Traces of uniform families of sets

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Abstract

The trace of a set F on a another set X is $F|_X = F \cap X$ and the trace of a family $\mathcal{F}$ of sets on X is $\mathcal{F}_X = \{F|_X : F \in \mathcal{F}\}$. In this note we prove that if a k-uniform family $\mathcal{F} \subset \binom{[n]}{k}$ has the property that for any k-subset X the trace $\mathcal{F}|_X$ does not contain a maximal chain (a family $C_0 \subset C_1 \subset ... \subset C_k$ with $|C_i| = i$), then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. This bound is sharp as shown by $\{F \in \binom{[n]}{k}, 1 \in F\}$. Our proof gives also the stability of the extremal family.

1 Introduction

Let $[n]$ denote the set of the first n positive integers $\{1, 2, ..., n\}$. Given a set X we write 2^X for its power set and $\binom{X}{l}$ for the set of all of its l-element subsets (l-*subsets* for short). Given a family $\mathcal{F} \subseteq 2^X$ of sets and an element $x \in X$ we write $\mathcal{F}_x$ for the subfamily of all the sets in $\mathcal{F}$ that contain x and $\mathcal{F}_{\overline{x}}$ for the family $\{F \setminus \{x\} : F \in \mathcal{F}_x\}$. The *degree* of x is the size of $\mathcal{F}_x$.

The trace of a set F on another set X is $F \cap X$ and is denoted by $F|_X$. The trace of a family $\mathcal{F}$ of sets is just the family of traces, i.e. $\mathcal{F}|_X = \{F|_X : F \in \mathcal{F}\}$. The following fundamental result concerning traces of families was proved in the early 1970s independently by Sauer [11], Shelah [12] and Vapnik and Chervonenkis [13].

Theorem 1.1. *If $\mathcal{F} \subseteq 2^{[n]}$ is a family with more than $\sum_{i=0}^{k-1}$ sets, then there exists a k-subset X of $[n]$ such that $\mathcal{F}|_X = 2^X$.*

The above theorem is sharp as shown by the families $\{F \subseteq [n] : |F| < k\}$ and $\{F \subseteq [n] : |F| > n-k\}$, but no characterization is known for the extremal families. Füredi and Quinn [7] constructed extremal families $\mathcal{F}_l$ of size $\sum_{i=0}^{k-1}$ for all l with $0 < l < k$ such that for any k-subset X of $[n]$ we have $\binom{X}{l} \not\subseteq \mathcal{F}|_X$.

Frankl and Pach [5] considered the k-uniform case of the problem. They proved the following upper bound.

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Theorem 1.2. *If $\mathcal{F} \subseteq \binom{[n]}{k}$ and $|\mathcal{F}| > \binom{n}{k-1}$, then there is a k-subset X of $[n]$ such that $\mathcal{F}|_X = 2^X$.*

Frankl and Pach conjectured $\{F \in \binom{[n]}{k} : 1 \in F\}$ to be an extremal family of size $\binom{n-1}{k-1}$, but Ahlswede and Khachatrian [1] disproved their conjecture by giving a counterexample of size $\binom{n-1}{k-1} + \binom{n-4}{k-3}$. Later Mubayi and Zhao [9] gave exponentially many pairwise non-isomorphic families of that size and improved the upper bound of Frankl and Pach, but the problem is still open.

Several papers [2], [3], [10] dealt with "Turán-type" problems of traces, i.e. given one or more families $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_s \subseteq 2^{[h]}$ what is the maximum size of a family $\mathcal{F} \subseteq 2^{[n]}$ such that for any h-subset X of $[n]$ and $1 \leq i \leq s$ the trace $\mathcal{F}|_X$ does not contain $\mathcal{H}_i$. With this formulation in Theorems 1.1 and 1.2 the excluded family is $2^{[k]}$.

In [10] it is proved (among others) that if we change $2^{[k]}$ to the maximal chain $\mathcal{C}_k = \{\emptyset, [1], [2], ..., [k]\}$ in Theorem 1.1, then the only extremal families are $\{F \subseteq [n] : |F| < k\}$ and $\{F \subseteq [n] : |F| > n-k\}$. In this note we consider the corresponding k-uniform problem and prove that the conjecture of Frankl and Pach becomes true in this scenario if again we change $2^{[k]}$ to $\mathcal{C}_k$. Furthermore we prove the stability of the extremal family $\{F \in \binom{[n]}{k} : 1 \in F\}$.

Theorem 1.3. *For every integer $2 \leq k$ and real $1/2 < c < 1$ there exists an $N_0(k,c)$ such that for any $n \geq N_0(k,c)$ if $\mathcal{F} \subseteq \binom{[n]}{k}$ has size larger than $c\binom{n-1}{k-1}$ and there is no subset X of $[n]$ with $|X| = k$ such that $\mathcal{C}_k \subseteq \mathcal{F}|_X$, then there exists an $x \in [n]$ such that $x \in F$ for all $F \in \mathcal{F}$.*

Clearly Theorem 1.3 is a generalization of the well-known Erdős-Ko-Rado theorem [4], therefore it is not surprising that our proof will use the following stability theorem of Hilton and Milner [8].

Theorem 1.4. *If $2k+1 \leq n$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family such that $\bigcap_{F \in \mathcal{F}} F = \emptyset$, then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.*

2 Proof of Theorem 1.3

First we prove a lemma stating that if we want to have an "almost" maximal chain $\mathcal{C}_k^- = \{[1], [2], ..., [k]\}$ as trace, then much smaller families suffice.

Lemma 2.1. *For every integer $2 \leq k$ and real $1/2 < c' < 1$ there exists an $N_0'(k,c')$ such that for any $n \geq N_0'(k,c')$ if $\mathcal{F} \subseteq \binom{[n]}{k}$ has size larger than $c'\binom{n-1}{k-1}$ then there exists a set $X \subset [n]$ with $|X| = k$ such that $\mathcal{C}_k^- \subseteq \mathcal{F}|_X$.*

Proof of Lemma: We proceed by induction on k. For $k = 2$, if there exists an intersecting pair of 2-sets $F_1, F_2 \in \mathcal{F}$, then $\emptyset \neq F_1|_{F_2} \subset F_2$ is a C_2^-. Therefore $\mathcal{F}$ is a pairwise disjoint family and thus $|\mathcal{F}| \leq n/2 < c'(n-1)$ for any $1/2 < c'$ if n is large enough.

Now suppose the lemma is proved for $k-1$ and any real between 1/2 and 1. For a real c' fix an $M > N'(k-1, \frac{c'+1/2}{2})$ such that the following inequalities hold for all $n \geq M$

$$\frac{c'-1/2}{2}\binom{n-2}{k-2} > \binom{n-2}{k-2} - \binom{n-k-2}{k-2}, \tag{1}$$

$$c'\left(\binom{n-2}{k-2} + \binom{n-3}{k-2}\right) > \binom{n-2}{k-2}. \tag{2}$$

The existence of such M for (1) follows from the fact that if we consider the two sides of (1) as polynomials of n, then the degree of the LHS is one larger than the degree of the RHS and for (2) from $c' > 1/2$ and from $\lim_{n\to\infty}\binom{n-2}{k-2}/\binom{n-3}{k-2} = 1$.

Let $N'(k,c') = M+1+2\binom{M+1}{k-1}$, $n \geq N'(k,c')$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ a family with $|\mathcal{F}| \geq c'\binom{n-1}{k-1}$. Let $x_1 \in [n]$ be an element with maximum degree which is at least the average degree $c'\binom{n-1}{k-1}\frac{k}{n} \geq c'\binom{n-2}{k-2}$ and consider $\mathcal{F}_{\overline{x}_1}$. By the inductive hypothesis there exists a $(k-1)$-subset $X \subset [n]\setminus\{x_1\}$ such that $\mathcal{F}_{\overline{x}_1}|_X$ contains $\mathcal{C}^-_{k-1}$. Just by removing these sets one after the other and repeatedly using the inductive hypothesis we get that $\mathcal{G} = \{X \in \mathcal{F}_{\overline{x}_1} : \mathcal{C}^-_{k-1} \subseteq \mathcal{F}_{\overline{x}_1}|_X\}$ has size at least $(c' - \frac{c'+1/2}{2})\binom{n-2}{k-2} = \frac{c'-1/2}{2}\binom{n-2}{k-2}$. If two sets $X_1, X_2 \in \mathcal{G}$ are disjoint, then writing $F_1 = X_1 \cup \{x_1\}, F_2 = X_2 \cup \{x_1\}$ both $\mathcal{F}|_{F_1}$ and $\mathcal{F}|_{F_2}$ contain $\mathcal{C}^-_k$ as $F_1|_{F_2} = F_1 \cap F_2 = F_2|_{F_1} = \{x\}$. Thus we may assume that $\mathcal{G}$ is intersecting and thus by Theorem 1.4 and (1) there exists an $x_2 \in [n]\setminus\{x_1\}$ such that $x_2 \in X$ for all $X \in \mathcal{G}$.

Let us assume that there is a set $F' \in \mathcal{F}_{x_1}$ with $x_2 \notin F'$. We claim that there is a set $X \in \mathcal{G}$ such that $F' \cap X = \emptyset$. Indeed, the number of $(k-1)$-sets containing x_2 and meeting F is $\binom{n-2}{k-2} - \binom{n-k-2}{k-2}$, thus again by (1) there is a set $X \in \mathcal{G}$ as claimed. By the definition of $\mathcal{G}$ there are sets $F_2, F_3, ..., F_k \in \mathcal{F}_{x_1}$ such that their traces on X form a $\mathcal{C}^-_{k-1}$. Writing $F = X \cup \{x\}$ we have $F'|_F = \{x_1\}$ and thus the traces of $F', F_2, F_3, ..., F_k$ on F form a $\mathcal{C}^-_k$ proving the lemma in this case.

Otherwise all sets in $\mathcal{F}_{x_1}$ contain x_2 and thus as x_1 is of maximum degree x_1 and x_2 are contained in the same sets of $\mathcal{F}$. The number of sets in $\mathcal{F}$ containing both x_1 and x_2 is at most $\binom{n-2}{k-2}$, thus removing these sets from $\mathcal{F}$ there remains a family $\mathcal{F}^1$ of subsets of $[n]\setminus\{x_1,x_2\}$ of size at least

$$\begin{aligned}
&c'\binom{n-1}{k-1} - \binom{n-2}{k-2} \\
&\qquad = c'\left(\binom{n-1}{k-1} - \binom{n-2}{k-1} + \binom{n-2}{k-1} - \binom{n-3}{k-1}\right) - \binom{n-2}{k-2} + c'\binom{n-3}{k-1} \\
&\qquad = c'\left(\binom{n-2}{k-2} + \binom{n-3}{k-2}\right) - \binom{n-2}{k-2} + c'\binom{n-3}{k-1} \geq c'\binom{n-3}{k-1} + 1,
\end{aligned}$$

where the last inequality follows by (2).

Let us consider an element $x_3 \in [n]\setminus\{x_1,x_2\}$ with maximum degree in $\mathcal{F}^1$. Repeating the above argument we either find a set $X \subset [n]\setminus\{x_1,x_2\}$ such that $\mathcal{C}^-_k \subset \mathcal{F}^1_{x_3}|_X \subset \mathcal{F}|_X$ or we have an element $x_4 \in [n]\setminus\{x_1,x_2,x_3\}$ such that x_3 and x_4 are contained in exactly the same sets of $\mathcal{F}^1$. Removing these sets from $\mathcal{F}^1$ we obtain a family $\mathcal{F}^2 \subset \binom{[n]\setminus\{x_1,x_2,x_3,x_4\}}{k}$ with size at least $|\mathcal{F}^1| - \binom{n-4}{k-2} \geq c'\binom{n-3}{k-1} - \binom{n-4}{k-2} + 1$ which is by (2) greater or equal to $c'\binom{n-5}{k-1} + 1 + 1 = c'\binom{n-5}{k-1} + 2$.

Repeating the above argument l times, we either find a set X such that $\mathcal{C}_k^- \subseteq \mathcal{F}^{l-1}|_X \subseteq \mathcal{F}|_X$ or subfamily $\mathcal{F}^l \subseteq \binom{[n]\setminus\{x_1,x_2,z_3,x_4,\dots,x_{2l-1},x_{2l}\}}{k} \cap \mathcal{F}^{l-1}$ with size at least $c'\binom{n-2l-1}{k-1}+l$. Thus we either find a set X such that $\mathcal{C}_k^- \subseteq \mathcal{F}^l|_X \subseteq \mathcal{F}|_X$ for some $l \leq \frac{n-M}{2}$ or as $n \geq M+1+2\binom{M+1}{k-1}$ we obtain a subfamily of $\mathcal{F}$ on M or $M+1$ elements (depending on the parity of n) with size larger than $\binom{M+1}{k-1}$, and thus by Theorem 1.2 we even find a $2^{[k]}$ as trace which proves the lemma. $\square$

To prove the theorem for some k and c, let us fix an integer $N(k,c)$ larger than $N'(k, \frac{c+1/2}{2})$ of the Lemma such that for any $n \geq N(k,c)$ the following inequality holds

$$\frac{c-1/2}{2}\binom{n-1}{k-1} > \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1. \tag{3}$$

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a family with size at least $c\binom{n-1}{k-1}$. We claim that the size of the set $\mathcal{H} = \{X \subset [n] : |X| = k, \mathcal{C}_k^- \subseteq \mathcal{F}_X\}$ is at least $\frac{c-1/2}{2}\binom{n-1}{k-1}$. Indeed, using Lemma 2.1 to $\mathcal{F}$ we obtain 1 set in $\mathcal{H}$, then removing this set from $\mathcal{F}$ and applying the Lemma again we get another set and so on until the remaining family contains less set than $\frac{c+1/2}{2}\binom{n-1}{k-1}$ sets. If there is a pair of disjoint sets $X_1, X_2 \in \mathcal{H}$, then $X_1 \cap X_2 = \emptyset$ extends this to a $\mathcal{C}_k$, thus we may assume that those sets form an intersecting family, therefore by Theorem 1.4 and (3) there must exist an element $x \in [n]$ such that $x \in X$ for all $X \in \mathcal{H}$. Any set $F \in \mathcal{F} \setminus \mathcal{H}$ must meet all sets in $\mathcal{H}$ as otherwise $F|_X = F \cap X = \emptyset$ would complete $\mathcal{C}_k^- \subseteq \mathcal{F}|_X$ to $\mathcal{C}_k$. But this can happen only if F contains x as otherwise the number of k-sets containing x and meeting F would be $\binom{n-1}{k-1} - \binom{n-k-1}{k-1}$ which is by (3) smaller than $\frac{c-1/2}{2}\binom{n-1}{k-1} \leq |\mathcal{H}|$. Thus all sets in $\mathcal{F}$ contain x which proves the theorem.

Remark

Frankl and Watanabe [6] strengthened the conjecture of Frankl and Pach to the following: for every $k \leq m$ there exists an $N = N(k,m)$ such that for any $n \geq N$ and family $\mathcal{F} \subseteq \binom{[n]}{m}$ with size larger than $\binom{n-m+k-1}{k-1}$ there is a k-subset X of $[n]$ such that $2^{[k]} = \mathcal{F}|_X$. The counterexample of Ahlswede and Khachatrian can be extended to the $k < m$ case. It is natural to ask what happens if we change again $2^{[k]}$ to $\mathcal{C}_k$. Our proof does not carry through mainly because of two reasons: $X_1 \cap X_2 = \emptyset$, $|X_1| = |X_2| = k$, $\mathcal{F} \subseteq \binom{[n]}{m}$, $\mathcal{C}_k^- \subseteq \mathcal{F}|_{X_1}, \mathcal{F}|_{X_2}$ does not imply $\mathcal{C}_k \subseteq \mathcal{F}|_{X_1}$ if $k < m$ and two different m-sets F_1, F_2 may have $F_1|_X = F_2|_X = X$ for some k-set X. However, we conjecture that an analogous statement for the $k < m$ case is true.

Conjecture 2.2. *For any pair of integers $2 \leq k \leq m$ and $\mathcal{F} \subseteq \binom{[n]}{m}$ with $|\mathcal{F}| > \binom{n-m+k-1}{k-1}$ there exists a k-subset X of $[n]$ such that $\mathcal{C}_k \subseteq \mathcal{F}|_X$.*

The bound (if true) would be sharp as shown by the family $\{F \in \binom{[n]}{m} : [m-k+1] \subset F\}$.

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