The inverse Erdős-Heilbronn problem

Van H. Vu*

Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA

vanvu@math.rutgers.edu

Philip Matchett Wood

Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA

matchett@math.rutgers.edu

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Abstract

The famous Erdős-Heilbronn conjecture (first proved by Dias da Silva and Hamidoune in 1994) asserts that if A is a subset of $\mathbb{Z}/p\mathbb{Z}$, the cyclic group of the integers modulo a prime p, then $|A \widehat{+} A| \geqslant \min\{2|A|-3, p\}$. The bound is sharp, as is shown by choosing A to be an arithmetic progression. A natural inverse result was proven by Karolyi in 2005: if $A \subset \mathbb{Z}/p\mathbb{Z}$ contains at least 5 elements and $|A \widehat{+} A| \leqslant 2|A|-3 < p$, then A must be an arithmetic progression.

We consider a large prime p and investigate the following more general question: what is the structure of sets $A \subset \mathbb{Z}/p\mathbb{Z}$ such that $|A \widehat{+} A| \leqslant (2+\epsilon)|A|$?

Our main result is an asymptotically complete answer to this question: there exists a function $\delta(p) = o(1)$ such that if $200 < |A| \leqslant (1-\epsilon')p/2$ and if $|A \widehat{+} A| \leqslant (2+\epsilon)|A|$, where $\epsilon' - \epsilon \geqslant \delta > 0$, then A is contained in an arithmetic progression of length $|A \widehat{+} A| - |A| + 3$.

With the extra assumption that $|A| \leqslant (\frac{1}{2} - \frac{1}{\log^c p})p$, our main result has Dias da Silva and Hamidoune's theorem and Karolyi's theorem as corollaries, and thus, our main result provides purely combinatorial proofs for the Erdős-Heilbronn conjecture and an inverse Erdős-Heilbronn theorem.

1 Introduction

For A a subset of an abelian group, we define the *sumset* of A to be the set of all sums of two elements in A, namely,

$$A + A := \{a + b : a, b \in A\};$$

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and we define the *restricted sumset* of A to be the set of all sums of two distinct elements of A, namely,

$$A \widehat{+} A := \{a + b : a, b \in A \text{ and } a \neq b\}.$$

Sumsets in a general abelian group have been extensively studied (see [31] for a survey), and we will focus on sumsets of $\mathbb{Z}/p\mathbb{Z}$, the integers modulo p, where p is a prime (see [29] for a survey). For variations on restricted sumset addition, see [25], [26], and [27].

Cauchy [8] and Davenport [9] proved independently that for every $A \subset \mathbb{Z}/p\mathbb{Z}$ we have $|A + A| \geqslant \min\{p, 2|A| - 1\}$. The problem of finding a lower bound for the cardinality of restricted sumsets in $\mathbb{Z}/p\mathbb{Z}$ is much harder. Erdős and Heilbronn made the following conjecture in 1964, which was proved by Dias da Silva and Hamidoune [10] thirty years later.

Theorem 1.1. [10] *For every $A \subset \mathbb{Z}/p\mathbb{Z}$, we have $|A \widehat{+} A| \geqslant \min\{p, 2|A| - 3\}$.*

The $2|A|-1$ term in the Cauchy-Davenport theorem and the $2|A|-3$ term in the Dias da Silva-Hamidoune theorem come from the extremal case when A is an arithmetic progression. For unrestricted sumsets, Vosper [40, 39] showed that an arithmetic progression is indeed the only extremal example:

Theorem 1.2. [40, 39] *For $A \subset \mathbb{Z}/p\mathbb{Z}$, if $|A + A| = 2|A| - 1 < p$, then A is an arithmetic progression.*

Though the situation with restricted sumsets is much more difficult, in 2005, Gyula Károlyi [24] proved a theorem that is just as strong as Vosper's:

Theorem 1.3. [24] *For $A \subset \mathbb{Z}/p\mathbb{Z}$, if $|A \widehat{+} A| = 2|A| - 3 < p$ and $5 \leqslant |A|$, then A is an arithmetic progression.*

Theorem 1.3 is notable in that Károlyi [24] succeeds in using an algebraic approach to prove a structural result, which has the added benefit that using ideas in [21, 22], Károlyi is able extend Theorem 1.3 to an arbitrary abelian group (see [24]).

Our goal is to investigate the following more general question:

Question 1.4. *For a constant $0 \leqslant c \leqslant 1$, classify all subsets $A \subset \mathbb{Z}/p\mathbb{Z}$ for which $|A| < p/(2 + c)$ and $|A \widehat{+} A| \leqslant (2 + c)|A|$.*

The $c = 1$ case of Question 1.4 is similar to a conjectural result suggested by Lev [25, page 29] (see Remark 5.1 for a comparison).

Partial answers for Question 1.4 were given by Bilu, Lev, and Ruzsa [5], by Freiman, Low, and Pitman [13], by Lev [25], and by Schoen [33]. To the best of our knowledge, the most current result is the following from [33]:

Theorem 1.5. [33] *For every $\epsilon > 0$, there exists a constant $n_0 = n_0(\epsilon)$ such that every set $A \subset \mathbb{Z}/p\mathbb{Z}$ satisfying $n_0 \leqslant |A| \leqslant p/35$ and satisfying*

$$|A \widehat{+} A| \leqslant (2.4 - \epsilon)|A|$$

is contained in an arithmetic progression in $\mathbb{Z}/p\mathbb{Z}$ of at most $|A \widehat{+} A| - |A| + 3$ terms.

Our main result is the following:

Theorem 1.6 (main theorem)**.** *There exist absolute constants* $p_0 \geqslant 2^{94}$ *and* $c > 0$ *such that the following holds for all* $p \geqslant p_0$ *and all* $0 \leqslant \epsilon < \epsilon' \leqslant 10^{-4}$ *satisfying* $\epsilon' - \epsilon \geqslant \frac{c(\log\log p)^2}{(\log p)^{2/3}}$*. If* $200 \leqslant |A| \leqslant \frac{p-3}{2(1+\epsilon')}$ *and if*

$$|A \widehat{+} A| \leqslant (2+\epsilon)\,|A|\,,$$

then A *is contained in an arithmetic progression of at most* $|A \widehat{+} A| - |A| + 3$ *terms.*

When $|A| \geqslant (p+3)/2$, it is trivial that $A \widehat{+} A$ is all of $\mathbb{Z}/p\mathbb{Z}$. Thus, Theorem 1.6 provides an asymptotically complete answer to Question 1.4 for small c via combinatorial methods. As corollaries to Theorem 1.6, it is easy to derive asymptotically complete versions of Theorem 1.1 and Theorem 1.3, thus providing alternate proofs for the Erdős-Heilbronn conjecture and an inverse Erdős-Heilbronn theorem, except for those A such that $(1-\delta)p/2 < |A| \leqslant (p+1)/2$ or $|A| < 200$, where δ goes to zero as p increases.

2 A combinatorial approach

There are two previous approaches to proving of the Erdős-Heilbronn conjecture. Dias da Silva and Hamidoune [10] used representation theory of the symmetric group, Young tableau, and exterior algebras in their proof. Later, Alon, Nathanson, and Ruzsa [3, 4] found another proof using the powerful Combinatorial Nullstellensatz (see [1, 2, 23] for surveys). Both proofs have a strong algebraic flavor, and in a remarkable step forward, Károlyi [24] used the Combinatorial Nullstellensatz and careful algebraic analysis to prove Theorem 1.3 ([24] also gives an alternate proof of Theorem 1.2).

A more combinatorial approach to the Erdős-Heilbronn conjecture (Theorem 1.1) is the rectification method, introduced by Freiman [12]. To apply the rectification method, one shows that if $|A \widehat{+} A|$ is sufficiently small then A can be viewed as a set of integers, and then one appeals to a version of Theorem 1.1 for subsets of integers (which is not hard to prove). The rectification method was used by Freiman, Low, and Pitman [13] in 1999 to prove Theorem 1.1 with the additional assumption that $60 \leqslant |A| \leqslant p/50$.

To prove our main result (Theorem 1.6), we will combine ideas from the rectification method with a strong new result due to Serra and Zémor [36] (see Subsection 4.2 for a discussion of the Serra-Zémor result). The first step in our proof, which we will carry out in the next section, is to reduce the study of restricted sumsets to non-restricted sumsets. This approach was first applied to the inverse Erdős-Heilbronn problem by Schoen [33] in 2002.

3 Translating between $A + A$ and $A \widehat{+} A$

Lemma 3.1. *There exists an absolute constant* c_0 *such that if* p *is sufficiently large and* $A \subset \mathbb{Z}/p\mathbb{Z}$*, then*

$$|A \widehat{+} A| > |A + A| - p\left(\frac{c_0(\log\log p)^2}{(\log p)^{2/3}}\right).$$

Proof. We proceed by bounding the cardinality of the set $E := \{z \in A : z + z \notin A \widehat{+} A\}$. Note that by the definition of sumset and restricted sumset, $|A + A| = |A \widehat{+} A| + |E|$. If

$$|E| \geqslant p \left(\frac{c_0 (\log \log p)^2}{(\log p)^{2/3}} \right)$$

for a particular constant c_0, then by [7] and the fact that p is sufficiently large, we have that the set E contains a non-trivial three-term arithmetic progression, say $a, b, c \in E$ such that $a \neq c$ and $a + c = 2b$. But then $b + b = 2b = a + c \in A \widehat{+} A$, a contradiction of the definition of the set E. Thus, we must have that

$$|E| < p \left(\frac{c_0 (\log \log p)^2}{(\log p)^{2/3}} \right).$$

Hence

$$|A + A| = |A \widehat{+} A| + |E| < |A \widehat{+} A| + p \left(\frac{c_0 (\log \log p)^2}{(\log p)^{2/3}} \right),$$

which is the desired inequality. □

Later, we found out that Schoen [33] proved a similar result to the above, using a different argument. Both arguments use results of Bourgain [6, 7] on integer sets containing no arithmetic progressions, and in the case when $|A|/p$ is bounded from below by a constant, our bound compares favorably to [33].

4 Background Results

4.1 Rectification

The rectification approach to sumset problems is to show that a subset $A \subset \mathbb{Z}/p\mathbb{Z}$ must behave the same way as a subset $B \subset \mathbb{Z}$, and then to appeal to a sumset result for the integers. For example, Schoen [33] proved Theorem 1.5 by passing to the integers and then applying a corollary of the following result, which is due to Lev (see [25, Theorem 1]).

Theorem 4.1. [25] *Let B be a set of $n \geqslant 3$ non-negative integers such that* $\gcd(B) = 1$ *and* $0 \in B$. *Then,*

$$\left|B \widehat{+} B\right| \geqslant \begin{cases} \max(B) + |B| - 2 & \text{if } \max(B) \leqslant 2\,|B| - 5, \\ 2.61\,|B| - 6 & \text{if } \max(B) \geqslant 2\,|B| - 4. \end{cases}$$

The rectification method was used by Freiman, Low, and Pitman [13, Theorem 2] to give the first partial answer to Question 1.4, and Lev [25] improved on their result to get the following theorem.

Theorem 4.2. [25] *Let A be a subset of $\mathbb{Z}/p\mathbb{Z}$ where $200 \leqslant |A| \leqslant p/50$. If*

$$|A \widehat{+} A| \leqslant 2.18\,|A| - 6,$$

then A is contained in an arithmetic progression of at most $|A \widehat{+} A| - |A| + 3$ terms.

We will use Theorem 4.2 to prove our main theorem (Theorem 1.6) in the case where A has cardinality $200 \leqslant |A| \leqslant p/50$.

4.2 The isoperimetric method

The isoperimetric method is an alternative to the rectification method, and it is used to indirectly show that a subset $A \subset \mathbb{Z}/p\mathbb{Z}$ behaves like a subset of the integers, typically by studying an extremal set that is constructed using the original set A. The isoperimetric method was introduced by Hamidoune [14] and was developed by the same author [15, 16] along with Serra and Zémor as coauthors [18, 19]. For a survey of the isoperimetric method, see [34].

The following is the main result from the isoperimetric method that we will use, and it was proven by Serra and Zémor [36, Theorem 3].

Theorem 4.3. [36] *There exist positive numbers p_0 and ϵ' such that for all primes $p > p_0$, any subset A of $\mathbb{Z}/p\mathbb{Z}$ such that*

(i) $|A + A| < (2 + \epsilon')\,|A|$ *and*

(ii) $m = |A + A| - 2\,|A|$ *satisfies* $m \leqslant \min\{|A| - 4, p - |A + A| - 3\}$

is contained in an arithmetic progression of at most $|A| + m + 1$ terms. Furthermore, one can take $\epsilon' = 10^{-4}$ and $p_0 = 2^{94}$.

Previous inverse theorems for sumsets focused on making the value of ϵ' as large as possible, even at the expense of requiring $|A|$ to be small. Serra and Zémor [36], on the other hand, proved the above result allowing $|A|$ to be as large as possible, at the expense of requiring ϵ' to be small.

5 Proof of the main theorem (Theorem 1.6)

By Theorem 4.2, we may assume that $|A| > p/50$. By hypothesis $|A \widehat{+} A| \leqslant (2 + \epsilon)\,|A|$, and so by Lemma 3.1,

$$\begin{aligned}(2+\epsilon)\,|A| \geqslant |A \widehat{+} A| &> |A + A| - p\left(\frac{c_0(\log\log p)^2}{(\log p)^{2/3}}\right)\\ &\geqslant |A + A|\left(1 - \frac{c'(\log\log p)^2}{(\log p)^{2/3}}\right),\end{aligned}$$

where, say, $c' = 50c_0$.

It is straightforward to verify condition (ii) of Theorem 4.3, and so we need to verify condition (i) by showing

$$\frac{2+\epsilon}{1 - \left(\frac{c'(\log\log p)^2}{(\log p)^{2/3}}\right)} \leqslant 2 + \epsilon'. \tag{1}$$

Setting $c = (2 + 10^{-4})c'$, we see that Inequality (1) is true if $\frac{c(\log \log p)^2}{(\log p)^{2/3}} \leqslant \epsilon' - \epsilon$, which holds by assumption.

Thus, we can apply Theorem 4.3 to show that A is contained in an arithmetic progression with at most $|A + A| - |A| + 1 \leqslant (1 + \epsilon') |A| + 1 \leqslant (p - 1)/2$ terms. The next step is to show that A is Freiman isomorphic of order 2 to a set integers satisfying the hypotheses of Theorem 4.1, which will allow us to conclude the result (see [38, Chapter 5.3] for a discussion of Freiman isomorphisms).

Let $L := \{a_0 + id \mod p : 0 \leqslant i \leqslant (1 + \epsilon') |A|\}$ be an arithmetic progression containing A, where i, a_0, and d are integers. Note that L is Freiman isomorphic or order 2 to the set of integers $M = \{0, 1, 2, \ldots, \lfloor (1 + \epsilon') |A| \rfloor\}$ and that A is Freiman isomorphic of order 2 to the set of integers $B = \{i \in M : a_0 + id \mod p \in A\}$. We may assume (by shifting L if necessary) that $a_0 \mod p \in A$, so that $0 \in B$ and B consists of non-negative integers. Since B is sufficiently dense in the interval M (recall, M contains at most $(1 + \epsilon') |B| + 1$ elements), we know that there exist two elements of B that differ by exactly 1, and so $\gcd(B) = 1$. Finally, we have $\left|B \hat{+} B\right| = |A \hat{+} A| \leqslant (2 + \epsilon) |A| = (2 + \epsilon) |B|$, and so by Theorem 4.1, we have that

$$\max(B) \leqslant \left|B \hat{+} B\right| - |B| + 2 = |A \hat{+} A| - |A| + 2.$$

Thus, B is contained in $M' := \{0, 1, 2, \ldots, |A \hat{+} A| - |A| + 2\}$, and so A is contained in $L' := \{a_0 + id \mod p : 0 \leqslant i \leqslant |A \hat{+} A| - |A| + 2\}$. We have thus shown that A is contained in an arithmetic progression of at most $|A \hat{+} A| - |A| + 3$ terms. ☐

Remark 5.1. It has been conjectured (see [25, page 29]) that a structure theorem along the lines of Theorem 1.6 may hold for a subset $A \subset \mathbb{Z}/p\mathbb{Z}$ satisfying $|A \hat{+} A| \leqslant 3|A| - 7$ and $|A| \leqslant (p - C)/2$, for some relatively small absolute constant C. However, it is possible to randomly construct sets A such that $|A|$ is slightly larger than $p/3$ and such that A has no arithmetic structure. Such a set A automatically satisfies $|A \hat{+} A| \leqslant 3|A| - 7$ (since $3|A| \geqslant p + 7$) and therefore violates the conjecture. In general, by the same random construction, any structure result derived from the hypothesis $|A \hat{+} A| \leqslant (2 + c)|A|$, where $0 \leqslant c \leqslant 1$ is a constant, must also include the hypothesis $|A| \leqslant p/(2 + c)$. For this reason, we include the hypothesis $|A| < p/(2 + c)$ in Question 1.4.

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