidean volume in $\mathbf{R}^n$. It is a well-known result of Herman Minkowski (see for instance [2]), that the functional $V_n(\lambda_1 K_1 + \cdots + \lambda_n K_n)$ is a homogeneous polynomial of degree $n$ with nonnegative coefficients, called the Minkowski polynomial. Here $''+''$ denotes *Minkowski* sum, and $\lambda K$ denotes the dilatation of $K$ with coefficient $\lambda \geq 0$. The coefficient $V(\mathbf{K}) =: (V(K_1, ..., K_n)$ of $\lambda_1 \cdot \lambda_2 \ldots \cdot \lambda_n$ is called the *mixed volume* of $K_1, ..., K_n$. Alternatively,

$$V(K_1, ..., K_n) = \frac{\partial^n}{\partial \lambda_1 ... \partial \lambda_n} V_n(\lambda_1 K_1 + \cdots + \lambda_n K_n),$$

and

$$V_n(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{r_1 + \cdots + r_n = n} \frac{V(\mathbf{K}_{\mathbf{r_1}, ..., \mathbf{r_n}})}{\prod_{1 \leq i \leq n} r_i!} (\prod_{1 \leq i \leq n} \lambda_i^{r_i}), \tag{6}$$

where the $n$-tuple $\mathbf{K}_{\mathbf{r_1}, ..., \mathbf{r_n}}$ consists of $r_i$ copies of $K_i, 1 \leq i \leq n$.

*The Alexandrov-Fenchel inequalities [1], [2] state that*

$$V(K_1, K_2, K_3, ..., K_n)^2 \geq V(K_1, K_1, K_3, ..., K_n) V(K_2, K_2, K_3, ..., K_n). \tag{7}$$

It follows from (6) that if $P, Q \subset R^n$ are convex compact sets then

$$Vol_n(tP + Q) = \sum_{0 \leq i \leq n} a_i t^i, t \geq 0;$$

where $a_0 = Vol_n(Q) = \frac{1}{n!} V(Q, \cdots, Q)$, $a_1 = \frac{1}{(n-1)!1!} V(P, Q, \cdots, Q), \ldots$, $a_n = Vol_n(Q) = \frac{1}{n!} V(P, \cdots, P)$.

Using the Alexandrov-Fenchel inequalities (7) we see that the sequence $(a_0, ..., a_n)$ is $ULC(n)$.

The next remarkable result was proved by G.S. Shephard in 1960:

**Theorem 2.3:** *A sequence $(a_0, ..., a_n)$ is $ULC(n)$ if and only if there exist two convex compact sets $P, Q \subset R^n$ such that*

$$\sum_{0 \leq i \leq n} a_i t^i = Vol_n(tP + Q), t \geq 0.$$

**Remark 2.4:** The "if" part in Theorem(2.3), which is a particular case of *the Alexandrov-Fenchel inequalities*, is not simple, but was proved seventy years ago [1]. The proof of the "only if" part in Theorem(2.3) is not difficult and short. G.S. Shephard first considers the case of positive coefficients, which is already sufficient for our application. In this positive case one chooses $Q = \{(x_1, ..., x_n) : \sum_{1 \le i \le n} x_i \le 1; x_i \ge 0\}$. In other words, the set $Q$ is the standard simplex in $R^n$. And the convex compact set

$$P = Diag(\lambda_1, ..., \lambda_n)Q = \{(x_1, ..., x_n) : \sum_{1 \le j \le n} \frac{x_j}{\lambda_j} \le 1; x_i \ge 0\}; \lambda_1 \ge ... \ge \lambda_n > 0.$$

The general nonnegative case is handled by the topological theory of convex compact subsets. ∎

# 3 Our proof of Theorem(1.1)

**Proof:** Let $\mathbf{a} = (a_0, ..., a_l)$ be $ULC(l)$ and $\mathbf{b} = (b_0, ..., b_d)$ be $ULC(d)$. Define two univariate polynomials $R_1(t) = \sum_{0 \le i \le l} a_i t^i$ and $R_2(t) = \sum_{0 \le j \le c} a_i t^j$.
Then the polynomial $R_1(t)R_2(t) := R_3(t) = \sum_{0 \le k \le l+d} c_k t^k$, where the sequence $\mathbf{c} = (c_0, ..., c_{l+d})$ is the convolution, $\mathbf{c} = \mathbf{a} \star \mathbf{b}$.
It follows from the "only if"!p part of Theorem(2.3) that

$$R_1(t) = Vol_l(tK_1 + K_2) \quad \text{and} \quad R_2(t) = Vol_d(tC_1 + C_2),$$

where $K_1, K_2, C_1, C_2$ are convex compact sets; $K_1, K_2 \subset R^l$ and $C_1, C_2 \subset R^d$.
Define the next two convex compact subsets of $R^{l+d}$:

$$P = K_1 \times C_1 \quad \text{and} \quad Q = K_2 \times C_2.$$

Here the cartesian product $A \times B$ of two subsets $A \subset R^l$ and $B \subset R^d$ is defined as

$$A \times B := \{(X, Y) \in R^{l+d} : X \in A, Y \in B\}.$$

By Fact(2.2), the *Minkowski* sum $tP + Q = (tK_1 + K_2) \times (tC_1 + C_2), t \ge 0$.
It follows that $Vol_l(tK_1+K_2)Vol_d(tC_1+C_2) = Vol_{l+d}(tP+Q)$. Therefore the polynomial $R_3(t) = Vol_{l+d}(tP + Q)$.
Finally, we get from *the Alexandrov-Fenchel inequalities* (the "if" part of Theorem(2.3)) that the sequence of its coefficients $\mathbf{c} = \mathbf{a} \star \mathbf{b}$ is $ULC(l + d)$. ∎

# 4 Final comments

1. Theorem(2.3) and a simple Fact(2.2) allowed us to use very basic (but powerful) representation of the convolution in terms of the product of the corresponding polynomials. The original Liggett's proof does not rely on this representation.

2. Let $\mathbf{a} = (a_0, ..., a_m)$ be a real sequence, satisfying the Newton inequalities (2) of order $m$. I.e. we dropped the condition of nonnegativity from the definition of *ultra-logconcavity*. It is not true that $\mathbf{c} = \mathbf{a} \star \mathbf{a}$ satisfies the Newton inequalities of order $2m$.

Indeed, consider $\mathbf{a} = (1, a, 0, -b, 1)$, where $a, b > 0$ . This real sequence clearly satisfies the Newton inequalities of order 4.
It follows that $c_6 = b^2, c_5 = 2a, c_4 = 2(1 - ab)$ and the number

$$\frac{c_5^2}{c_4 c_6} = 2\frac{a^2}{b^2(1 - ab)}$$

converges to zero if the positive numbers $a, b, \frac{a}{b}$ converge to zero.

3. The reader can find further implications (and their generalizations) of Theorem(2.3) in [4].


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