The spectral gap of random graphs with given expected degrees*

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Abstract

We investigate the Laplacian eigenvalues of a random graph $G(n, \boldsymbol{d})$ with a given expected degree distribution $\boldsymbol{d}$. The main result is that w.h.p. $G(n, \boldsymbol{d})$ has a large subgraph $\mathrm{core}(G(n, \boldsymbol{d}))$ such that the spectral gap of the normalized Laplacian of $\mathrm{core}(G(n, \boldsymbol{d}))$ is $\geqslant 1 - c_0 \bar{d}_{\min}^{-1/2}$ with high probability; here $c_0 > 0$ is a constant, and $\bar{d}_{\min}$ signifies the minimum expected degree. The result in particular applies to *sparse* graphs with $\bar{d}_{\min} = O(1)$ as $n \to \infty$. The present paper complements the work of Chung, Lu, and Vu [Internet Mathematics **1**, 2003].

1 Introduction and Results

1.1 Spectral Techniques for Graph Problems

Numerous heuristics for graph partitioning problems are based on *spectral methods*: the heuristic sets up a matrix that represents the input graph and reads information on the *global structure* of the graph out of the eigenvalues and eigenvectors of the matrix. Since there are rather efficient methods for computing eigenvalues and -vectors, spectral techniques are very popular in various applications [22, 23].

Though in many cases there are worst-case examples known showing that certain spectral heuristics perform badly on general instances (e.g., [16]), spectral methods are in

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common use and seem to perform well on many "practical" inputs. Therefore, in order to gain a better theoretical understanding of spectral methods, quite a few papers deal with rigorous analyses of spectral heuristics on suitable classes of *random* graphs. For example, Alon and Kahale [2] suggested a spectral heuristic for Graph Coloring, Alon, Krivelevich, and Sudakov [3] dealt with a spectral method for Maximum Clique, and McSherry [20] studied a spectral heuristic for recovering a "latent" partition.

However, a crucial problem with most known spectral methods is that their use is limited to essentially *regular* graphs, where all vertices have (approximately) the same degree. The reason is that most of these algorithms rely on the spectrum of the *adjacency matrix*, which is quite susceptible to fluctuations of the vertex degrees. In fact, as Mihail and Papadimitriou [21] pointed out, in the case of irregular graphs the eigenvalues of the adjacency matrix just mirror the tails of the degree distribution, and thus do not reflect any *global* graph properties.

Nevertheless, in the recent years it has emerged that many interesting types of graphs actually share two peculiar properties. The first one is that the distribution of the vertex degrees is *extremely irregular*. In fact, 'power law' degree distributions where the number of vertices of degree d is proportional to $d^{-\gamma}$ for a constant $\gamma > 1$ are ubiqutuous [1, 12]. The second property is *sparsity*, i.e., the average degree remains bounded as the size of the graph/network grows over time. Concrete examples include the www and further graphs related to the Internet [12].

Therefore, the goal of this paper is to study the use of spectral methods on a simple model of *sparse* and *irregular* random graphs. More precisely, we are going to work with the following model of random graphs with a given expected degree sequence from Chung and Lu [7].

> Let $V = \{1, \dots, n\}$, and let $\boldsymbol{d} = (\bar{d}(v))_{v \in V}$, where each $\bar{d}(v)$ is a positive real. Let $\bar{d} = \frac{1}{n}\sum_{v \in V} \bar{d}(v)$ and suppose that $\bar{d}(w)^2 = o(\sum_{v \in V} \bar{d}(v))$ for all $w \in V$. Then $G(n, \boldsymbol{d})$ has the vertex set V, and for any two distinct vertices $v, w \in V$ the edge $\{v, w\}$ is present with probability $p_{vw} = \bar{d}(v)\bar{d}(w)(\bar{d}n)^{-1}$ independently of all others.

Of course, the random graph model $G(n, \boldsymbol{d})$ is simplistic in that edges occur independently. Other models (e.g., the 'preferential attachment model') are arguably more meaningful in many contexts as they actually provide a process that naturally entails an irregular degree distribution [4]. By contrast, in $G(n, \boldsymbol{d})$ the degree distribution is given a priori. Hence, one could say that this paper merely to provides a 'proof of concept': spectral methods can be adapted so as to be applicable to sparse irregular graphs.

Let us point out a few basic properties of $G(n, \boldsymbol{d})$. Assuming that $\bar{d}(v) \ll \bar{d}n$ for all $v \in V$, we see that the *expected* degree of each vertex $v \in V$ is $\sum_{w \in V - \{v\}} p_{vw} = \bar{d}(v)(1 - (\bar{d}n)^{-1}) \sim \bar{d}(v)$, and the expected average degree is $(1 - o(1))\bar{d}$. In other words, $G(n, \boldsymbol{d})$ is a random graph with a given expected degree sequence $\boldsymbol{d}$. We say that $G(n, \boldsymbol{d})$ has some property $\mathcal{E}$ *with high probability (w.h.p.)* if the probability that $\mathcal{E}$ holds tends to one as $n \to \infty$.

While Mihail and Papadimitriou [21] proved that in general the spectrum of the *adjacency matrix* of $G(n, \boldsymbol{d})$ does not yield any information about global graph properties but is just determined by the upper tail of the degree sequence $\boldsymbol{d}$, Chung, Lu, and Vu [8] studied the eigenvalue distribution of the *normalized Laplacian* of $G(n, \boldsymbol{d})$. To state their result precisely, we recall that the normalized Laplacian $\mathcal{L}(G)$ of a graph $G = (V, E)$ is defined as follows. Letting $d_G(v)$ denote the degree of v in G, we set

$$\ell_{vw} = \begin{cases} 1 & \text{if } v = w \text{ and } d_G(v) > 0, \\ -1/\sqrt{d_G(v)d_G(w)} & \text{if } \{v, w\} \in E, \\ 0 & \text{otherwise} \end{cases} \qquad (v, w \in V(G)) \tag{1}$$

and define $\mathcal{L}(G) = (\ell_{vw})_{v,w\in V}$. Then $\mathcal{L}(G)$ is singular and positive semidefinite, and its largest eigenvalue is $\leqslant 2$. Letting $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{\#V}$ denote the eigenvalues of $\mathcal{L}(G)$, we call $\lambda(G) = \min\{\lambda_2, 2 - \lambda_{\#V}\}$ the *spectral gap* of $\mathcal{L}(G)$. Now, setting $\bar{d}_{\min} = \min_{v\in V} \bar{d}(v)$ and assuming $\bar{d}_{\min} \gg \ln^2 n$, Chung, Lu, and Vu proved that

$$\lambda(G(n, \boldsymbol{d})) \geqslant 1 - (1 + o(1))4\bar{d}^{\,-\frac{1}{2}} - \bar{d}_{\min}^{-1} \ln^2 n \tag{2}$$

w.h.p. As for general graphs with average degree $\bar{d}$ the spectral gap is at most $1 - 4\bar{d}^{\,-\frac{1}{2}}$, the bound (2) is essentially best possible.

The spectral gap is directly related to various combinatorial graph properties. To see this, we let $e(X, Y) = e_G(X, Y)$ signify the number of X-Y-edges in G for any two sets $X, Y \subset V$, and we set $d_G(X) = \sum_{v\in X} d_G(v)$. We say that G has (α, β)-*low discrepancy* if for *any* two disjoint sets $X, Y \subset V$ we have

$$\left|e_G(X, Y) - d_G(X)d_G(Y)(2\#E)^{-1}\right| \leqslant (1 - \alpha)\sqrt{d_G(X)d_G(Y)} + \beta \qquad \text{and} \tag{3}$$

$$\left|2e_G(X, X) - d_G(X)^2(2\#E)^{-1}\right| \leqslant (1 - \alpha)d_G(X) + \beta. \tag{4}$$

An easy computation shows that $d_G(X)d_G(Y)(2\#E)^{-1}$ is the number of X-Y-edges that we would *expect* if G were a random graph with expected degree sequence $\boldsymbol{d} = (d_G(v))_{v\in V}$. Similarly, $d_G(X)^2(4\#E)^{-1}$ is the expected number of edges inside of X in such a random graph. Thus, the closer $\alpha < 1$ is to 1 and the smaller $\beta \geqslant 0$, the more G "resembles" a random graph if (3) and (4) hold. Finally, if $\lambda(G) \geqslant \gamma$, then G has $(\gamma, 0)$-low discrepancy [6]. Hence, the larger the spectral gap, the more G "looks like" a random graph.

As a consequence, the result (2) of Chung, Lu, and Vu shows that the spectrum of the Laplacian does reflect the global structure of the random graph $G(n, \boldsymbol{d})$ (namely, the low discrepancy property), provided that $\bar{d}_{\min} = \min_{v\in V} \bar{d}(v) \gg \ln^2 n$, i.e., the graph is *dense enough*. Studying the normalized Laplacian of *sparse* random graphs $G(n, \boldsymbol{d})$ (e.g., with average degree $\bar{d} = O(1)$ as $n \to \infty$), we complement this result.

1.2 Results

Observe that (2) is void if $\bar{d}_{\min} \leqslant \ln^2 n$, because in this case the r.h.s. is negative. In fact, the following proposition shows that if $\bar{d}$ is "small", then in general the spectral gap of $\mathcal{L}(G(n, \boldsymbol{d}))$ is just 0, even if the expected degrees of all vertices coincide.

Proposition 1.1 *Let $d > 0$ be arbitrary but constant, set $d_v = d$ for all $v \in V$, and let $\boldsymbol{d} = (d_v)_{v \in V}$. Let $0 = \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant 2$ be the eigenvalues of $\mathcal{L}(G(n, \boldsymbol{d}))$. Then w.h.p. the following holds.*

1. *There are numbers $k, l = \Omega(n)$ such that $\lambda_k = 0$ and $\lambda_{n-l} = 2$; in other words, the eigenvalues 0 and 2 have multiplicity $\Omega(n)$, and thus the spectral gap is 0.*
2. *For each fixed $k \geqslant 2$ there exist $\Omega(n)$ of indices j such that $\lambda_j = 1 - k^{-1/2} + o(1)$.*
3. *Similarly, for any fixed $k \geqslant 2$ there are $\Omega(n)$ of indices j so that $\lambda_j = 1 + k^{-1/2} + o(1)$.*

Nonetheless, the main result of the paper shows that even in the sparse case w.h.p. $G(n, \boldsymbol{d})$ has a *large subgraph* $\mathrm{core}(G)$ on which a similar statement as (2) holds.

Theorem 1.2 *There are constants $c_0, d_0 > 0$ such that the following holds. Suppose that $\boldsymbol{d} = (\bar{d}(v))_{v \in V}$ satisfies*

$$d_0 \leqslant \bar{d}_{\min} = \min_{v \in V} \bar{d}(v) \leqslant \max_{v \in V} \bar{d}(v) \leqslant n^{0.99}. \tag{5}$$

Then w.h.p. the random graph $G = G(n, \boldsymbol{d})$ has an induced subgraph $\mathrm{core}(G)$ that enjoys the following properties.

1. *We have $\sum_{v \in G - \mathrm{core}(G)} \bar{d}(v) + d_G(v) \leqslant n \exp(-\bar{d}_{\min}/c_0)$.*
2. *Moreover, the spectral gap satisfies $\lambda(\mathrm{core}(G)) \geqslant 1 - c_0 \bar{d}_{\min}^{-1/2}$.*

The first part of Theorem 1.2 says that w.h.p. $\mathrm{core}(G)$ constitutes a "huge" subgraph of G. Moreover, by the second part the spectral gap of the core is close to 1 if $\bar{d}_{\min}$ exceeds a certain constant. An important aspect is that the theorem applies to very general degree distributions, including but not limited to the case of power laws.

It is instructive to compare Theorem 1.2 with (2), cf. Remark 3.7 below. Further, in Remark 3.6 we point out that the bound on the spectral gap given in Theorem 1.2 is best possible up to the precise value of c_0.

Theorem 1.2 has a few interesting algorithmic implications. Namely, we can extend a couple of algorithmic results for random graphs in which all expected degrees are equal to the irregular case.

Corollary 1.3 *There is a polynomial time algorithm `LowDisc` that satisfies the following two conditions.*

Correctness. *For any input graph G `LowDisc` outputs two numbers $\alpha, \beta \geqslant 0$ such that G has (α, β)-low discrepancy.*

Completeness. *If $G = G(n, \boldsymbol{d})$ is a random graph such that $\boldsymbol{d}$ satisfies the assumption (5) of Theorem 1.2, then $\alpha \geqslant 1 - c_0 \bar{d}_{\min}^{-1/2}$ and $\beta \leqslant n \exp(-\bar{d}_{\min}/(2c_0))$ w.h.p.*

`LowDisc` relies on the fact that for a given graph G the subgraph core(G) can be computed efficiently. Then, `LowDisc` computes the spectral gap of $\mathcal{L}(\mathrm{core}(G))$ to bound the discrepancy of G. If $G = G(n, \boldsymbol{d})$, then Theorem 1.2 entails that the spectral gap is large w.h.p., so that the bound (α, β) on the discrepancy of $G(n, \boldsymbol{d})$ is "small". Hence, `LowDisc` shows that spectral techniques do yield information on the global structure of the random graphs $G(n, \boldsymbol{d})$.

One might argue that we could just derive by probabilistic techniques such as the "first moment method" that $G(n, \boldsymbol{d})$ has low discrepancy w.h.p. However, such arguments just show that "most" graphs $G(n, \boldsymbol{d})$ have low discrepancy. By contrast, the statement of Corollary 1.3 is much stronger: for a *given outcome* $G = G(n, \boldsymbol{d})$ of the random experiment we can find a *proof* that G has low discrepancy *in polynomial time*. This can, of course, not be established by the "first moment method" or the like.

Since the discrepancy of a graph is closely related to quite a few prominent graph invariants that are (in the worst case) NP-hard to compute, we can apply Corollary 1.3 to obtain further algorithmic results on random graphs $G(n, \boldsymbol{d})$. For instance, we can bound the *independence number* $\alpha(G(n, \boldsymbol{d}))$ efficiently.

Corollary 1.4 *There exists a polynomial time algorithm* `BoundAlpha` *that satisfies the following conditions.*

Correctness. *For any input graph G* `BoundAlpha` *outputs an upper bound $\alpha \geqslant \alpha(G)$ on the independence number.*

Completeness. *If $G = G(n, \boldsymbol{d})$ is a random graph such that $\boldsymbol{d}$ satisfies (5), then $\alpha \leqslant c_0 n \bar{d}_{\min}^{-1/2}$ w.h.p.*

1.3 Related Work

The Erdős-Rényi model $G_{n,p}$ of random graphs, which is the same as $G(n, \boldsymbol{d})$ with $\bar{d}(v) = np$ for all v, has been studied thoroughly. Concerning the eigenvalues $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ of its adjacency matrix $A = A(G_{n,p})$, Füredi and Komlós [15] showed that if $np(1-p) \gg \ln^6 n$, then $\max\{-\lambda_1(A), \lambda_{n-1}(A)\} \leqslant (2 + o(1))(np(1-p))^{1/2}$ and $\lambda_n(A) \sim np$. Feige and Ofek [13] showed that $\max\{-\lambda_1(A), \lambda_{n-1}(A)\} \leqslant O(np)^{1/2}$ and $\lambda_n(A) = \Theta(np)$ also holds w.h.p. under the weaker assumption $np \geqslant \ln n$.

By contrast, in the sparse case $\bar{d} = np = O(1)$, neither

$$\lambda_n(A) = \Theta(\bar{d}) \text{ nor } \max\{-\lambda_1(A), \lambda_{n-1}(A)\} \leqslant O(\bar{d})^{1/2}$$

is true w.h.p. For if $\bar{d} = O(1)$, then the vertex degrees of $G = G_{n,p}$ have (asymptotically) a Poisson distribution with mean $\bar{d}$. Consequently, the degree distribution features a fairly heavy upper tail. Indeed, the maximum degree is $\Omega(\ln n / \ln \ln n)$ w.h.p., and the highest degree vertices induce both positive and negative eigenvalues as large as $\Omega(\ln n / \ln \ln n)^{1/2}$ in absolute value [19]. Nonetheless, following an idea of Alon and Kahale [2] and building on the work of Kahn and Szemerédi [14], Feige and Ofek [13] showed that the graph $G' = (V', E')$ obtained by removing all vertices of degree, say, $> 2\bar{d}$ from G w.h.p.

satisfies $\max\{-\lambda_1(A(G')), \lambda_{\#V'-1}(A(G'))\} = O(\bar{d}^{1/2})$ and $\lambda_{\#V(G')}(A(G')) = \Theta(\bar{d})$. The articles [13, 15] are the basis of several papers dealing with rigorous analyses of spectral heuristics on random graphs. For instance, Krivelevich and Vu [18] proved (among other things) a similar result as Corollary 1.4 for the $G_{n,p}$ model. Further, the first author [10] used [13, 15] to investigate the Laplacian of $G_{n,p}$.

The graphs we are considering in this paper may have a significantly more general (i.e., irregular) degree distribution than even the sparse random graph $G_{n,p}$. In fact, irregular degree distributions such as *power laws* occur in real-world networks, cf. Section 1.1. While such networks are frequently modeled best by *sparse* graphs (i.e., $\bar{d} = O(1)$ as $n \to \infty$), the *maximum* degree may very well be as large as $n^{\Omega(1)}$, i.e., not only logarithmic but even polynomial in n. As a consequence, the eigenvalues of the adjacency matrix are determined by the upper tail of the degree distribution rather than by global graph properties [21]. Furthermore, the idea of Feige and Ofek [13] of just deleting the vertices of degree $\gg \bar{d}$ is not feasible, because the high degree vertices constitute a significant share of the graph. Thus, the adjacency matrix is simply not appropriate to represent power law graphs.

As already mentioned in Section 1.1, Chung, Lu, and Vu [8] were the first to obtain rigorous results on the normalized Laplacian (in the case $\bar{d}_{\min} \gg \ln^2 n$). In addition to (2), they also proved that the global distribution of the eigenvalues follows the semicircle law. Their proofs rely on the "trace method" of Wigner [24], i.e., Chung, Lu, and Vu (basically) compute the trace of $\mathcal{L}(G(n,\boldsymbol{d}))^k$ for a large even number k. Since this equals the sum of the k'th powers of the eigenvalues of $\mathcal{L}(G(n,\boldsymbol{d}))$, they can thus infer the distribution of the eigenvalues. However, the proofs in [8] hinge upon the assumption that $\bar{d}_{\min} \gg \ln^2 n$, and indeed there seems to be no easy way to extend the trace method to the sparse case. Furthermore, a matrix closely related to the normalized Laplacian was used by Dasgupta, Hopcroft, and McSherry [11] to devise a spectral heuristic for partitioning sufficiently dense irregular graphs (with minimum expected degree $\gg \ln^6 n$). The spectral analysis in [11] also relies on the trace method.

The techniques of this paper can be used to obtain further algorithmic results. For example, in [9] we present a spectral partitioning algorithm for sparse irregular graphs.

1.4 Techniques and Outline

After introducing some notation and stating some auxiliary lemmas on the $G(n,\boldsymbol{d})$ model in Section 2, we prove Proposition 1.1 and define the subgraph $\mathrm{core}(G(n,\boldsymbol{d}))$ in Section 3. The proof of Proposition 1.1 shows that the basic reason why the spectral gap of a sparse random graph $G(n,\boldsymbol{d})$ is small actually is the existence of vertices of degree $\ll \bar{d}_{\min}$, i.e., of "atypically small" degree. Therefore, the subgraph $\mathrm{core}(G(n,\boldsymbol{d}))$ is essentially obtained by removing such vertices. The construction of the core is to some extent inspired by the work of Alon and Kahale [2] on coloring random graphs.

In Section 4 we analyze the spectrum of $\mathcal{L}(\mathrm{core}(G(n,\boldsymbol{d})))$. Here the main difficulty turns out to be the fact that the entries ℓ_{vw} of $\mathcal{L}(\mathrm{core}(G(n,\boldsymbol{d})))$ are mutually dependent random variables (cf. (1)). Therefore, we shall consider a modified matrix $\mathcal{M}$ with entries $(\bar{d}(v)\bar{d}(w))^{-\frac{1}{2}}$ if v, w are adjacent, and 0 otherwise ($v, w \in V$). That is, we replace the actual vertex degrees by their expectations, so that we obtain a matrix with mutually independent entries (up to the trivial dependence resulting from symmetry, of course). Then, we show that $\mathcal{M}$ provides a "reasonable" approximation of $\mathcal{L}(\mathrm{core}(G(n,\boldsymbol{d})))$.

Furthermore, in Section 5 we prove that the spectral gap of $\mathcal{M}$ is large w.h.p., which finally implies Theorem 1.2. The analysis of $\mathcal{M}$ in Section 5 follows a proof strategy of Kahn and Szemerédi [14]. While Kahn and Szemerédi investigated random regular graphs, we modify their method rather significantly so that it applies to irregular graphs. Moreover, Section 6 contains the proofs of Corollaries 1.3 and 1.4. Finally, in Section 7 we prove a few auxiliary lemmas.

2 Preliminaries

Throughout the paper, we let $V=\{1,\ldots,n\}$. Since our aim is to establish statements that hold with probability tending to 1 as $n\to\infty$, we may and shall assume throughout that n is a sufficiently large number. Moreover, we assume that $d_0>0$ and $c_0>0$ signify sufficiently large constants satisfying $c_0\ll d_0$. In addition, we assume that the expected degree sequence $\boldsymbol{d}=(\bar{d}(v))_{v\in V}$ satisfies

$$\begin{aligned} d_0 &\leqslant \bar{d}_{\min}=\min_{v\in V}\bar{d}(v)\leqslant\max_{v\in V}\bar{d}(v)\leqslant n^{0.99}, \qquad \textit{which implies} && (6)\\ \mathrm{Vol}(Q) &= \sum_{v\in Q}\bar{d}(v)\geqslant d_0\#Q \qquad \textit{for all } Q\subset V. && (7)\end{aligned}$$

No attempt has been made to optimize the constants involved in the proofs.

If $G=(V,E)$ is a graph and $U,U'\subset V$, then we let $e(U,U')=e_G(U,U')$ signify the number of U-U'-edges in G. Moreover, we let $\mu(U,U')$ denote the expectation of $e(U,U')$ in a random graph $G=G(n,\boldsymbol{d})$. In addition, we set $\mathrm{Vol}(U)=\sum_{v\in U}\bar{d}(v)$. For a vertex $v\in V$, we let $N_G(v)=\{w\in V:\{v,w\}\in E\}$.

If $M=(m_{vw})_{v,w\in V}$ is a matrix and $A,B\subset V$, then $M_{A\times B}$ denotes the matrix obtain from M by replacing all entries m_{vw} with $(v,w)\notin A\times B$ by 0. Moreover, if $A=B$, then we briefly write M_A instead of $M_{A\times B}$. Further, $\boldsymbol{E}$ signifies the identity matrix (in any dimension). If $x_1,\ldots,x_k$ are numbers, then $\mathrm{diag}(x_1,\ldots,x_k)$ denotes the $k\times k$ matrix with $x_1,\ldots,x_k$ on the diagonal, and zeros everywhere else. For a set X we denote by $\mathbf{1}_X\in\mathbf{R}^X$ the vector with all entries equal to 1. In addition, if $Y\subset X$, then $\mathbf{1}_{X,Y}\in\mathbf{R}^X$ denotes the vector whose entries are 1 on Y, and 0 on $X-Y$.

We frequently need to estimate the probability that a random variable deviates from its mean significantly. Let ϕ denote the function

$$\phi:(-1,\infty)\to\mathbf{R},\quad x\mapsto(1+x)\ln(1+x)-x. \tag{8}$$

Then it is easily verified via elementary calculus that $\phi(x)\leqslant\phi(-x)$ for $0\leqslant x<1$, and that

$$\phi(x)\geqslant\frac{x^2}{2(1+x/3)}\qquad(x\geqslant 0),\qquad\text{cf. [17, p. 27].}\tag{9}$$

A proof of the following Chernoff bound can be found in [17, pages 26–29].

Lemma 2.1 *Let $X = \sum_{i=1}^{N} X_i$ be a sum of mutually independent Bernoulli random variables with variance $\sigma^2 = \mathrm{Var}(X)$. Then for any $t > 0$ we have*

$$\begin{aligned}\max\{\mathrm{P}(X \leqslant \mathrm{E}(X) - t), \mathrm{P}(X \geqslant \mathrm{E}(X) + t)\} &\leqslant \exp\left(-\sigma^2 \phi\left(\frac{t}{\sigma^2}\right)\right) \\ &\leqslant \exp\left(-\frac{t^2}{2(\sigma^2 + t/3)}\right). \qquad (10)\end{aligned}$$

A further type of tail bound that we will use repeatedly concerns functions X from graphs to reals that satisfy the following *Lipschitz condition*:

> Let $G = (V, E)$ be a graph. Let $v, w \in V$, $v \neq w$, and let G^+ (resp. G^-) denote the graph obtained from G by adding (resp. deleting) the edge $\{v, w\}$. Then $|X(G^{\pm}) - X(G)| \leqslant 1$. (11)

Lemma 2.2 *Let $0 < \gamma \leqslant 0.01$ be an arbitrarily small constant. If X satisfies (11), then*

$$\mathrm{P}\left[|X(G(n, \boldsymbol{d})) - \mathrm{E}(X(G(n, \boldsymbol{d})))| \geqslant (\bar{d}n)^{\frac{1}{2}+\gamma}\right] \leqslant \exp(-(\bar{d}n)^{\gamma}/300).$$

Combining (10) and Lemma 2.2, we obtain the following bound on the "empirical variance" of the degree distribution of $G(n, \boldsymbol{d})$.

Corollary 2.3 *W.h.p. $G = G(n, \boldsymbol{d})$ satisfies $\sum_{v \in V}(d_G(v) - \bar{d}(v))^2/\bar{d}(v) \leqslant 10^6 n$.*

A crucial property of $G(n, \boldsymbol{d})$ is that w.h.p. for *all* subsets $U, U' \subset V$ the number $e(U, U')$ of U-U'-edges does not exceed its mean $\mu(U, U')$ to much. More precisely, we have the following estimate.

Lemma 2.4 *W.h.p. $G = G(n, \boldsymbol{d})$ enjoys the following property.*

> *Let $U, U' \subset V$ be subsets of size $u = \#U \leqslant u' = \#U' \leqslant \frac{n}{2}$. Then at least one of the following conditions holds.*
>
> 1. *$e_G(U, U') \leqslant 300\mu(U, U')$.* (12)
> 2. *$e_G(U, U') \ln(e_G(U, U')/\mu(U, U')) \leqslant 300u' \ln(n/u')$.*

If $Q \subset V$ has a "small" volume $\mathrm{Vol}(Q)$, we expect that most vertices in Q have most of their neighbors outside of Q. The next corollary shows that this is in fact the case for all Q simultaneously w.h.p.

Corollary 2.5 *Let $c' > 0$ be a constant. Suppose that $\bar{d}_{\min} \geqslant d_0$ for a sufficiently large number $d_0 = d_0(c')$. Then the random graph $G = G(n, \boldsymbol{d})$ enjoys the following two properties w.h.p.*

> *Let $1 \leqslant \zeta \leqslant \bar{d}^{\frac{1}{2}}$. If the volume of $Q \subset V$ satisfies*
>
> $$\exp(2c'\bar{d}_{\min})\zeta\#Q \leqslant \mathrm{Vol}(Q) \leqslant \exp(-3c'\bar{d}_{\min})n, \qquad (13)$$
>
> *then $e_G(Q) \leqslant 0.001\zeta^{-1}\exp(-c'\bar{d}_{\min})\mathrm{Vol}(Q)$.*
>
> *If $\mathrm{Vol}(Q) \leqslant \bar{d}^{\frac{1}{2}}\#Q^{5/8}n^{3/8}$ and $\#Q \leqslant n/2$, then $e_G(Q) \leqslant 3000\#Q$.* (14)

Finally, the following two lemmas relate to volume $\mathrm{Vol}(Q) = \sum_{v\in Q} \bar{d}(v)$ of a set $Q \subset V$ to the actual sum $\sum_{v\in Q} d_G(v)$.

Lemma 2.6 *The random graph $G = G(n, \boldsymbol{d})$ enjoys the following property w.h.p.*

Let $Q \subset V$, $\#Q \leqslant n/2$. If $\mathrm{Vol}(Q) > 1000\#Q^{5/8}n^{3/8}$, then

$$\sum_{v\in Q} d_G(v) \geqslant \frac{1}{4}\mathrm{Vol}(Q). \tag{15}$$

Lemma 2.7 *Let $C > 0$ be a sufficiently large constant. Let $G = G(n, \boldsymbol{d})$. Then w.h.p. for any set $X \subset V$ such that $\mathrm{Vol}(X) \leqslant n\exp(-\bar{d}_{\min}/C)$ we have*

$$\sum_{v\in X} d_G(v) \leqslant n\exp(-\bar{d}_{\min}/(4C)).$$

We defer the proofs of Lemmas/Corollaries 2.2–2.7 to Section 7.

3 The Core

In Section 3.1 we prove Proposition 1.1. Then, in Section 3.2 we present the construction of the subgraph $\mathrm{core}(G(n, \boldsymbol{d}))$ and establish the first part of Theorem 1.2.

3.1 Why can the Spectral Gap be Small?

To motivate the definition of the core, we discuss the reasons that may cause the spectral gap of $\mathcal{L}(G(n, \boldsymbol{d}))$ to be "small", thereby proving Proposition 1.1. To keep matters simple, we assume that $d_0 \leqslant \bar{d}(v) = \bar{d} = O(1)$ for all $v \in V$. Then $G(n, \boldsymbol{d})$ is just an Erdős-Rényi graph $G_{n,p}$ with $p = \bar{d}/n$. Therefore, the following result follows from the study of the component structure of $G_{n,p}$ (cf. [17]).

Lemma 3.1 *Let $K = O(1)$ as $n \to \infty$, and let T be a tree on K vertices. Then w.h.p. $G(n, \boldsymbol{d})$ features $\Omega(n)$ connected components that are isomorphic to T. Moreover, the largest component of $G(n, \boldsymbol{d})$ contains $\Omega(n)$ induced vertex disjoint copies of T.*

Lemma 3.1 readily yields the first part of Proposition 1.1.

Lemma 3.2 *Let $\mathcal{C}$ be a tree component of G. Then $\mathcal{C}$ induces eigenvalues* 0 *and* 2 *in the spectrum of $\mathcal{L}(G)$.*

Proof. We recall the simple proof of this fact from [5]. Define a vector $\xi = (\xi_v)_{v\in V}$ by letting $\xi_v = d_G(v)^{\frac{1}{2}}$ for $v \in \mathcal{C}$, and $\xi_v = 0$ for $v \in V - \mathcal{C}$. Then $\mathcal{L}(G)\xi = 0$. Furthermore, let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ be a bipartition of $\mathcal{C}$. Let $\eta = (\eta_v)_{v\in V}$ have entries $\eta_v = d_G(v)^{\frac{1}{2}}$ for $v \in \mathcal{C}_1$, $\eta_v = -d_G(v)^{\frac{1}{2}}$ for $v \in \mathcal{C}_2$, and $\eta_v = 0$ for $v \in V - \mathcal{C}$. Then $\mathcal{L}(G)\eta = 2\eta$. □

Hence, the fact that $G(n, \boldsymbol{d})$ contains a large number of tree components w.h.p. yields the "trivial" eigenvalues 0 and 2 (both with multiplicity $\Omega(n)$). In addition, there is a "local" structure that affects the spectral gap, namely the existence of vertices of "atypically small" degree. More precisely, we call a vertex v of G a (d, d, ε)-*star* if

- v has degree d,
- its neighbors $v_1, \ldots, v_d$ have degree d as well and $\{v_1, \ldots, v_d\}$ is an independent set,
- all neighbors $w \neq v$ of v_i have degree $1/\varepsilon$ and have only one neighbor in $\{v_1, \ldots, v_d\}$.

The following lemma shows that (d, d, ε)-stars with $d < \bar{d}_{\min}$ and $\varepsilon > 0$ small induce eigenvalues "far apart" from 1.

Lemma 3.3 *If G has a (d, d, ε)-star, then $\mathcal{L}(G)$ has eigenvalues λ, λ' such that*

$$|1 - d^{-\frac{1}{2}} - \lambda|, |1 + d^{-\frac{1}{2}} - \lambda'| \leqslant \sqrt{\varepsilon}.$$

Proof. Let v be a (d, d, ε)-star and consider the vector $\xi = (\xi_u)_{u \in V}$ with entries $\xi_v = d^{\frac{1}{2}}$, $\xi_{v_i} = 1$ for $1 \leqslant i \leqslant d$, and $\xi_w = 0$ for $w \in V - \{v, v_1, \ldots, v_d\}$. Moreover, let $\eta = \xi - \mathcal{L}(G)\xi$. Then $\eta_v = 1$, $\eta_{v_i} = d^{-\frac{1}{2}}$, $\eta_w = \sqrt{\varepsilon/d}$ for all $v \neq w \in N(v_i)$ $(1 \leqslant i \leqslant d)$, and $\eta_u = 0$ for all other vertices u. Hence, $\|\mathcal{L}(G)\xi - (1 - d^{-\frac{1}{2}})\xi\|^2 \cdot \|\xi\|^{-2} = \|\eta - d^{-\frac{1}{2}}\xi\|^2/(2d) \leqslant \varepsilon$. Consequently, ξ is "almost" an eigenvector with eigenvalue $1 - d^{-\frac{1}{2}}$, which implies that $\mathcal{L}(G)$ has an eigenvalue λ such that $|1 - d^{-\frac{1}{2}} - \lambda| \leqslant \sqrt{\varepsilon}$. Similarly, considering the vector $\xi' = (\xi'_u)_{u \in V}$ with $\xi'_v = -\sqrt{d}$, $\xi'_{v_i} = 1$, and $\xi'_w = 0$ for all other w, we see that there is an eigenvalue λ' such that $|1 + d^{-\frac{1}{2}} - \lambda'| \leqslant \sqrt{\varepsilon}$. □

Lemma 3.1 implies that w.h.p. $G = G(n, \boldsymbol{d})$ contains (d, d, ε)-stars for any fixed d and ε. Therefore, Lemma 3.3 entails that $\mathcal{L}(G)$ has eigenvalues $1 \pm d^{-\frac{1}{2}} + o(1)$ w.h.p., and thus yields the second and the third part of Proposition 1.1. Setting $d < \bar{d}_{\min}$, we thus see that w.h.p. "low degree vertices" (namely, v and $v_1, \ldots, v_d$) cause eigenvalues rather close to 0 and 2. In fact, in a sense such (d, d, ε)-stars are a "more serious" problem than the existence of tree components (cf. Lemma 3.2), because by Lemma 3.1 an abundance of such (d, d, ε)-stars also occur inside of the largest component. Hence, we cannot get rid of the eigenvalues $1 \pm d^{-\frac{1}{2}}$ by just removing the "small" components of $G(n, \boldsymbol{d})$.

3.2 The construction of $\mathrm{core}(G(n, \boldsymbol{d}))$

As we have seen in Section 3.1, to obtain a subgraph H of $G = G(n, \boldsymbol{d})$ such that $\mathcal{L}(H)$ has a large spectral gap, we need to get rid of the small degree vertices of G. More precisely, we should ensure that for each vertex $v \in H$ the degree $d_H(v)$ of v *inside of* H is not "much smaller" than $\bar{d}_{\min}$. To this end, we consider the following construction.

CR1. Initially, let $H = G - \{v : d_G(v) \leqslant 0.01\bar{d}_{\min}\}$.

Thus, CR1 just removes all vertices of degree much smaller than $\bar{d}_{\min}$. However, it is *not* true in general that $d_H(v) \geqslant 0.01\bar{d}_{\min}$ for all $v \in H$; for some vertices $v \in H$ may have plenty of neighbors outside of H. Therefore, in the second step CR2 of the construction we keep removing such vertices as well.

CR2. While there is a vertex $v \in H$ that has $\geqslant \max\{c_0, \exp(-\bar{d}_{\min}/c_0)\bar{d}^{-\frac{1}{2}} d_G(v)\}$ neighbors in $G - H$, remove v from H.

The final outcome H of the process is $\mathrm{core}(G)$. Observe that by (6) for all $v \in \mathrm{core}(G)$

$$d_{\mathrm{core}(G)}(v) \geqslant \frac{\bar{d}_{\min}}{200},\ e(v, G - \mathrm{core}(G)) < \max\{c_0, \exp(-\bar{d}_{\min}/c_0)\bar{d}^{-\frac{1}{2}} d_G(v)\}. \tag{16}$$

Additionally, in the analysis of the spectral gap of $\mathcal{L}(\mathrm{core}(G))$ in Section 4.1, we will need to consider the following subgraph $\mathcal{S}$, which is defined by a "more picky" version of CR1–CR2.

S1. Initially, let $\mathcal{S} = \mathrm{core}(G) - \{v \in V : |d_{\mathrm{core}(G)}(v) - \bar{d}(v)| \geqslant 0.01\bar{d}(v)\}$.

S2. While there is a vertex $v \in \mathcal{S}$ so that

$$e_G(v, G - \mathcal{S}) \geqslant \max\{c_0, d_G(v)\bar{d}^{-\frac{1}{2}} \exp(-\bar{d}_{\min}/c_0)\},$$

remove v from $\mathcal{S}$.

Then by (5) after the process S1–S2 has terminated, every vertex $v \in \mathcal{S}$ satisfies

$$\begin{aligned} \max\{e(v, H - \mathcal{S}), e(v, V - H)\} &\leqslant \max\{c_0, \exp(-\bar{d}_{\min}/c_0)\bar{d}^{-\frac{1}{2}} d_G(v)\}, \text{ and} \\ \left|d_{\mathcal{S}}(v) - \bar{d}(v)\right| &\leqslant \frac{1}{50}\bar{d}(v). \end{aligned} \tag{17}$$

Moreover, we emphasize that $\mathcal{S} \subset \mathrm{core}(G)$.

An important property of $\mathrm{core}(G)$ is that given just $\bar{d}_{\min}$, G (and c_0), we can compute $\mathrm{core}(G)$ efficiently (without any further information about $\boldsymbol{d}$). This fact is the basis of the algorithmic applications (Corollaries 1.3 and 1.4). By contrast, while $\mathcal{S}$ will be useful in the *analysis* of $\mathcal{L}(\mathrm{core}(G))$, it cannot be computed without explicit knowledge of $\boldsymbol{d}$.

In Section 3.3 we shall analyze the processes CR1–CR2 and S1–S2 in detail in order to show that w.h.p. both $\mathcal{S}$ and $\mathrm{core}(G)$ constitute a huge fraction of G.

Proposition 3.4 *W.h.p. we have*

$$\mathrm{Vol}(V - \mathrm{core}(G)) \leqslant \mathrm{Vol}(V - \mathcal{S}) \leqslant \exp(-100\bar{d}_{\min}/c_0)n$$

and

$$\sum_{v \in V - \mathrm{core}(G)} d_G(v) \leqslant \sum_{v \in V - \mathcal{S}} d_G(v) \leqslant \exp(-2\bar{d}_{\min}/c_0)n.$$

In addition, the following bound will be of significance in Section 4.

Proposition 3.5 *W.h.p. we have*

$$\sum_{v\in\mathcal{S}} e(v, G-\mathrm{core}(G))^2/\bar{d}(v) \leqslant \sum_{v\in\mathcal{S}} e(v, G-\mathcal{S})^2/\bar{d}(v) \leqslant \frac{n}{2}.$$

Proof. Since $\mathrm{core}(G)\subset\mathcal{S}$, we have $e(v, G-\mathrm{core}(G)) \leqslant e(v, G-\mathcal{S})$ for all $v\in\mathcal{S}$, so that the left inequality in Proposition 3.5 is clear. To prove the right inequality, recall that each $v\in\mathcal{S}$ satisfies $e(v,G-\mathcal{S}) \leqslant \max\{c_0, 2\bar{d}(v)\bar{d}^{-\frac{1}{2}}\exp(-\bar{d}_{\min}/c_0)\}$ by S2. Therefore,

$$\begin{aligned}\sum_{v\in\mathcal{S}} \frac{e(v,G-\mathcal{S})^2}{\bar{d}(v)} &\leqslant \sum_{v\in\mathcal{S}} \frac{c_0^2 + 4\bar{d}(v)^2\bar{d}^{-1}\exp(-2\bar{d}_{\min}/c_0)}{\bar{d}(v)} \\ &\leqslant \frac{c_0^2 n}{\bar{d}_{\min}} + 4\bar{d}^{-1}\exp(-2\bar{d}_{\min}/c_0)\sum_{v\in\mathcal{S}}\bar{d}(v) \leqslant \frac{n}{2},\end{aligned}$$

provided that the lower bound d_0 on $\bar{d}_{\min}$ is large enough (say, $d_0 \geqslant 10c_0^2$). □

Remark 3.6 *Letting $d=\bar{d}_{\min}$ and assuming that $\bar{d}=O(1)$ as $n\to\infty$, one can derive that w.h.p.* $\mathrm{core}(G_{n,p})$ *contains a (d,d,ε)-star ($\varepsilon>0$ arbitrarily small but fixed as $n\to\infty$). Hence, by Lemma 3.3 the spectral gap of $\mathcal{L}(\mathrm{core}(G(n,\boldsymbol{d})))$ is at most $1-\bar{d}_{\min}^{-1/2}+o(1)$. Thus, Theorem 1.2 best possible up to the precise values of the constants c_0, d_0.*

Remark 3.7 *While the result (2) of Chung, Lu, and Vu [8] is void if $\bar{d}_{\min}\leqslant \ln^2 n$, in the case $\bar{d}_{\min}\gg\ln^2 n$ its dependence on $\boldsymbol{d}$ is better than the estimate provided by Theorem 1.2. In the light of Remark 3.6, this shows that in the dense case $\bar{d}_{\min}\gg\ln^2 n$ "bad" local structures such as $(\bar{d}_{\min},\bar{d}_{\min},\varepsilon)$-stars do not occur w.h.p.*

3.3 Proof of Proposition 3.4

To establish Proposition 3.4, we consider the following additional process to generate a subgraph K of $G=G(n,\boldsymbol{d})$.

K1. Initially, let $K = G - \{v\in V : |d_G(v)-\bar{d}(v)| \geqslant 0.001\bar{d}(v)\}$.

K2. While there is a $v\in K$ such that $e(v, V-K)\geqslant \frac{1}{2}\max\{c_0, \bar{d}(v)\bar{d}^{-\frac{1}{2}}\exp(-\bar{d}_{\min}/c_0)\}$, remove v from K.

The main difference between K1–K2 and S1–S2 is that K2 refers to the *expected* degree $\bar{d}(v)$, while S2 is phrased in terms of the *empirical* degree $d_G(v)$. In effect, K1–K2 will be a little easier to analyze. The three processes are related as follows.

Lemma 3.8 *We have $K\subset\mathcal{S}\subset H$.*

Proof. The assumption (6) ensures that all $v \in K$ satisfy $d_G(v) \geqslant 0.01\bar{d}_{\min}$, so that $K \subset H$. Hence, if $v \in K$, then (6) and (16) entail that $\left|d_H(v) - \bar{d}(v)\right| \leqslant \left|d_G(v) - \bar{d}(v)\right| + e(v, G-H) \leqslant \frac{1}{500}\bar{d}(v)$. Consequently, K is contained in the subgraph of H defined in the first step S1 of the construction of $\mathcal{S}$ (cf. Section 4.1). Thus, K2 ensures that $K \subset \mathcal{S}$. $\square$

By Lemma 3.8, it suffices to prove that

$$\mathrm{Vol}(G-K) \leqslant n\exp(-100\bar{d}_{\min}/c_0) \qquad \text{w.h.p.} \tag{18}$$

To establish (18), we first bound the volume of the set of vertices removed by K1.

Lemma 3.9 *W.h.p.* $R = \{v \in V : |d_G(v) - \bar{d}(v)| \geqslant 0.001\bar{d}(v)\}$ *has volume*

$$\mathrm{Vol}(R) \leqslant n\exp(-10^{-9}\bar{d}_{\min}).$$

We defer the proof of Lemma 3.9 to Section 3.4. Furthermore, to facilitate the analysis of step K2, we show that w.h.p. there are only few vertices that have plenty of neighbors inside of R.

Lemma 3.10 *Let*

$$\kappa_v = 0.01\max\{c_0, \bar{d}(v)\bar{d}^{-\frac{1}{2}}\exp(-\bar{d}_{\min}/c_0)\} \text{ and } Q = \{v \in V : e(v,R) \geqslant \kappa_v\}.$$

Then $\mathrm{Vol}(Q) \leqslant \exp(-\bar{d}_{\min})\bar{d}^{-2}n$ *w.h.p.*

We prove Lemma 3.10 in Section 3.5. Finally, we are in a position to analyze the volume of the set of vertices removed during the iterative procedure in step K2.

Lemma 3.11 *Let T be the set of all vertices removed during the second step K2 of the construction of K. Then* $\mathrm{Vol}(T) \leqslant n\exp(-101\bar{d}_{\min}/c_0)$ *w.h.p.*

Proof. If $\bar{d}_{\min} \geqslant \ln n$, then Lemma 3.10 entails that $Q = \emptyset$ w.h.p., so that step K2 does not remove any vertices at all, i.e., $T = \emptyset$. Thus, let us assume in the sequel that $\bar{d}_{\min} < \ln n$. In addition, suppose that $\mathrm{Vol}(R) \leqslant n\exp(-10^{-9}\bar{d}_{\min})$, and that $\mathrm{Vol}(Q)$ obeys the bound from Lemma 3.10.

Suppose that $\mathrm{Vol}(T) \geqslant n\exp(-101\bar{d}_{\min}/c_0)$. Let $z_1, \ldots, z_k$ be the vertices deleted from K by K2 (in this order). The basic idea of the proof is to exhibit a set $Z \subset T = \{z_1, \ldots, z_k\}$ that violates one of the two properties (13), (14). In other words, Z will be an "atypically dense" set of "small volume". As Corollary 2.5 implies that (13), (14) actually are true for all subsets of V w.h.p., this implies that w.h.p. $\mathrm{Vol}(T) < n\exp(-101\bar{d}_{\min}/c_0)$, as desired.

To define the set Z, let j^* be the maximum index such that $\mathrm{Vol}(\{z_1, \ldots, z_{j^*}\}) < n\exp(-103\bar{d}_{\min}/c_0)$, and set $Z = \{z_1, \ldots, z_{j^*+1}\}$. Since we assume that $\bar{d}(w) \leqslant n^{0.99}$ for all $w \in V$, that $\bar{d}_{\min} \leqslant \ln n$, and that c_0 is a large enough constant, we obtain

$$\begin{aligned} n\exp(-103\bar{d}_{\min}/c_0) &\leqslant \mathrm{Vol}(Z) < n\exp(-103\bar{d}_{\min}/c_0) + \bar{d}(z_{j^*+1}) \\ &\leqslant n\exp(-102\bar{d}_{\min}/c_0). \end{aligned} \tag{19}$$

Further, as

$$e(z_j, R \cup \{z_1, \ldots, z_{j-1}\}) \geqslant 0.5 \max\{c_0, \exp(-\bar{d}_{\min}/c_0)\bar{d}^{\,-\frac{1}{2}}\bar{d}(z_j)\}$$

for all j, we have

$$e(Z, R \cup Z) \geqslant \max\{c_0 \# Z, \mathrm{Vol}(Z)\bar{d}^{\,-\frac{1}{2}} \exp(-\bar{d}_{\min}/c_0)\}. \tag{20}$$

1st case: $\mathrm{Vol}(Z) \geqslant \bar{d}^{\,\frac{1}{2}} \exp(2\bar{d}_{\min}/c_0)\#Z$**.** Consider

$$Z' = \{v \in Z : e(v, R) \geqslant 0.1 e(v, R \cup Z)\}.$$

Then Lemma 3.10 implies in combination with (19) that

$$\mathrm{Vol}(Z') \leqslant 0.1 \mathrm{Vol}(Z),$$

whence

$$\begin{aligned} 2e(Z) &\geqslant 2e(Z - Z', Z) \overset{(20)}{\geqslant} 0.9\bar{d}^{\,-\frac{1}{2}} \exp(-\bar{d}_{\min}/c_0)\mathrm{Vol}(Z - Z') \\ &\geqslant 0.5\bar{d}^{\,-\frac{1}{2}} \exp(-\bar{d}_{\min}/c_0)\mathrm{Vol}(Z). \end{aligned} \tag{21}$$

Hence, setting $c' = c_0^{-1}$ and $\zeta = \bar{d}^{\,\frac{1}{2}}$, we conclude that

$$\begin{aligned} \mathrm{Vol}(Z) &\leqslant \exp(-3c'\bar{d}_{\min})n && \text{[by (19)]}, \\ \mathrm{Vol}(Z) &\geqslant \zeta \exp(2c'\bar{d}_{\min})\#Z && \text{[because the 1st case occurs]}, \\ e(Z) &> 0.25 \exp(-c'\bar{d}_{\min})\zeta^{-1}\mathrm{Vol}(Z) && \text{[due to (21)]}. \end{aligned}$$

Thus, Z violates (13).

2nd case: $\mathrm{Vol}(Z) < \bar{d}^{\,\frac{1}{2}} \exp(2\bar{d}_{\min}/c_0)\#Z$**.** Let

$$Z' = \{v \in Z : e(v, R) \geqslant 0.01c_0 \wedge \bar{d}(v) \leqslant 100\bar{d}^{\,\frac{1}{2}} \exp(\bar{d}_{\min}/c_0)\}.$$

Then by Lemma 3.10 and because the 2nd case occurs,

$$\#Z' \;\leqslant\; \mathrm{Vol}(Z') \leqslant \exp(-\bar{d}_{\min})\bar{d}^{-2}n \overset{(19)}{\leqslant} \frac{\mathrm{Vol}(Z)}{\bar{d}^2 \exp(2\bar{d}_{\min}/c_0)} \leqslant 0.1\#Z.$$

Furthermore, letting $Z'' = \{v \in Z : \bar{d}(v) \leqslant 100\bar{d}^{\,\frac{1}{2}} \exp(\bar{d}_{\min}/c_0)\}$, we have $\#Z'' \geqslant 0.99\#Z$, and thus $\#Z'' - Z' \geqslant 0.5\#Z$. Therefore, due to (20)

$$\begin{aligned} e(Z) &\geqslant e(Z'' - Z', Z) \geqslant \frac{1}{2} \sum_{v \in Z'' - Z'} [e(v, R \cup Z) - 0.01c_0] \\ &\geqslant 0.2c_0\#Z'' - Z' > 3000\#Z. \end{aligned} \tag{22}$$

Moreover, (19) implies that $\#Z \leqslant \mathrm{Vol}(Z) \leqslant \exp(-\frac{16}{3}\bar{d}_{\min}/c_0)n$, whence

$$\mathrm{Vol}(Z) \leqslant \bar{d}^{\,\frac{1}{2}}\#Z^{5/8}n^{3/8}.$$

Consequently, (22) shows that Z violates (14).

□

Corollary 3.12 *W.h.p. we have* $\mathrm{Vol}(G-K) \leqslant \exp(-100\bar{d}_{\min}/c_0)n$.

Proof. This is an immediate consequence of Lemmas 3.9 and 3.11. □

Finally, Corollary 3.12 establishes (18), so that Proposition 3.4 follows from Lemmas 2.7 and 3.8.

3.4 Proof of Lemma 3.9

The Chernoff bound (10) implies that

$$\mathrm{P}\left[v \in R\right] = \mathrm{P}\left[\left|d_G(v) - \bar{d}(v)\right| \geqslant 0.001\bar{d}(v)\right] \leqslant \exp\left[-10^{-7}\bar{d}(v)\right] \qquad \text{for all } v \in V. \quad (23)$$

Therefore, assuming that $\min_{v\in V} \bar{d}(v) \geqslant \bar{d}_{\min} \geqslant d_0$ for a large enough d_0 (cf. (6)), we get

$$\mathrm{E}(\mathrm{Vol}(R)) \leqslant \sum_{v\in V} \bar{d}(v)\mathrm{P}\left[v \in R\right] \leqslant n\exp(-10^{-8}\bar{d}_{\min}). \quad (24)$$

Consequently, the remaining task is to show that $\mathrm{Vol}(R)$ does not exceed its expectation by too much w.h.p.

If $\bar{d}_{\min} \geqslant \ln\ln n$, then we just apply Markov's inequality and obtain that

$$\mathrm{P}\left[\mathrm{Vol}(R) > n\exp(-10^{-8}\bar{d}_{\min}/2)\right] \leqslant \exp(-10^{-8}\bar{d}_{\min}/2) = o(1),$$

as desired.

Now, let us assume that $\bar{d}_{\min} < \ln\ln n$. Set

$$V_j = \left\{v \in V : 2^j\bar{d}_{\min} \leqslant \bar{d}(v) < 2^{j+1}\bar{d}_{\min}\right\} \text{ for } j \geqslant 0$$

and let $L = 2\cdot 10^7 \bar{d}_{\min}^{-1}\ln n$. Setting

$$X_1 = \sum_{0\leqslant j\leqslant \log_2 L} 2^{j+1}\bar{d}_{\min}\#V_j\cap R, \qquad X_2 = \sum_{j>\log_2 L} 2^{j+1}\bar{d}_{\min}\#V_j\cap R,$$

we have $\mathrm{Vol}(R) \leqslant X_1 + X_2$. Moreover, by our choice of L

$$\begin{aligned}\mathrm{P}\left[X_2 > 0\right] &\leqslant \sum_{j>\log_2 L}\mathrm{P}\left[V_j \neq \emptyset\right] \overset{(23)}{\leqslant} \sum_{j>\log_2 L}\#V_j\exp\left[-10^{-7}2^j\bar{d}_{\min}\right] \\ &\leqslant n\exp(-2\ln n) \leqslant n^{-1}.\end{aligned} \quad (25)$$

Furthermore, we claim that the random variable $X = \frac{X_1}{10^8\ln n}$ satisfies the Lipschitz condition (11). To see this, let $G = G(n, \boldsymbol{d})$, let $v, w \in V$, and let G^+ (resp. G^-) denote the graph obtained from G by adding (removing) the edge $e = \{v, w\}$. Of course, adding or removing e only affects the degrees of v, w. Thus, if $v \in V_{j_1}$ and $w \in V_{j_2}$, then

$$|X_1(G^\pm) - X_1(G)| \leqslant 2\bar{d}_{\min}\min\{2^{j_1}, L\} + 2\bar{d}_{\min}\min\{2^{j_2}, L\} \leqslant 4\bar{d}_{\min}L \leqslant 10^8\ln n,$$

whence $|X(G^{\pm}) - X(G)| \leqslant 1$. Therefore, as $\bar{d}_{\min} \leqslant \ln\ln n$, Lemma 2.2 entails that

$$\begin{aligned} \mathrm{P}\left[|X_1 - \mathrm{E}(X_1)| > n\exp(-\bar{d}_{\min})\right] &\leqslant \mathrm{P}\left[|X - \mathrm{E}(X)| > \frac{n\exp(-\bar{d}_{\min})}{10^8 \ln n} = n^{1-o(1)}\right] \\ &\leqslant \mathrm{P}\left[|X - \mathrm{E}(X)| > (\bar{d}n)^{0.501}\right] \leqslant \exp(-(\bar{d}n)^{0.001}/300) \leqslant n^{-1}, \qquad (26) \end{aligned}$$

where the second inequality sign is due to our assumption that $\bar{d} \leqslant n^{0.99}$, and the last one follows from (6). Finally, since $\mathrm{E}(X_1) \leqslant 2\mathrm{E}(\mathrm{Vol}(R)) \leqslant 2n\exp(-10^{-8}\bar{d}_{\min})$ by (24), combining (25) and (26), we conclude that $\mathrm{Vol}(R) \leqslant X_1 + X_2 \leqslant 3n\exp(-10^{-8}\bar{d}_{\min}) \leqslant n\exp(-10^{-8}\bar{d}_{\min}/2)$ w.h.p.

3.5 Proof of Lemma 3.10

Our goal is to estimate $\mathrm{Vol}(Q)$, where $Q = \{v \in V : e(v, R) \geqslant \kappa_v\}$. Since it is a bit intricate to work with Q directly, we shall actually work with a different set Q'. Let us call $v \in V$ *critical* if there is a set $T' \subset N_G(v)$ of size $\#T' = \kappa_v/2$ such that $\bar{d}(w) \leqslant \ln^2 n$ and $|e(w, V - T') - \bar{d}(w)| \geqslant 10^{-4}\bar{d}(w)$ for all $w \in T'$. Now Q' is the set of all critical vertices.

Lemma 3.13 *We have $Q \subset Q'$ w.h.p.*

To prove Lemma 3.13, we need the following observation.

Lemma 3.14 *W.h.p. all sets $T \subset V$ such that $t = \#T \leqslant n^{0.998}$ and $\bar{d}(w) \leqslant \ln^2 n$ for all $w \in T$ satisfy $e(T) \leqslant 10^6 t$.*

Proof. Since

$$\mu(T) \leqslant \frac{\mathrm{Vol}(T)^2}{\bar{d}n} \leqslant \frac{t^2 \ln^4 n}{\bar{d}n} \leqslant tn^{-0.001},$$

in the case $e(T) > 10^6 t$ we have

$$e(T)\ln\frac{e(T)}{\mu(T)} \geqslant 1000t\ln(n) > 300t\ln(n/t).$$

Hence, if $e(T) > 10^6 t$, then T violates property (12), and thus Lemma 2.4 implies that $e(T) \leqslant 10^6 t$ w.h.p. □

Proof of Lemma 3.13. Consider a vertex w such that $\bar{d}(w) > \ln^2 n$. Then by the Chernoff bound (10) we have

$$\mathrm{P}[w \in R] = \mathrm{P}\left[\left|\bar{d}(w) - d_G(w)\right| > 0.001\bar{d}(w)\right] \leqslant 2\exp(-\phi(0.001)\ln^2 n) = o(n^{-1}).$$

Thus, by the union bound all $w \in R$ satisfy $\bar{d}(w) \leqslant \ln^2 n$ w.h.p.

Now, consider a vertex $v \in Q$ and let T be a set of κ_v vertices in R that are adjacent with v. Since we assume that $\bar{d}(v) \leqslant n^{0.99}$ and $\kappa_v \leqslant \bar{d}(v)$, we can apply Lemma 3.14 to obtain that $e(T) \leqslant 10^6\kappa_v$ w.h.p. Hence, there exists a subset $T' \subset T$ of size $\kappa_v/2$ such that $e(w,T) \leqslant 10^7$ for all $w \in T'$. Since all $w \in T'$ satisfy $\bar{d}(w) \geqslant d_0$ for a large enough d_0, we conclude that

$$|e(w,V-T') - \bar{d}(w)| \geqslant |d_G(w) - \bar{d}(w)| - e(w,T) \geqslant 0.001\bar{d}(w) - 10^7 \geqslant 10^{-4}\bar{d}(w),$$

where the second inequality follows from $w \in R$. Thus, we obtain $v \in Q'$, as desired. □

Thus, in order to estimate $\mathrm{Vol}(Q)$ it suffices to bound $\mathrm{Vol}(Q')$. As a first step, we estimate $\mathrm{E}(\mathrm{Vol}(Q'))$.

Lemma 3.15 *We have* $\mathrm{E}(\mathrm{Vol}(Q')) \leqslant \exp(-2\bar{d}_{\min})\bar{d}^{-3}n$.

The proof of Lemma 3.15 relies on the following bound.

Lemma 3.16 *Let $T' \subset V$ be a set of volume $\mathrm{Vol}(T') \leqslant n\ln^{-3} n$ such that $\bar{d}(w) \leqslant \ln^2 n$ for all $w \in T'$. Then*

$$\mathrm{P}\left[\forall w \in T' : |e(w,V-T') - \bar{d}(w)| \geqslant 10^{-4}\bar{d}(w)\right] \leqslant \exp(-2\cdot 10^{-9}\mathrm{Vol}(T')).$$

Proof. Since for $w \in T'$ the random variables $e(w,V-T')$ are mutually independent with expectation $\frac{\bar{d}(w)\mathrm{Vol}(V-T')}{\bar{d}n} \sim \bar{d}(w)$, the assertion follows from the Chernoff bound (10). □

Proof of Lemma 3.15. Lemma 3.16 entails that

$$\mathrm{P}\left[v \text{ is critical}\right] \leqslant \sum_{T'} \mathrm{P}\left[T' \subset N_G(v)\right] \exp(-2\cdot 10^{-9}\mathrm{Vol}(T'))$$

for any $v \in V$, where the sum ranges over all sets $T' \subset V$ of size $\kappa_v/2$ such that $\bar{d}(w) \leqslant \ln^2 n$ for all $w \in T'$. Moreover, for any such set T' we have $\mathrm{P}\left[T' \subset N_G(v)\right] \leqslant \prod_{w\in T'} \frac{\bar{d}(w)\bar{d}(v)}{\bar{d}n}$. Hence,

$$\begin{aligned}
\mathrm{P}\left[v \text{ is critical}\right] &\leqslant \sum_{T'} \exp(-2\cdot 10^{-9}\mathrm{Vol}(T')) \prod_{w\in T'} \frac{\bar{d}(w)\bar{d}(v)}{\bar{d}n} \\
&\leqslant \left(\frac{\bar{d}(v)}{\bar{d}n}\right)^{\kappa_v/2} \sum_{T'} \exp(-10^{-9}\mathrm{Vol}(T')) \\
&\leqslant \binom{n}{\kappa_v/2} \left(\frac{\bar{d}(v)}{\bar{d}n\exp(10^{-9}\bar{d}_{\min})}\right)^{\frac{\kappa_v}{2}}. \qquad (27)
\end{aligned}$$

1st case: $\bar{d}(v) \leqslant c_0\bar{d}^{\frac{1}{2}}\exp(\bar{d}_{\min}/c_0)$. In this case we have $\kappa_v = 0.01c_0$. Hence, (27) yields

$$\mathrm{P}\left[v \text{ is critical}\right] \leqslant \exp(-10^{-12}c_0\bar{d}_{\min})\bar{d}^{-4}, \qquad (28)$$

provided that c_0 is large enough.

2nd case: $\bar{d}(v) > c_0\bar{d}^{\frac{1}{2}}\exp(\bar{d}_{\min}/c_0)$**.** Then

$$\kappa_v = 0.01\bar{d}(v)\bar{d}^{-\frac{1}{2}}\exp(-\bar{d}_{\min}/c_0) \geqslant 0.01c_0,$$

and thus (27) implies

$$\mathrm{P}\left[v \text{ is critical}\right] \leqslant \left(\frac{2\mathrm{e}\bar{d}(v)}{\kappa_v\bar{d}\exp(10^{-9}\bar{d}_{\min})}\right)^{\kappa_v/2} \leqslant \left[\exp(10^{-10}\bar{d}_{\min})\sqrt{\bar{d}}\right]^{-\kappa_v/2}, \quad (29)$$

where we once more assume that the constant c_0 is sufficiently large.

Finally, letting

$$I_j = \{v \in V : 2^{j-1} < 100c_0^{-1}\kappa_v \leqslant 2^j\},$$

we conclude that

$$\begin{aligned}
\mathrm{E}(\mathrm{Vol}(Q')) &= \sum_{v\in V}\mathrm{P}\left[v \text{ is critical}\right]\bar{d}(v) \\
&= \sum_{j\geqslant 0}\sum_{v\in I_j}\mathrm{P}\left[v \text{ is critical}\right]\bar{d}(v) \\
&\overset{(28)}{\leqslant} \frac{0.01c_0\bar{d}^{\frac{1}{2}}\exp(\bar{d}_{\min}/c_0)n}{\exp(10^{-12}c_0\bar{d}_{\min})\bar{d}^4} + \sum_{j\geqslant 1}\sum_{v\in I_j}\mathrm{P}\left[v \text{ is critical}\right]\bar{d}(v) \\
&\overset{(29)}{\leqslant} n\exp(-10^{-13}c_0\bar{d}_{\min})\bar{d}^{-3} \\
&\qquad + \sum_{j\geqslant 1}\#I_j \cdot c_0\bar{d}^{\frac{1}{2}}\exp(\bar{d}_{\min}/c_0)2^j\left[\exp(10^{-10}\bar{d}_{\min})\sqrt{\bar{d}}\right]^{-0.01\cdot 2^{j-2}c_0} \\
&\leqslant \exp(-2\bar{d}_{\min})\bar{d}^{-3}n,
\end{aligned}$$

provided that the constant c_0 is chosen large enough. □

Proof of Lemma 3.10. Due to Lemma 3.13, it suffices to bound $\mathrm{Vol}(Q')$. If $\bar{d} \geqslant \ln\ln n$ Lemma 3.15 implies in combination with Markov's inequality that

$$\mathrm{Vol}(Q') \leqslant \exp(-2c_0^2\bar{d}_{\min})\bar{d}^{-2}n \text{ w.h.p.}$$

Thus, let us assume in the sequel that $\bar{d}_{\min} \leqslant \bar{d} < \ln\ln n$. We call $v \in V$ *bad* if $\bar{d}(v) \leqslant \ln^2 n$ and there is a set $T'' \subset N_G(v)$ of size $\kappa_v/2$ such that all $w \in T'$ satisfy $\bar{d}(w) \leqslant \ln^2 n$, $d_G(w) \leqslant 2\ln^2 n$, and $|e(w, V - T') - \bar{d}(w)| \geqslant 10^{-4}\bar{d}(w)$. Moreover, let Q'' be the set of all bad vertices. As every bad vertex is critical, we have $Q'' \subset Q'$.

Furthermore, we claim that $Q'' = Q'$ w.h.p. To see this, note that the Chernoff bound (10) implies that w.h.p. $d_G(w) \leqslant 2\ln^2 n$ for all $w \in V$ with $\bar{d}(w) \leqslant \ln^2 n$ (cf. the proof of Lemma 3.13). Hence, the condition that $d_G(w) \leqslant 2\ln^2 n$ in the definition of "bad" is void w.h.p. In addition, since $\bar{d} \leqslant \ln\ln n$, (29) implies that w.h.p. all $v \in Q'$ satisfy $\bar{d}(v) \leqslant \ln^2 n$. Thus, w.h.p. the notions "bad" and "critical" coincide.

Therefore, in order to establish the lemma it suffices to bound $\mathrm{Vol}(Q'')$. To this end, we basically just need to bound $\#Q''$, because $\bar{d}(v)$ is "small" for all $v \in Q''$. In order to estimate $\#Q''$, we observe that the random variable $X = \#Q'' \ln^{-3} n$ satisfies the Lipschitz condition (11). For let us consider the graph $\hat{G}$ obtained from $G = G(n, \boldsymbol{d})$ by adding or removing a single edge $e = \{u, u'\}$, and let $\hat{Q}''$ be the set of all vertices v that are bad in $\hat{G}$. To bound $|\#Q'' - \#\hat{Q}''|$, let $N_u = N_G(u)$ if $d_G(u) \leqslant 1 + 2\ln^2 n$, and set $N_u = \emptyset$ otherwise; we define $N_{u'}$ analogously. Moreover, let $U = \{u, u'\} \cup N_u \cup N_{u'}$. Since a vertex v that is adjacent with neither u nor u' is bad in G iff it is bad in $\hat{G}$, and because the definition of "bad" ignores vertices of degree $> 2\ln^2 n$, we conclude that $Q'' - U = \hat{Q}'' - U$. Consequently, $|\#Q'' - \#\hat{Q}''| \leqslant \#U \leqslant 4 + 2\ln^2 n \leqslant \ln^3 n$, so that X satisfies (11).

In effect, as $\mathrm{Vol}(Q'') \leqslant \#Q'' \ln^2 n = X \ln^5 n$, we conclude that $\mathrm{Vol}(Q'') \ln^{-5} n$ satisfies (11) as well. Therefore, Lemma 2.2 entails that w.h.p.$|\mathrm{Vol}(Q'') - \mathrm{E}(\mathrm{Vol}(Q''))| \leqslant n^{0.999} \leqslant \exp(-2\bar{d}_{\min})\bar{d}^{-3} n$ (recall that we are assuming $\bar{d}_{\min} \leqslant \bar{d} \leqslant \ln\ln n$). Thus, Lemma 3.15 implies that $\mathrm{Vol}(Q') = \mathrm{Vol}(Q'') \leqslant \exp(-\bar{d}_{\min})\bar{d}^{-2} n$ w.h.p. □

4 The Spectral Gap of the Laplacian

4.1 Outline of the Proof

We let $G = (V, E) = G(n, \boldsymbol{d})$, $H = \mathrm{core}(G)$, and we let $\mathcal{S}$ denote the outcome of the process S1–S2 (cf. Section 3.3). Furthermore, consider the diagonal matrices

$$\begin{aligned} D = \mathrm{diag}(d_H(v)^{-\frac{1}{2}})_{v \in H}, &\qquad \bar{D} = \mathrm{diag}(\bar{d}(v)^{-\frac{1}{2}})_{v \in H}, \text{ and define vectors} \\ \omega = D^{-1}\mathbf{1}_H = (d_H(v)^{1/2})_{v \in H}, &\qquad \bar{\omega} = \bar{D}^{-1}\mathbf{1}_H = (\bar{d}(v)^{1/2})_{v \in H}, \\ \omega_{\mathcal{S}} = D^{-1}\mathbf{1}_{H,\mathcal{S}}, &\qquad \bar{\omega}_{\mathcal{S}} = \bar{D}^{-1}\mathbf{1}_{H,\mathcal{S}}. \end{aligned}$$

Thus, the entries of $\omega_{\mathcal{S}}$ (resp. $\bar{\omega}_{\mathcal{S}}$) are $d_G(v)^{\frac{1}{2}}$ (resp. $\bar{d}(v)^{\frac{1}{2}}$) for $v \in \mathcal{S}$, and 0 for $v \in H - \mathcal{S}$. In addition, we let

$$M = \boldsymbol{E} - \mathcal{L}(H) = D \cdot A(H) \cdot D.$$

Since $\mathcal{L}(H)\omega = 0$, our task is to estimate $\sup_{0 \neq \xi \perp \omega} \|M\xi\| \cdot \|\xi\|^{-1}$. A crucial issue is that the entries of M are not independent. For if two vertices $v, w \in H$ are adjacent, then the vw'th entry of M is $(d_H(v)d_H(w))^{-1/2}$, and of course $d_H(v)$, $d_H(w)$ are neither mutually independent nor independent of the presence of the edge $\{v, w\}$. To deal with the dependence of the matrix entries, we consider the matrix $\mathcal{M} = \bar{D} \cdot A(H) \cdot \bar{D}$, whose vw'th entry is $(\bar{d}(v)\bar{d}(w))^{-\frac{1}{2}}$ if v, w are adjacent in H, and 0 otherwise. Thus, in $\mathcal{M}$ the 'weights' of the entries are in terms of *expected* degrees $\bar{d}(v), \bar{d}(w)$ rather than the actual degrees $d_H(v), d_H(w)$. Furthermore, to relate M and $\mathcal{M}$, we decompose M into four blocks

$$M = M_{\mathcal{S}} + M_{H-\mathcal{S}} + M_{(H-\mathcal{S}) \times \mathcal{S}} + M_{\mathcal{S} \times (H-\mathcal{S})}. \tag{30}$$

Then we expect that $M_{\mathcal{S}}$ should be "similar" to $\mathcal{M}_{\mathcal{S}}$, because by (17) for all $v \in \mathcal{S}$ the degree $d_H(v)$ is close to its mean $\bar{d}(v)$. Thus, to analyze $M_{\mathcal{S}}$, we investigate $\mathcal{M}_{\mathcal{S}}$ on the orthogonal complement of $\bar{\omega}_{\mathcal{S}}$.

Proposition 4.1 *There is a constant $c_1 > 0$ such that*

$$\sup_{\substack{0 \neq \xi \perp \bar{\omega}_{\mathcal{S}} \\ 0 \neq \chi \perp \bar{\omega}_{\mathcal{S}}}} \frac{|\langle \mathcal{M}_{\mathcal{S}} \xi, \chi \rangle|}{\|\xi\| \cdot \|\chi\|} \leqslant c_1 \bar{d}_{\min}^{-\frac{1}{2}} \qquad w.h.p.$$

The proof of Proposition 4.1 can be found in Section 5. Further, in Section 4.2 we combine Propositions 3.4 and 4.1 to bound $\|M_{\mathcal{S}}\eta\|$ for $\eta \perp \omega$.

Corollary 4.2 *There is a constant $c_2 > 0$ such that*

$$\sup_{\eta \perp \omega,\, \|\eta\|=1} \|M_{\mathcal{S}}\eta\| \leqslant c_2 \bar{d}_{\min}^{-1/2} \;\; w.h.p.$$

Corollary 4.2 bounds the first part of the decomposition (30). To bound $\|M_{H-\mathcal{S}}\|$, we show that $H - \mathcal{S}$ "is tree-like": we can decompose the vertex set into classes $Z_1, \ldots, Z_K$ such that every vertex $v \in Z_j$ has "only few" neighbors in the classes Z_i with index $i \geqslant j$.

Lemma 4.3 *W.h.p. $H - \mathcal{S}$ has a decomposition $V(H - \mathcal{S}) = \bigcup_{j=1}^{K} Z_j$ such that for all $j = 1, \ldots, K$ and all $v \in Z_j$ we have*

$$e\left(v, \bigcup_{i=j}^{K} Z_i\right) \leqslant \max\{c_0, \exp(-\bar{d}_{\min}/c_0) d_H(v)\}. \tag{31}$$

We defer the proof of Lemma 4.3 to Section 4.3. Using Lemma 4.3, in Section 4.4 we derive the following bound on $\|M_{H-\mathcal{S}}\|$.

Proposition 4.4 *W.h.p. $\|M_{H-\mathcal{S}}\| \leqslant 21 c_0^{1/2} \bar{d}_{\min}^{-1/2}$.*

Finally, using just the construction S1–S2 of $\mathcal{S}$ and some elementary estimates, in Section 4.5 we shall bound the third and the fourth part of the decomposition (30) as follows.

Proposition 4.5 *We have $\|M_{\mathcal{S}\times(H-\mathcal{S})}\| = \|M_{(H-\mathcal{S})\times\mathcal{S}}\| \leqslant 2 c_0^{1/2} \bar{d}_{\min}^{-1/2}$ w.h.p.*

Proof of Theorem 1.2. The first assertion follows directly from Proposition 3.4. Moreover, due to the decomposition (30) of M, Corollary 4.2, Proposition 4.4, and Proposition 4.5 entail the bound $\sup_{\eta \perp \omega,\, \|\eta\|=1} \|M\eta\| \leqslant c_0 \bar{d}_{\min}^{-1/2}$ w.h.p. As $\mathcal{L}(H) = \boldsymbol{E} - M$, we thus obtain the second part of the theorem. $\square$

4.2 Proof of Corollary 4.2

Since $\mathcal{M}$ is obtained from M by replacing the actual degrees $d_H(v)$ by the expected degrees $\bar{d}(v)$, to prove the proposition we basically need to investigate how much $d_H(v)$ and $\bar{d}(v)$ differ ($v \in \mathcal{S}$). More precisely, we need to investigate how the vectors $\omega_{\mathcal{S}}$ and $\bar{\omega}_{\mathcal{S}}$ relate to each other.

Lemma 4.6 *There is a constant $C > 0$ such that w.h.p. the following bounds hold.*

1. $\|\omega - \omega_{\mathcal{S}}\|^2 \leqslant Cn$.
2. $\|\omega_{\mathcal{S}} - D\bar{D}^{-1}\bar{\omega}_{\mathcal{S}}\|^2 \leqslant Cn$.
3. $\|\bar{\omega}_{\mathcal{S}}\|^2 \geqslant \bar{d}n/2$.
4. $\|\mathcal{M}_{\mathcal{S}}\bar{\omega}_{\mathcal{S}} - \bar{\omega}_{\mathcal{S}}\|^2 \leqslant Cn$.

Proof. By Proposition 3.4 we have $\|\omega_{\mathcal{S}} - \omega\|^2 = \sum_{v \in H - \mathcal{S}} d_H(v) \leqslant \sum_{v \in V - \mathcal{S}} d_G(v) \leqslant n$, whence the first assertion follows. With respect to the second one, we have

$$\left\|D\bar{D}^{-1}\bar{\omega}_{\mathcal{S}} - \omega_{\mathcal{S}}\right\|^2 = \sum_{v \in \mathcal{S}} \left(\frac{\bar{d}(v) - d_H(v)}{\sqrt{d_H(v)}}\right)^2 \overset{(17)}{\leqslant} 2\sum_{v \in \mathcal{S}} \frac{(\bar{d}(v) - d_H(v))^2}{\bar{d}(v)}$$
$$\leqslant 4\sum_{v \in \mathcal{S}} \frac{e(v, G - \mathcal{S})^2}{\bar{d}(v)} + 4\sum_{v \in \mathcal{S}} \frac{(\bar{d}(v) - d_G(v))^2}{\bar{d}(v)}. \tag{32}$$

Invoking Corollary 2.3 and Proposition 3.5, we conclude that the right hand side of (32) is $\leqslant Cn$ w.h.p. Furthermore, the third part of the lemma follows simply from Proposition 3.4: we have $\|\bar{\omega}_{\mathcal{S}}\|^2 = \mathrm{Vol}(S) \geqslant n\bar{d}/2$. Finally, as $\mathcal{M}_{\mathcal{S}} = \bar{D}_{\mathcal{S}} A(G)_{\mathcal{S}} \bar{D}_{\mathcal{S}}$, the entries of $\xi = \mathcal{M}_{\mathcal{S}}\bar{\omega}_{\mathcal{S}}$ are $\xi_v = d_{\mathcal{S}}(v)\bar{d}(v)^{-\frac{1}{2}}$ for $v \in \mathcal{S}$, and $\xi_v = 0$ for $v \notin \mathcal{S}$. Hence, Proposition 3.5 entails that

$$\|\mathcal{M}_{\mathcal{S}}\bar{\omega}_{\mathcal{S}} - \bar{\omega}_{\mathcal{S}}\|^2 = \sum_{v \in \mathcal{S}} \frac{(d_{\mathcal{S}}(v) - \bar{d}(v))^2}{\bar{d}(v)} \leqslant 2\sum_{v \in \mathcal{S}} \frac{e(v, G - \mathcal{S})^2}{\bar{d}(v)} + 2\sum_{v \in V} \frac{(\bar{d}(v) - d_G(v))^2}{\bar{d}(v)}.$$

Applying Corollary 2.3 and Proposition 3.5 once more, we obtain the fourth assertion. □

Proof of Corollary 4.2. Let $\eta \perp \omega$ be a unit vector. Since (17) implies that $\|D_{\mathcal{S}}\bar{D}_{\mathcal{S}}^{-1}\| \leqslant 2$, we have

$$\|M_{\mathcal{S}}\eta\| = \|D_{\mathcal{S}}\bar{D}_{\mathcal{S}}^{-1}\mathcal{M}_{\mathcal{S}}D_{\mathcal{S}}\bar{D}_{\mathcal{S}}^{-1}\eta\| \leqslant \|D_{\mathcal{S}}\bar{D}_{\mathcal{S}}^{-1}\| \cdot \|\mathcal{M}_{\mathcal{S}}D_{\mathcal{S}}\bar{D}_{\mathcal{S}}^{-1}\eta\| \leqslant 2\|\mathcal{M}_{\mathcal{S}}D_{\mathcal{S}}\bar{D}_{\mathcal{S}}^{-1}\eta\|. \tag{33}$$

Let $\zeta = D_{\mathcal{S}}\bar{D}_{\mathcal{S}}^{-1}\eta$. Then we can decompose $\zeta = \alpha\xi + \beta\|\bar{\omega}_{\mathcal{S}}\|^{-1}\bar{\omega}_{\mathcal{S}}$ such that $\alpha^2 + \beta^2 = \|\zeta\| \leqslant 2$ and $\xi \perp \bar{\omega}_{\mathcal{S}}$ is a unit vector. Hence,

$$\|\mathcal{M}_{\mathcal{S}}D_{\mathcal{S}}\bar{D}_{\mathcal{S}}^{-1}\eta\| = \|\mathcal{M}_{\mathcal{S}}\zeta\| \leqslant 2\|\mathcal{M}_{\mathcal{S}}\xi\| + |\beta| \cdot \|\bar{\omega}_{\mathcal{S}}\|^{-1} \cdot \|\mathcal{M}_{\mathcal{S}}\bar{\omega}_{\mathcal{S}}\|$$
$$\leqslant 2\|\mathcal{M}_{\mathcal{S}}\xi\| + |\beta| \cdot \|\mathcal{M}_{\mathcal{S}}\|. \tag{34}$$

Furthermore, $\|M\| = \|\boldsymbol{E} - \mathcal{L}(H)\| \leqslant 1$. In addition, $\|D_{\mathcal{S}}\bar{D}_{\mathcal{S}}^{-1}\| \leqslant 2$ by (17). Consequently,

$$\|\mathcal{M}_{\mathcal{S}}\| = \|\bar{D}_{\mathcal{S}}D_{\mathcal{S}}^{-1}M_{\mathcal{S}}\bar{D}_{\mathcal{S}}D_{\mathcal{S}}^{-1}\| \leqslant \|M_{\mathcal{S}}\| \cdot \|\bar{D}_{\mathcal{S}}D_{\mathcal{S}}^{-1}\|^2 \leqslant 4\|M\| \leqslant 4,$$

whence (33) and (34) yield

$$\|M_{\mathcal{S}}\eta\| \leqslant 2\|\mathcal{M}_{\mathcal{S}}\xi\| + 4|\beta|. \tag{35}$$

Thus, to bound $\|M_{\mathcal{S}}\eta\|$, we need to estimate $\|\mathcal{M}_{\mathcal{S}}\xi\|$. Since $\xi \perp \bar{\omega}_{\mathcal{S}}$, the third and the fourth part of Lemma 4.6 imply that

$$\left|\left\langle \mathcal{M}_{\mathcal{S}}\xi, \frac{\bar{\omega}_{\mathcal{S}}}{\|\bar{\omega}_{\mathcal{S}}\|}\right\rangle\right| = \|\bar{\omega}_{\mathcal{S}}\|^{-1} \cdot |\langle \xi, \mathcal{M}_{\mathcal{S}}\bar{\omega}_{\mathcal{S}}\rangle| \leqslant \|\bar{\omega}_{\mathcal{S}}\|^{-1}\|\mathcal{M}_{\mathcal{S}}\bar{\omega}_{\mathcal{S}} - \bar{\omega}_{\mathcal{S}}\| \leqslant 2C\bar{d}^{\,-\frac{1}{2}} \tag{36}$$

w.h.p. Furthermore, Proposition 4.1 entails that w.h.p.

$$\left\|\mathcal{M}_{\mathcal{S}}\xi - \left\langle \mathcal{M}_{\mathcal{S}}\xi, \frac{\bar{\omega}_{\mathcal{S}}}{\|\bar{\omega}_{\mathcal{S}}\|}\right\rangle \frac{\bar{\omega}_{\mathcal{S}}}{\|\bar{\omega}_{\mathcal{S}}\|}\right\| \leqslant \sup_{\chi \perp \bar{\omega}_{\mathcal{S}},\, \|\chi\|=1} |\langle \mathcal{M}_{\mathcal{S}}\xi, \chi\rangle| \leqslant c_1\bar{d}_{\min}^{-1/2}. \tag{37}$$

Combining (36) and (37), we conclude that

$$\|\mathcal{M}_{\mathcal{S}}\xi\| \leqslant 2(C + c_1)\bar{d}_{\min}^{-1/2} \qquad \text{w.h.p.} \tag{38}$$

To complete the proof, we show that $|\beta| \leqslant 4C\bar{d}^{\,-\frac{1}{2}}$. As $\eta \perp \omega$ by assumption, we obtain

$$\begin{aligned} |\beta| \cdot \|\bar{\omega}_{\mathcal{S}}\| &= |\langle \zeta, \bar{\omega}_{\mathcal{S}}\rangle| = \left|\left\langle \eta, D\bar{D}^{-1}\bar{\omega}_{\mathcal{S}}\right\rangle\right| = \left|\left\langle \eta, D\bar{D}^{-1}\bar{\omega}_{\mathcal{S}} - \omega\right\rangle\right| \\ &\leqslant \left\|D\bar{D}^{-1}\bar{\omega}_{\mathcal{S}} - \omega\right\| \leqslant \|\omega - \omega_{\mathcal{S}}\| + \left\|D\bar{D}^{-1}\bar{\omega}_{\mathcal{S}} - \omega_{\mathcal{S}}\right\| \overset{\text{Lemma 4.6}}{\leqslant} 2(Cn)^{1/2}. \end{aligned} \tag{39}$$

Finally, because $\|\bar{\omega}_{\mathcal{S}}\|^2 \geqslant n\bar{d}/2$ by the third part of Lemma 4.6, (39) implies that $|\beta| \leqslant 4C\bar{d}^{\,-\frac{1}{2}}$. Therefore, (35) and (38) yield $\|M_{\mathcal{S}}\eta\| \leqslant 100(c_1 + C)\bar{d}_{\min}^{-1/2}$, as desired. $\square$

4.3 Proof of Lemma 4.3

To prove Lemma 4.3, we consider the following process.

P0. Determine $H = \text{core}$.

P1. Let $Q_0' = \{v \in H : |d_H(v) - \bar{d}(v)| \geqslant 0.01\bar{d}(v)\}$.

P2. While there is a $v \in H - Q_0'$ such that

$$e(v, R) \geqslant \max\{c_0, \exp(-\bar{d}_{\min}/c_0)\bar{d}^{\,-\frac{1}{2}}d_G(v)\},$$

add v to Q_0'.

P3. Let $j = 0$.
While $Q'_j \neq \emptyset$, let

$$Z'_{j+1} = \{v \in Q'_j : e(v, Q'_j) \leqslant \max\{c_0, \exp(-\bar{d}_{\min}/c_0) d_H(v)\}, \quad Q'_{j+1} = Q'_j - Z'_{j+1},$$

and increase j by 1.

Observe that the set Q'_0 obtained in P0–P2 is just $V - \mathcal{S}$. Hence, if P3 terminates, then it produces a decomposition $Z'_1, \ldots, Z'_k$ of $H - \mathcal{S}$ that enjoys the property stated in Lemma 4.3. Thus, Lemma 4.3 is an immediate consequence of the following statement.

Lemma 4.7 *The process P0–P3 terminates w.h.p.*

To prove Lemma 4.7, we consider a further process that is a little easier to analyze than P0–P3.

A1. Let $Q_0 = \{v \in V : |d_G(v) - \bar{d}(v)| \geqslant 0.001\bar{d}(v)\}$.

A2. While there is a $v \in Q_0 - R$ such that

$$e(v, Q_0) \geqslant \frac{1}{2} \max\{c_0, \exp(-\bar{d}_{\min}/c_0)\bar{d}^{-\frac{1}{2}}\bar{d}(v)\},$$

let $Q_0 = Q_0 \cup \{v\}$.

A3. Set $j = 0$.
While $Q_j \neq \emptyset$, let

$$Z_{j+1} = \{v \in Q_j : e(v, Q_j) \leqslant \frac{1}{2} \max\{c_0, \exp(-\bar{d}_{\min}/c_0) d_G(v)\},$$

$Q_{j+1} = Q_j - Z_{j+1}$, and increase j by 1.

Recalling the process K1–K2 from Section 3.3, we note that $Q_0 = G - K$.

Lemma 4.8 *If the process A1–A3 terminates, then so does P0–P3.*

Proof. Lemma 3.8 implies that $Q_0 = G - K \supset H - \mathcal{S} = Q'_0$. Hence, by induction we have $Q_j \supset Q'_j$ for all $j \geqslant 1$. □

Due to Lemma 4.8, in order to prove Lemma 4.7 we just need to show the following.

Lemma 4.9 *The process A1–A3 terminates w.h.p.*

Proof. Let J be the total number of sets generated by A3 (possibly $J = \infty$), and let $Q = \bigcap_{j=1}^{J} Q_j$; our objective is to show that $Q = \emptyset$ w.h.p. Since $Q_0 = G - K$, Corollary 3.12 yields

$$\#Q \overset{(7)}{\leqslant} \mathrm{Vol}(Q) \leqslant \mathrm{Vol}(Q_0) \leqslant \exp(-100\bar{d}_{\min}/c_0)n \tag{40}$$

w.h.p. Furthermore, step A3 ensures that

$$4e(Q) \geqslant \max\left\{c_0\#Q, \exp(-\bar{d}_{\min}/c_0) \sum_{v \in Q} d_G(v)\right\}. \tag{41}$$

Thus, if (40) is true and $Q \neq \emptyset$, then one of the following two conditions holds:

- either $\sum_{v\in Q} d_G(v)$ is much smaller than $\mathrm{Vol}(Q)$ – say, smaller than $\frac{1}{4}\mathrm{Vol}(Q)$;
- or (41) implies that $e(Q)$ is "large", although $\mathrm{Vol}(Q)$ is "small".

Loosely speaking, the first situation is unlikely due to Lemma 2.6, and the second one does not occur w.h.p. by Corollary 2.5. More precisely, assuming that $Q \neq \emptyset$, and that (40) holds, we shall prove that one of the properties (13), (14), (15) is violated.

1st case: $\mathrm{Vol}(Q) \leqslant 1000\#Q^{5/8}n^{3/8}$**.** Then (41) shows that $e(Q) > 10^4\#Q$, so that (14) is false.

2nd case: $\mathrm{Vol}(Q) > 1000\#Q^{5/8}n^{3/8}$ **and** $e(Q) < \frac{1}{20}\exp(-\bar{d}_{\min}/c_0)\mathrm{Vol}(Q)$**.** Then by (41)

$$\frac{1}{4}\exp(-\bar{d}_{\min}/c_0)\sum_{v\in Q} d_G(v) \leqslant e(Q) < \frac{1}{20}\exp(-\bar{d}_{\min}/c_0)\mathrm{Vol}(Q),$$

and thus (15) is violated.

3rd case: $\mathrm{Vol}(Q) > 1000\#Q^{5/8}n^{3/8}$ **and** $e(Q) \geqslant \frac{1}{20}\exp(-\bar{d}_{\min}/c_0)\mathrm{Vol}(Q)$**.** Then we obtain from (40) that

$$\mathrm{Vol}(Q) > \#Q^{5/8}n^{3/8} \geqslant \#Q\exp(2\bar{d}_{\min}/c_0). \tag{42}$$

As $e(Q) \geqslant \frac{1}{20}\exp(-\bar{d}_{\min}/c_0)\mathrm{Vol}(Q)$, (40) and (42) imply that (13) is violated.

Thus, in all three cases either (13), (14), or (15) is false, whence Corollary 2.5 and Lemma 2.6 imply that $Q = \emptyset$ w.h.p. $\square$

4.4 Proof of Proposition 4.4

By Lemma 4.3 $H - \mathcal{S}$ has a decomposition $Z_1, \ldots, Z_K$ that satisfies (31) w.h.p. We set $Z_{\geqslant j} = \bigcup_{i=j}^{K} Z_i$ and define $Z_{<j}$, $Z_{>j}$ analogously. Let $\xi = (\xi_v)_{v\in H}$ be a unit vector, and set $\eta = (\eta_w)_{w\in H} = M_{H-\mathcal{S}}\xi$. Our objective is to bound $\|\eta\|$.

The entries of η are

$$\eta_v = \sum_{w\in N_H(v)\cap Z_{\geqslant j}} \frac{\xi_w}{(d_H(v)d_H(w))^{1/2}} + \sum_{w\in N_H(v)\cap Z_{<j}} \frac{\xi_w}{(d_H(v)d_H(w))^{1/2}} \quad (v \in Z_j,\ j \geqslant 1),$$

and $\eta_v = 0$ for $v \in \mathcal{S}$. Let

$$\begin{aligned}\alpha_j &= \sum_{v\in Z_j}\left[\sum_{w\in N_H(v)\cap Z_{\geqslant j}} \frac{\xi_w}{(d_H(v)d_H(w))^{1/2}}\right]^2,\\ \beta_j &= \sum_{v\in Z_j}\left[\sum_{w\in N_H(v)\cap Z_{<j}} \frac{\xi_w}{(d_H(v)d_H(w))^{1/2}}\right]^2.\end{aligned}$$

Then

$$\|\eta\|^2 \leqslant 2\sum_{j=1}^{K} \alpha_j + \beta_j. \tag{43}$$

With respect to α_j, the Cauchy-Schwarz inequality yields

$$\alpha_j \;=\; \sum_{v\in Z_j}\left[\sum_{w\in N_H(v)\cap Z_{\geqslant j}} \frac{\xi_w}{\sqrt{d_H(v)d_H(w)}}\right]^2 \leqslant \sum_{v\in Z_j}\sum_{w\in N_H(v)\cap Z_{\geqslant j}} \frac{e(v,Z_{\geqslant j})\xi_w^2}{d_H(v)d_H(w)}. \tag{44}$$

Since by (31) for all $v \in Z_j$ we have $e(v, Z_{\geqslant j}) \leqslant \max\{c_0, \exp(-\bar{d}_{\min}/c_0)d_H(v)\}$, (44) entails

$$\begin{aligned}
\alpha_j &\leqslant \sum_{v\in Z_j}\sum_{w\in N_H(v)\cap Z_{\geqslant j}} \frac{\max\{c_0,\exp(-\bar{d}_{\min}/c_0)d_H(v)\}\xi_w^2}{d_H(v)d_H(w)}\\
&\leqslant \left(\frac{c_0}{\min_{v\in H} d_H(v)} + \exp(-\bar{d}_{\min}/c_0)\right)\sum_{v\in Z_j}\sum_{w\in N_H(v)\cap Z_{\geqslant j}}\frac{\xi_w^2}{d_H(w)}
\end{aligned} \tag{45}$$

As $d_H(v) \geqslant \frac{1}{2}d_G(v) \geqslant \bar{d}_{\min}/200$ for all $v \in H$ by (16), (45) implies

$$\sum_{j=1}^{K}\alpha_j \leqslant \frac{201c_0}{\bar{d}_{\min}}\sum_{j=1}^{K}\sum_{w\in Z_j}\frac{e(w,Z_{\leqslant j})\xi_w^2}{d_H(w)} \leqslant \frac{201c_0}{\bar{d}_{\min}}\sum_{w\in H-\mathcal{S}}\xi_w^2 \leqslant \frac{201c_0}{\bar{d}_{\min}}\|\xi\|^2 \leqslant \frac{201c_0}{\bar{d}_{\min}}. \tag{46}$$

Furthermore, once more due to the Cauchy-Schwarz inequality, we have

$$\beta_j \;=\; \sum_{v\in Z_j}\left[\sum_{w\in N_H(v)\cap Z_{<j}} \frac{\xi_w}{\sqrt{d_H(v)d_H(w)}}\right]^2 \leqslant \sum_{v\in Z_j}\sum_{w\in N_H(v)\cap Z_{<j}} \frac{e(v,Z_{<j})\xi_w^2}{d_H(v)d_H(w)}. \tag{47}$$

Hence, as

$$e(w, Z_{>j}) \leqslant \max\{c_0, \exp(-\bar{d}_{\min}/c_0)d_H(w)\}$$

for all $w \in Z_j$ due to (31), (47) yields

$$\begin{aligned}
\sum_{j=1}^{K}\beta_j &\leqslant \sum_{j=1}^{K}\sum_{v\in Z_j}\sum_{w\in N_H(v)\cap Z_{<j}}\frac{\xi_w^2}{d_H(w)} = \sum_{j=1}^{K}\sum_{w\in Z_j} e(w,Z_{>j})\frac{\xi_w^2}{d_H(w)}\\
&\leqslant \sum_{j=1}^{K}\sum_{w\in Z_j}\max\{c_0,\exp(-\bar{d}_{\min}/c_0)d_H(w)\}\frac{\xi_w^2}{d_H(w)}\\
&\leqslant \left(\frac{c_0}{\min_{w\in H} d_H(w)} + \exp(-\bar{d}_{\min}/c_0)\right)\sum_{w\in H-S}\xi_w^2\\
&\overset{(16)}{\leqslant} \frac{201c_0}{\bar{d}_{\min}}\|\xi\|^2 \leqslant \frac{201c_0}{\bar{d}_{\min}}.
\end{aligned} \tag{48}$$

Combining (43), (46), and (48), we conclude that $\|M_{H-\mathcal{S}}\xi\| = \|\eta\| \leqslant 21c_0^{1/2}\bar{d}_{\min}^{-1/2}$. Since this holds for all unit vectors ξ, we obtain $\|M_{H-\mathcal{S}}\| \leqslant 21c_0^{1/2}\bar{d}_{\min}^{-1/2}$, thereby completing the proof.

4.5 Proof of Proposition 4.5

Let $\xi = (\xi_w)_{w \in V(H)}$ be a unit vector. Set $\eta = (\eta_v)_{v \in V(H)} = M_{\mathcal{S} \times (H-\mathcal{S})} \xi$. Then

$$\eta_v = \sum_{w \in N_H(v) - \mathcal{S}} \frac{\xi_w}{\sqrt{d_H(v) d_H(w)}} \qquad \text{for } v \in \mathcal{S}, \text{ and } \eta_v = 0 \text{ for } v \in V(H) - \mathcal{S}.$$

Therefore, applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|\eta\|^2 &= \sum_{v \in \mathcal{S}} \left[\sum_{w \in N_H(v) - \mathcal{S}} \frac{\xi_w}{\sqrt{d_H(v) d_H(w)}} \right]^2 \\ &\leqslant \sum_{v \in \mathcal{S}} \left[\sum_{w \in N_H(v) - \mathcal{S}} \frac{\xi_w^2}{d_H(w)} \right] \left[\sum_{w \in N_H(v) - \mathcal{S}} \frac{1}{d_H(v)} \right]. \end{aligned} \tag{49}$$

As for all $v \in \mathcal{S}$ we have $e(v, H - \mathcal{S}) \leqslant 2 \max\{c_0, \exp(-\bar{d}_{\min}/c_0) d_H(v)\}$ and $d_H(v) \geqslant \frac{\bar{d}_{\min}}{2}$ by (6) and (17), we conclude that

$$\sum_{w \in N_H(v) - \mathcal{S}} \frac{1}{d_H(v)} = \frac{e(v, H - \mathcal{S})}{d_H(v)} \leqslant 2 \max\left\{ \frac{2c_0}{\bar{d}_{\min}}, \exp(-\bar{d}_{\min}/c_0) \right\} \overset{(6)}{\leqslant} \frac{4c_0}{\bar{d}_{\min}}. \tag{50}$$

Plugging (50) into (49), we obtain

$$\begin{aligned} \|\eta\|^2 &\leqslant \frac{4c_0}{\bar{d}_{\min}} \sum_{v \in \mathcal{S}} \sum_{w \in N_H(v) - \mathcal{S}} \frac{\xi_w^2}{d_H(w)} \\ &= \frac{4c_0}{\bar{d}_{\min}} \sum_{w \in H - \mathcal{S}} e(w, \mathcal{S}) \frac{\xi_w^2}{d_H(w)} \leqslant \frac{4c_0}{\bar{d}_{\min}} \sum_{w \in H - \mathcal{S}} \xi_w^2 \leqslant \frac{4c_0}{\bar{d}_{\min}}, \end{aligned}$$

because $\|\xi\| \leqslant 1$. Thus, $\|M_{\mathcal{S} \times (H-\mathcal{S})}\|^2 = \sup_{\|\xi\|=1} \|M_{\mathcal{S} \times (H-\mathcal{S})} \xi\|^2 \leqslant 4c_0 \bar{d}_{\min}^{-1}$, as desired.

5 Proof of Proposition 4.1

Throughout this section, we assume that (6) is satisfied. Moreover, we let $G = G(n, \boldsymbol{d})$, set $H = \mathrm{core}(G)$, and let $\mathcal{S}$ be the set constructed via the process S1–S2 from Section 3.2. Further, by Proposition 3.4 we may assume that $\#\mathcal{S} \geqslant \frac{n}{2}$.

5.1 Outline of the Proof

Instead of the matrix $\mathcal{M} = (m_{uv})_{u,v \in H}$ we shall mostly study a slightly modified matrix $\mathcal{M}' = (m'_{vw})_{v,w \in V}$, whose entries are defined as follows: let

$$m'_{vw} = \begin{cases} (\bar{d}(v)\bar{d}(w))^{-1/2} & \text{if } \{v, w\} \in E(G) \\ 0 & \text{otherwise} \end{cases} \qquad (v, w \in V,\ v \neq w).$$

Furthermore, for all $v \in V$ we let $m'_{vv} = \bar{d}(v)^{-1}$ with probability $p_{vv} = \bar{d}(v)^2(\bar{d}n)^{-1}$, and $m'_{vv} = 0$ with probability $1 - p_{vv}$, where the entries m'_{vv} are mutually independent and independent of choice of G. Then

$$\mathrm{E}(m'_{vw}) = \left(\bar{d}(v)\bar{d}(w)\right)^{1/2} (\bar{d}n)^{-1} \qquad \text{for all } v, w \in V. \tag{51}$$

The difference between $\mathcal{M}$ and $\mathcal{M}'$ is just that in $\mathcal{M}'$ we add entries corresponding to vertices $v \in V - H$, and we also add entries on the diagonal. Therefore, the matrix $\mathcal{M}$ is a minor of $\mathcal{M}' - \mathrm{diag}(m'_{vv})_{v\in V}$, where

$$\|\mathrm{diag}(m'_{vv})_{v\in V}\| \overset{(6)}{\leqslant} \bar{d}_{\min}^{-1/2}. \tag{52}$$

Thus, setting $\tilde{\omega} = (\bar{d}(u)^{1/2})_{u\in V}$ and

$$S = \left\{x \in \mathbf{R}^V : \|x\| \leqslant 1,\ x \perp \tilde{\omega}\right\}, \qquad S' = \{x = (x_v)_{v\in V} \in S : x_v = 0 \text{ for all } v \in V - \mathcal{S}\},$$

our aim is to prove that there is a constant $c_1 > 0$ such that w.h.p.

$$\max\{|\langle \mathcal{M}'x, y\rangle| : x, y \in S'\} \leqslant c_1 \bar{d}_{\min}^{-1/2}. \tag{53}$$

Then Proposition 4.1 will follow from (52) and (53).

To establish (53), we shall replace the infinite set S' by a finite set T' such that

$$\max_{x,y\in S'} |\langle \mathcal{M}'x, y\rangle| \leqslant 5 \max_{x,y\in T'} |\langle \mathcal{M}'x, y\rangle| + 8. \tag{54}$$

Then, we show that

$$\max_{x,y\in T'} |\langle \mathcal{M}'x, y\rangle| \leqslant c_2 \bar{d}_{\min}^{-1/2} \text{ w.h.p.}, \tag{55}$$

where c_2 is a suitable constant, so that (53) will follow from (54) and (55).

To define T', set $\varepsilon = 0.01$, and let $\varepsilon n^{-1/2}\mathbf{Z}$ signify the set of all integer multiples of $\varepsilon n^{-1/2}$. Let

$$\begin{aligned} T &= \left\{x \in \left[\varepsilon n^{-1/2}\mathbf{Z}\right]^n : |\langle \tilde{\omega}, x\rangle| \leqslant \bar{d}^{\,\frac{1}{2}} n^{-1/2},\ \|x\| \leqslant 1\right\}, \\ T' &= \left\{x = (x_1, \ldots, x_n)^T \in T : x_v = 0 \text{ for all } v \in V - \mathcal{S}\right\}. \end{aligned}$$

Lemma 5.1 *The set T' satisfies (54) and there is a constant $c_3 > 0$ such that $\#T \leqslant c_3^n$.*

We prove Lemma 5.1 in Section 5.2. Our next goal is to establish (55). Given vectors $x = (x_u)_{u\in V}, y = (y_v)_{v\in V} \in \mathbf{R}^V$, we define

$$B(x, y) = \left\{(u, v) \in V^2 : n^2\bar{d}_{\min} |x_u y_v|^2 \leqslant \bar{d}(u)\bar{d}(v)\right\}, \qquad X_{x,y} = \sum_{(u,v)\in B(x,y)} m'_{uv} x_u y_v.$$

We shall prove that there exist constants $c_4, c_5 > 0$ such that w.h.p.

$$\begin{aligned} \max_{x,y\in T} |X_{x,y}| &\leqslant c_4 \bar{d}_{\min}^{-1/2}, & (56) \\ \max_{x,y\in T'} \sum_{(u,v)\notin B(x,y)} |m'_{uv} x_u y_v| &\leqslant c_5 \bar{d}_{\min}^{-1/2}. & (57) \end{aligned}$$

Then (55) will follow from (56) and (57) (with $c_2 = c_4 + c_5$).

In order to show (56), we proceed in two steps. First, we bound the expectation of $X_{x,y}$.

Lemma 5.2 *There is a constant $c_6 > 0$ such that $|\mathrm{E}(X_{x,y})| \leqslant c_6 \bar{d}_{\min}^{-1/2}$ for all $x, y \in T$.*

Secondly, we bound the probability that $X_{x,y}$ deviates from its expectation significantly.

Lemma 5.3 *Let $x, y \in \mathbf{R}^n$, $\|x\|, \|y\| \leqslant 1$. Then for any constant $C > 0$ there exists a constant $K > 0$ such that $\mathrm{P}\left[|X_{x,y} - \mathrm{E}(X_{x,y})| > K\bar{d}_{\min}^{-1/2}\right] \leqslant C^{-n}$.*

Combining Lemmas 5.2 and 5.3, we conclude that there is a constant $c_4 > 0$ such that

$$\mathrm{P}\left[|X_{x,y}| > c_4 \bar{d}_{\min}^{-1/2}\right] \leqslant (2c_3^2)^{-n}$$

for any two points $x, y \in T$. Therefore, invoking Lemma 5.1 and applying the union bound, we conclude

$$\mathrm{P}\left[\max_{x,y \in T} |X_{x,y}| > c_5 \bar{d}_{\min}^{-1/2}\right] \leqslant \#T \cdot (2c_3)^{-n} \leqslant 2^{-n},$$

thereby proving that (56) is true w.h.p. The proofs of Lemmas 5.2 and 5.3 can be found in Sections 5.3 and 5.4.

The remaining task is to show that (57) holds w.h.p. To this end, in Section 5.5 we show the following.

Lemma 5.4 *If G enjoys the property (12), then*

$$\max_{x,y \in \mathbf{R}^V - \{0\}} \sum_{(u,v) \notin B(x,y),\, u \neq v} \frac{|m_{uv} x_u y_v|}{\|x\| \cdot \|y\|} \leqslant c_5 \bar{d}_{\min}^{-1/2} \qquad \textit{for a certain constant } c_5 > 0.$$

Thus, Lemmas 2.4 and 5.4 imply that (57) holds w.h.p., and Proposition 4.1 follows.

5.2 Proof of Lemma 5.1

To prove Lemma 5.1, we observe that every vector $x \in S'$ can be approximated by a point in the slightly "stretched" grid $(1 - 2\varepsilon)^{-1} T'$.

Lemma 5.5 *For each $x \in S'$ there is $y \in (1 - 2\varepsilon)^{-1} T'$ such that $\|x - y\| \leqslant \frac{1}{2}$.*

Proof. Relabeling the vertices as necessary, we may assume that $\mathcal{S} = \{1, \ldots, s\}$, $s \geqslant n/2$. Let $x' = (x'_i)_{i=1,\ldots,n} = (1-2\varepsilon)x$. We construct a vector $y'' = (y''_i)_{i=1,\ldots,n} \in \left[\varepsilon n^{-1/2}\mathbf{Z}\right]^n$ inductively as follows. Let $1 \leqslant i \leqslant s$, and assume that we have defined $y''_1, \ldots, y''_{i-1}$ already. There are two points $p_i, q_i \in \varepsilon n^{-1/2}\mathbf{Z}$ such that $|p_i - x'_i|, |q_i - x'_i| \leqslant \varepsilon n^{-1/2}$; choose $y''_i \in \{p_i, q_i\}$ so that $\left|\sum_{j=1}^{i} \tilde{\omega}_j (x'_j - y''_j)\right|$ is minimal. Further, set $y''_i = 0$ for $s < i \leqslant n$. By construction, we have

$$\|x' - y''\| \leqslant \left[\sum_{i=1}^{n} |x'_i - y''_i|^2\right]^{1/2} \leqslant \varepsilon, \tag{58}$$

$$\left|\sum_{j=1}^{i} \tilde{\omega}_j (x'_j - y''_j)\right| \leqslant \varepsilon n^{-1/2} \cdot \max_j \tilde{\omega}_j = o(1) \quad (1 \leqslant i \leqslant n). \tag{59}$$

Let $I = \{j \in \mathcal{S} : \bar{d}(j) \leqslant 100\bar{d}\}$. Since $s \geqslant \frac{n}{2}$, we have $\#I \geqslant n/10$. Furthermore, as $\tilde{\omega} \perp x'$, (59) implies $|\langle y'', \tilde{\omega}\rangle| = o(1)$. Therefore, there is a set $J \subset I$ such that

$$\left| |\langle y'', \tilde{\omega}\rangle| - \sum_{j \in J} \varepsilon \tilde{\omega}_j n^{-1/2} \right| \leqslant \bar{d}^{\,\frac{1}{2}} n^{-1/2}. \tag{60}$$

Now, define $y'_j = y''_j$ for $j \in V - J$, and set

$$y'_j = y''_j + \varepsilon n^{-1/2} \times \begin{cases} 1 & \text{if } \langle y'', \tilde{\omega}\rangle < 0 \\ -1 & \text{otherwise} \end{cases} \qquad \text{for } j \in J.$$

Then (60) implies that $|\langle y', \tilde{\omega}\rangle| \leqslant \bar{d}^{\,\frac{1}{2}} n^{-1/2}$. Moreover, (58) yields $\|x' - y'\| \leqslant \|x' - y''\| + \|y' - y''\| \leqslant 2\varepsilon$. Hence, $\|y'\| \leqslant \|x'\| + \|x' - y'\| \leqslant 1$, so that $y' \in T'$. Thus, setting $y = (1-2\varepsilon)^{-1} y'$ completes the proof. □

Proof of Lemma 5.1. To prove (54), consider a vector $x \in S'$. We want to approximate x by a linear combination of vectors $t_0, t_1, t_2, \ldots$ in $(1-2\varepsilon)^{-1}T'$. Let $x_0 = x$. For each $i \geqslant 0$, we define a vector $x_{i+1} \perp \tilde{\omega}$ of norm $\leqslant 1$ and a vector t_i as follows. By Lemma 5.5, there exists a vector $t_i \in (1-2\varepsilon)^{-1}T'$ such that $\|x_i - t_i\| \leqslant \frac{1}{2}$. If $x_i = t_i$, then we set $x_{i+1} = 0$ and $t_j = z_{j-1} = 0$ for all $j > i$. Otherwise, let $x'_{i+1} = \|x_i - t_i\|^{-1}(x_i - t_i)$, define $z_i = -\|\tilde{\omega}\|^{-2} \langle t_i, \tilde{\omega}\rangle \tilde{\omega}$, and set $x_{i+1} = x'_{i+1} - \|x_i - t_i\|^{-1} z_i$. Then $\|x_{i+1}\| \leqslant 1$, $x_{i+1} \perp \tilde{\omega}$, and $\|z_i\| \leqslant 2n^{-1}$. Thus, we obtain a representation

$$x = \sum_{i=0}^{\infty} a_i (t_i + z_i), \text{ where } a_0 = 1, \text{ and } a_i = \prod_{0 \leqslant j < i} \|x_i - t_i\| \leqslant 2^{-i} \text{ for } i \geqslant 1.$$

Hence, $\|x - \sum_{i=0}^{\infty} a_i t_i\| \leqslant \sum_{i=0}^{\infty} a_i \|z_i\| \leqslant 4/n$.

Now, let $y \in S'$, and let $s_i \in (1-2\varepsilon)^{-1}T$ and $0 \leqslant b_i \leqslant 2^{-i}$ be such that

$$\left\| y - \sum_{i=0}^{\infty} b_i s_i \right\| \leqslant 2/n.$$

Then we obtain

$$
\begin{aligned}
|\langle \mathcal{M}'x, y\rangle| &= \left|\sum_{i,j=0}^{\infty} a_i b_j \langle \mathcal{M}'t_i, s_i\rangle\right| + \|\mathcal{M}'\| \left[\left\|x - \sum_{i=0}^{\infty} a_i t_i\right\| + \left\|y - \sum_{i=0}^{\infty} b_i s_i\right\|\right] \\
&\leqslant \left[\sum_{i,j=0}^{\infty} a_i b_j\right] (1-2\varepsilon)^{-2} \max\{|\langle \mathcal{M}'t, s\rangle| : s, t \in T'\} + \frac{8}{n}\|\mathcal{M}'\| \\
&\leqslant \frac{4}{(1-2\varepsilon)^2} \max\{|\langle \mathcal{M}'t, s\rangle| : s, t \in T'\} + 8,
\end{aligned}
$$

thereby establishing (54).

In order to estimate $\#T$, consider

$$
T^+ = \left\{t = (t_i)_{1 \leqslant i \leqslant n} \in \left[\varepsilon n^{-1/2}\mathbf{Z}\right]^n : t_i > 0 \text{ for all } i\right\}.
$$

Clearly, it suffices to exhibit a constant $\hat{c}_3 > 0$ such that $\#T^+ \leqslant \hat{c}_3^n$. To this end, let us assign to each $t \in T^+$ the cube $Q_t = \left\{(x_1, \ldots, x_n) \in [0,1]^n : t_i - \varepsilon n^{-1/2} < x_i \leqslant t_i \text{ for all } i\right\}$. Then Q_t is contained in the unit ball in $\mathbf{R}^n$, and the volume of Q_t equals $\varepsilon^n n^{-n/2}$. Moreover, if $s, t \in T^+$ are distinct, then Q_s and Q_t are disjoint. Therefore, letting V_n signify the volume of the unit ball in $\mathbf{R}^n$, we conclude that $\#T^+ \left(\frac{\varepsilon}{n}\right)^{n/2} \leqslant V_n \sim (\pi n)^{-1/2} \left(\frac{2e\pi}{n}\right)^{n/2}$. □

5.3 Proof of Lemma 5.2

To prove Lemma 5.2, we employ the following bound.

Lemma 5.6 *If $x, y \in \mathbf{R}^n$ have norm $\leqslant 1$, then*

$$
\sum_{(u,v) \in V^2 - B(x,y)} \left|(\bar{d}(u)\bar{d}(v))^{1/2} x_u y_v\right| \leqslant \bar{d}_{\min}^{1/2} n.
$$

Proof. The definition of $B(x,y)$ implies that $(\bar{d}(u)\bar{d}(v))^{1/2} \leqslant \bar{d}_{\min}^{1/2} n \cdot |x_u y_v|$ for all $(u,v) \notin B(x,y)$. Therefore,

$$
\sum_{(u,v) \in V^2 - B(x,y)} \left|(\bar{d}(u)\bar{d}(v))^{1/2} x_u y_v\right| \leqslant \bar{d}_{\min}^{1/2} n \sum_{(u,v) \notin B(x,y)} x_u^2 y_v^2 \leqslant \bar{d}_{\min}^{1/2} n \cdot \|x\|^2 \|y\|^2 \leqslant \bar{d}_{\min}^{1/2} n,
$$

as claimed. □

Proof of Lemma 5.2. Let $x, y \in T$. Then

$$
\begin{aligned}
|\mathrm{E}(X_{x,y})| &\overset{(51)}{=} \left|\sum_{(u,v) \in B(x,y)} \frac{(\bar{d}(u)\bar{d}(v))^{1/2} \cdot x_u y_v}{\bar{d}n}\right| \\
&\leqslant \left|\sum_{(u,v) \in V^2} \frac{(\bar{d}(u)\bar{d}(v))^{1/2} \cdot x_u y_v}{\bar{d}n}\right| + \left|\sum_{(u,v) \in V^2 - B(x,y)} \frac{(\bar{d}(u)\bar{d}(v))^{1/2} \cdot x_u y_v}{\bar{d}n}\right| \\
&\overset{\text{Lemma 5.6}}{\leqslant} \frac{|\langle x, \tilde{\omega}\rangle \cdot \langle y, \tilde{\omega}\rangle|}{\bar{d}n} + \bar{d}_{\min}\bar{d}^{-\frac{3}{2}} \leqslant n^{-2} + \bar{d}_{\min}\bar{d}^{-\frac{3}{2}} \leqslant \bar{d}_{\min}^{-1/2},
\end{aligned}
$$

because $|\langle\tilde{\omega}, x\rangle|, |\langle\tilde{\omega}, y\rangle| \leqslant (\bar{d}/n)^{1/2}$ by the definition of T. $\square$

5.4 Proof of Lemma 5.3

We shall prove below that

$$\mathrm{E}\left[\exp(n\bar{d}_{\min}^{1/2}(X_{x,y} - \mathrm{E}(X_{x,y})))\right] \leqslant \exp(16n). \tag{61}$$

Then Markov's inequality implies that

$$\mathrm{P}\left[X_{x,y} - \mathrm{E}(X_{x,y}) \geqslant K\bar{d}_{\min}^{-1/2}\right] \leqslant \mathrm{P}\left[\exp\left[n\bar{d}_{\min}^{1/2}X_{x,y}\right] \geqslant \exp\left[Kn\right]\right] \leqslant \exp\left[(16-K)n\right].$$

Hence, choosing K large enough, we can ensure that the right hand side is $\leqslant \frac{1}{2}\exp(-Cn)$. As a similar estimate holds for $-X_{x,y} = X_{-x,y}$, we obtain

$$\mathrm{P}\left[|X_{x,y}| \geqslant K\bar{d}_{\min}^{-1/2}\right] \leqslant \exp(-Cn),$$

as desired.

To prove (61), we set $\lambda = n\bar{d}_{\min}^{1/2}$ and let

$$\alpha_{uv} = (\bar{d}(u)\bar{d}(v))^{-1/2} \times \begin{cases} 0 & \text{if } (u,v),(v,u) \notin B(x,y), \\ x_u y_v & \text{if } (u,v) \in B(x,y) \wedge (v,u) \notin B(x,y), \\ x_v y_u & \text{if } (u,v) \notin B(x,y) \wedge (v,u) \in B(x,y), \\ x_u y_v + x_v y_u & \text{if } (u,v),(v,u) \in B(x,y) \end{cases}$$

signify the possible contribution of the edge $\{u,v\}$ to $X_{x,y}$ $(u,v \in V)$. Moreover, we define a random variable $X_{x,y}(u,v)$ by letting $X_{x,y}(u,v) = \alpha_{uv}$ if $m'_{uv} > 0$, and $X_{x,y}(u,v) = 0$ otherwise. Finally, let $\mathcal{E} = \{\{u,v\} : u,v \in V\}$. Then

$$X_{x,y} = \sum_{\{u,v\}\in\mathcal{E}} X_{x,y}(u,v).$$

Since $(X_{x,y}(u,v))_{\{u,v\}\in\mathcal{E}}$ is a family of mutually independent random variables, we have

$$\mathrm{E}(\exp(\lambda(X_{x,y} - \mathrm{E}(X_{x,y}))) = \prod_{\{u,v\}\in\mathcal{E}} \mathrm{E}\left[\exp(\lambda(X_{x,y}(u,v) - \mathrm{E}(X_{x,y}(u,v))))\right]. \tag{62}$$

Moreover, by the definition of $B(x,y)$, for all $(u,v) \in B(x,y)$ we have

$$\lambda(\bar{d}(u)\bar{d}(v))^{-1/2}x_u y_v \leqslant 1,$$

whence $\lambda\alpha_{uv} \leqslant 2$ for all $u,v \in V$. Therefore, as $\exp(t) - 1 \leqslant t + 4t^2$ if $|t| \leqslant 4$, (62) yields

$$\begin{aligned} \mathrm{E}(\exp(\lambda X_{x,y})) &\leqslant \prod_{\{u,v\}\in\mathcal{E}} 1 + \lambda\left[\mathrm{E}(X_{x,y}(u,v) - \mathrm{E}(X_{x,y}(u,v))\right] + 4\lambda^2\mathrm{Var}(X_{x,y}(u,v)) \\ &\leqslant \prod_{\{u,v\}\in\mathcal{E}} 1 + 4p_{uv}\lambda^2\alpha_{uv}^2 \leqslant \exp\left[4\lambda^2 \sum_{\{u,v\}\in\mathcal{E}} p_{uv}\alpha_{uv}^2\right]. \end{aligned} \tag{63}$$

Furthermore,

$$\begin{aligned}\lambda^2 \sum_{\{u,v\}\in\mathcal{E}} p_{uv}\alpha_{uv}^2 &\leqslant \sum_{\{u,v\}\in\mathcal{E}} \lambda^2 \cdot \frac{\bar{d}(u)\bar{d}(v)}{\bar{d}n} \cdot \frac{2\left(x_u^2 y_v^2 + x_v^2 y_u^2\right)}{\bar{d}(u)\bar{d}(v)} \\ &\leqslant 2n \sum_{\{u,v\}\in\mathcal{E}} \left(x_u^2 y_v^2 + x_v^2 y_u^2\right) \leqslant 4n\|x\|^2\|y\|^2 \leqslant 4n. \end{aligned} \tag{64}$$

Plugging (64) into (63), we conclude that $\mathrm{E}(\exp(\lambda X_{x,y})) \leqslant \exp(16n)$, thereby establishing (61).

5.5 Proof of Lemma 5.4

Let $x, y \in \mathbf{R}^n$ be vectors of norm $\leqslant 1$. After decomposing x, y into differences of vectors with non-negative entries, we may assume that $x_u, y_v \geqslant 0$ for all $u, v \in V$. Moreover, splitting x and y into sums of two vectors each, we may assume that at most $\frac{n}{2}$ coordinates of each vector are non-zero. We partition the relevant coordinates $\mathcal{S}$ into a few pieces on which the entries of x (resp. y) are roughly the same: for $i, j \in \mathbf{Z}$ we set

$$\begin{aligned} \mathcal{A}_i &= \left\{u \in \mathcal{S} : 2^{i-1}n^{-1/2} \leqslant (\bar{d}_{\min}/\bar{d}(u))^{1/2}x_u \leqslant 2^i n^{-1/2}\right\},\ a_i = \#\mathcal{A}_i \leqslant \frac{n}{2}, & (65)\\ \mathcal{B}_j &= \left\{v \in \mathcal{S} : 2^{j-1}n^{-1/2} \leqslant (\bar{d}_{\min}/\bar{d}(v))^{1/2}y_v \leqslant 2^j n^{-1/2}\right\},\ b_j = \#\mathcal{B}_j \leqslant \frac{n}{2}, & (66)\\ e_{ij} &= e(\mathcal{A}_i, \mathcal{B}_j), \qquad \mu_{ij} = \mu(\mathcal{A}_i, \mathcal{B}_j). & (67) \end{aligned}$$

If $u \in \mathcal{A}_i$, $v \in \mathcal{B}_j$, then $2^{i+j-2}n^{-1} \leqslant \bar{d}_{\min}(\bar{d}(u)\bar{d}(v))^{-1/2}x_u y_v \leqslant 2^{i+j}n^{-1}$. Hence,

$$2^{i+j-2}n^{-1}e_{ij} \leqslant \sum_{(u,v)\in\mathcal{A}_i\times\mathcal{B}_j,\, u\neq v} \bar{d}_{\min} m'_{uv} x_u y_v \leqslant 2^{i+j}n^{-1}e_{ij}, \tag{68}$$

so that basically $\sum_{(u,v)\in\mathcal{A}_i\times\mathcal{B}_j,\, u\neq v} m'_{uv}x_u y_v$ is determined by i, j, and e_{ij}.

Now, let us single out those indices i, j such that $(\mathcal{A}_i \times \mathcal{B}_j) - B(x,y) \neq \emptyset$. If $(u,v) \notin B(x,y)$ are such that $u \in \mathcal{A}_i$ and $v \in B_j$, then by the definition of $B(x,y)$, (65), and (66) we have

$$2^{i+j} \geqslant \frac{n\bar{d}_{\min}x_u y_v}{\sqrt{\bar{d}(u)\bar{d}(v)}} > \bar{d}_{\min}^{1/2}.$$

Therefore, setting

$$\mathcal{Q} = \{(i,j) : 2^{i+j} > \bar{d}_{\min}^{1/2},\, a_i \leqslant b_j\}, \quad \mathcal{Q}' = \{(i,j) : 2^{i+j} > \bar{d}_{\min}^{1/2},\, a_i > b_j\},$$

we obtain

$$\sum_{(u,v)\notin B(x,y),\, u\neq v} m'_{uv}x_u y_v \leqslant \sum_{(i,j)\in\mathcal{Q}\cup\mathcal{Q}'}\ \sum_{(u,v)\in\mathcal{A}_i\times\mathcal{B}_j,\, u\neq v} m'_{uv}x_u y_v.$$

Hence, by symmetry, it suffices to show that

$$\sum_{(i,j)\in\mathcal{Q}}\ \sum_{(u,v)\in\mathcal{A}_i\times\mathcal{B}_j,\, u\neq v} m'_{uv}x_u y_v \leqslant c_5\bar{d}_{\min}^{-1/2} \tag{69}$$

for some constant $c_5 > 0$.

To show (69), we split $\mathcal{Q}$ into two parts: let

$$\mathcal{Q}_1 = \{(i,j) \in \mathcal{Q} : e_{ij} \leqslant 300 \cdot \mu_{ij}\}$$

and $\mathcal{Q}_2 = \mathcal{Q} - \mathcal{Q}_1$.

Lemma 5.7 *We have* $\sum_{(i,j)\in\mathcal{Q}_1} \sum_{(u,v)\in\mathcal{A}_i\times\mathcal{B}_j,\, u\neq v} m'_{uv} x_u y_v \leqslant 4800\bar{d}_{\min}^{-1/2}$.

Proof. Let $\bar{B} = \bigcup_{(i,j)\in\mathcal{Q}_1} \mathcal{A}_i \times \mathcal{B}_j$. If $(i,j) \in \bar{B}$ and $(u,v) \in \mathcal{A}_i \times \mathcal{B}_j$, then by (65), (66), and the definition of $\mathcal{Q}_1$ we have

$$\frac{x_u y_v}{\left(\bar{d}(u)\bar{d}(v)\right)^{1/2}} \geqslant \frac{2^{i+j}}{4\bar{d}_{\min} n} > \frac{1}{4n}\bar{d}_{\min}^{-1/2},$$

and thus

$$\sum_{(u,v)\in\bar{B},\, u\neq v} \left(\bar{d}(u)\bar{d}(v)\right)^{1/2} x_u y_v \leqslant 4\bar{d}_{\min}^{1/2} n \|x\|^2 \|y\|^2 \leqslant 4\bar{d}_{\min}^{1/2} n. \tag{70}$$

Therefore, we obtain

$$\begin{aligned}
\sum_{(i,j)\in\mathcal{Q}_1} \sum_{(u,v)\in\mathcal{A}_i\times\mathcal{B}_j,\, u\neq v} \bar{d}_{\min} m'_{uv} x_u y_v &\overset{(68)}{\leqslant} \sum_{(i,j)\in\mathcal{Q}_1} e_{ij} 2^{i+j} n^{-1} \\
&\leqslant 300 \sum_{(i,j)\in\mathcal{Q}_1} \mu_{ij} 2^{i+j} n^{-1} \text{ [by the definition of } \mathcal{Q}_1] \\
&\leqslant 300 \sum_{(i,j)\in\mathcal{Q}_1} \sum_{(u,v)\in\mathcal{A}_i\times\mathcal{B}_j,\, u\neq v} p_{uv} 2^{i+j} n^{-1} \\
&= 300 \sum_{(i,j)\in\mathcal{Q}_1} \sum_{(u,v)\in\mathcal{A}_i\times\mathcal{B}_j,\, u\neq v} \frac{\bar{d}(u)\bar{d}(v)}{\bar{d}n} 2^{i+j} n^{-1} \\
&\overset{(65),(66)}{\leqslant} \frac{1200\bar{d}_{\min}}{\bar{d}n} \sum_{(i,j)\in\mathcal{Q}_1} \sum_{(u,v)\in\mathcal{A}_i\times\mathcal{B}_j,\, u\neq v} \left(\bar{d}(u)\bar{d}(v)\right)^{1/2} x_u y_v \\
&\leqslant \frac{1200}{n} \sum_{(u,v)\in\bar{B},\, u\neq v} \left(\bar{d}(u)\bar{d}(v)\right)^{1/2} x_u y_v \overset{(70)}{\leqslant} 4800\bar{d}_{\min}^{1/2},
\end{aligned}$$

thereby completing the proof. □

Thus, the remaining task is to estimate the contribution of the pairs $(i,j) \in \mathcal{Q}_2$.

Lemma 5.8 *There is a constant* $c_8 > 0$ *such that* $\sum_{(i,j)\in\mathcal{Q}_2} \sum_{(u,v)\in\mathcal{A}_i\times\mathcal{B}_j,\, u\neq v} m'_{uv} x_u y_v \leqslant c_8 \bar{d}_{\min}^{-1/2}$.

Proof. We decompose $\mathcal{Q}_2$ into several sets: let

$$\begin{aligned}
D_1 &= \{(i,j) \in \mathcal{Q}_2 : \bar{d}_{\min}^{1/2} e_{ij} < \mu_{ij} 2^{i+j}\},\\
D_2 &= \left\{(i,j) \in \mathcal{Q}_2 : \bar{d}_{\min}^{1/2} 2^j < 2^i\right\} - D_1,\\
D_3 &= \{(i,j) \in \mathcal{Q}_2 : \ln(n/b_j) \leqslant 4\ln(e_{ij}/\mu_{ij})\} - (D_1 \cup D_2),\\
D_4 &= \{(i,j) \in \mathcal{Q}_2 : n/b_j \leqslant 2^{4j}\} - (D_1 \cup D_2 \cup D_3),\\
D_5 &= \{(i,j) \in \mathcal{Q}_2 : n/b_j > 2^{4j}\} - (D_1 \cup D_2 \cup D_3);
\end{aligned}$$

then $\mathcal{Q}_2 = \bigcup_{k=1}^5 D_k$. We shall bound $\sum_{(i,j)\in D_k} e_{ij} 2^{i+j} n^{-1}$ separately for $k = 1, \ldots, 5$. To this end, we stress that for all $(i,j) \in \mathcal{Q}_2$

$$e_{ij} > 300\mu_{ij} \quad \text{and} \tag{71}$$

$$e_{ij}\ln(e_{ij}/\mu_{ij}) \leqslant cb_j \ln(n/b_j), \tag{72}$$

for a certain constant $c > 0$, because we are assuming that (12) holds.

With respect to D_1, we observe that $\mu_{ij} \leqslant \sum_{v\in\mathcal{A}_i,\, w\in\mathcal{B}_j} \bar{d}(v)\bar{d}(w)(\bar{d}n)^{-1}$. Hence, the definitions (65), (66) of $\mathcal{A}_i$, $\mathcal{B}_j$ imply that

$$\begin{aligned}
\mu_{ij} 2^{2(i+j)} n^{-1} &\leqslant 16\bar{d}_{\min}^2 \sum_{v\in\mathcal{A}_i,\, w\in\mathcal{B}_j} \frac{\bar{d}(v)\bar{d}(w)}{\bar{d}} \left[(\bar{d}(v)\bar{d}(w))^{-1/2} x_v y_w\right]^2\\
&\leqslant 16\bar{d}_{\min}^2 \bar{d}^{-1} \sum_{v\in\mathcal{A}_i,\, w\in\mathcal{B}_j} x_v^2 y_w^2
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum_{(i,j)\in D_1} e_{ij} 2^{i+j} n^{-1} &\leqslant \bar{d}_{\min}^{-1/2} \sum_{(i,j)\in D_1} \mu_{ij} 2^{2(i+j)} n^{-1}\\
&\leqslant 16(\bar{d}_{\min}/\bar{d})^{3/2} \bar{d}^{\frac{1}{2}} \sum_{(i,j)\in D_1} \sum_{(v,w)\in\mathcal{A}_i\times\mathcal{B}_j} x_v^2 y_w^2\\
&\leqslant 16(\bar{d}_{\min}/\bar{d})^{3/2} \bar{d}^{\frac{1}{2}} \|x\|^2\|y\|^2 \leqslant 16(\bar{d}_{\min}/\bar{d})^{3/2} \bar{d}^{\frac{1}{2}}.
\end{aligned} \tag{73}$$

Regarding D_2, we recall that $\sum_{j\in\mathbf{Z}} e(v, \mathcal{B}_j) \leqslant e(v, \mathcal{S}) \leqslant 2\bar{d}(v)$ for all $v \in \mathcal{S}$ by (17). Therefore, for all $i \in \mathbf{Z}$ we have

$$\sum_{j\in\mathbf{Z}} e_{ij} 2^{2i} n^{-1} \overset{(65)}{\leqslant} 4 \sum_{v\in\mathcal{A}_i} \frac{\bar{d}_{\min} x_v^2}{\bar{d}(v)} \sum_{j\in\mathbf{Z}} e(v, \mathcal{B}_j) \leqslant 8\bar{d}_{\min} \sum_{v\in\mathcal{A}_i} x_v^2. \tag{74}$$

Thus, by the definition of D_2

$$\sum_{(i,j)\in D_2} e_{ij} 2^{i+j} n^{-1} \leqslant \bar{d}_{\min}^{-1/2} \sum_{(i,j)\in D_2} e_{ij} 2^{2i} n^{-1} \overset{(74)}{\leqslant} 8\bar{d}_{\min}^{1/2} \|x\|^2 \leqslant 8\bar{d}_{\min}^{1/2}. \tag{75}$$

Concerning D_3, we have

$$\sum_{(i,j)\in D_3} e_{ij}2^{i+j}n^{-1} \overset{(72)}{\leqslant} c\sum_{(i,j)\in D_3} \frac{b_j\ln(n/b_j)}{\ln(e_{ij}/\mu_{ij})}2^{i+j}n^{-1} \leqslant 4c\sum_{(i,j)\in D_3} 2^{i+j}b_jn^{-1}. \tag{76}$$

Furthermore, if $(i,j)\in D_3$, then $(i,j)\notin D_2$, so that $2^i \leqslant 2^j\bar{d}_{\min}^{1/2}$. In addition, we generally assume that $\bar{d}(v)\geqslant \bar{d}_{\min}$ for all v. In effect,

$$\begin{aligned}\sum_{(i,j)\in D_3} e_{ij}2^{i+j}/n &\overset{(76)}{\leqslant} 4c\sum_{j\in\mathbf{Z}}\sum_{i\,:\,2^i\leqslant 2^j\bar{d}_{\min}^{1/2}} b_j2^{i+j}n^{-1} \leqslant 8c\bar{d}_{\min}^{1/2}\sum_{j\in\mathbf{Z}} b_j2^{2j}n^{-1}\\ &\overset{(66)}{\leqslant} 32c\bar{d}_{\min}^{1/2}\sum_{j\in\mathbf{Z}}\sum_{v\in\mathcal{B}_j}\frac{\bar{d}_{\min}}{\bar{d}(v)}y_v^2 \leqslant 32c\bar{d}_{\min}^{1/2}\|y\|^2 \leqslant 32c\bar{d}_{\min}^{1/2}.\end{aligned} \tag{77}$$

Moreover,

$$\sum_{(i,j)\in D_4} e_{ij}2^{i+j}/n \overset{(72)}{\leqslant} c\sum_{(i,j)\in D_4}\frac{b_j\ln(n/b_j)}{\ln(e_{ij}/\mu_{ij})}2^{i+j}/n \overset{(71)}{\leqslant} 4c\sum_{(i,j)\in D_4} jb_j2^{i+j}/n. \tag{78}$$

If $(i,j)\in D_4$, then

$$\begin{aligned}2^i &\leqslant \frac{e_{ij}\bar{d}_{\min}^{1/2}}{\mu_{ij}2^j} && [\text{because } (i,j)\notin D_1]\\ &\leqslant \bar{d}_{\min}^{1/2}\left(\frac{n}{b_j}\right)^{1/4}2^{-j} && [\text{because } (i,j)\notin D_3]\\ &\leqslant \bar{d}_{\min}^{1/2} && [\text{because } (i,j)\in D_4].\end{aligned} \tag{79}$$

Combining (78) and (79) and observing that $j\geqslant 0$ for all $(i,j)\in D_4$, we obtain

$$\sum_{(i,j)\in D_4} e_{ij}2^{i+j}/n \leqslant 8c\sum_{j\geqslant 0}\bar{d}_{\min}^{1/2}jb_j2^j/n = 8c\bar{d}_{\min}^{1/2}\sum_{j\geqslant 0}2^{-j}j(b_j2^{2j}n^{-1}). \tag{80}$$

Furthermore, as $\bar{d}(v)\geqslant\bar{d}_{\min}$ for all v,

$$b_j2^{2j}n^{-1} \overset{(66)}{\leqslant} 4\sum_{v\in\mathcal{B}_j}\frac{\bar{d}_{\min}}{\bar{d}(v)}y_v^2 \leqslant 4\|y\|^2 \leqslant 4. \tag{81}$$

Thus, plugging (81) into (80), we conclude that

$$\sum_{(i,j)\in D_4} e_{ij}2^{i+j}/n \leqslant 32c\bar{d}_{\min}^{1/2}\sum_{j\geqslant 1} j2^{-j} = 64c\bar{d}_{\min}^{1/2}. \tag{82}$$

Let $(i,j)\in D_5$. Then $\ln(n/b_j)\leqslant 2\ln(nb_j^{-1}2^{-2j})$, so that

$$e_{ij}2^j \overset{(72)}{\leqslant} \frac{cb_j\ln(n/b_j)}{\ln(e_{ij}/\mu_{ij})}\cdot 2^j \overset{(71)}{\leqslant} cb_j2^{2j}\cdot\ln(n/b_j)\cdot 2^{-j} \leqslant cb_j2^{2j}\cdot 2\ln\left(nb_j^{-1}2^{-2j}\right)\cdot 2^{-j}.$$

Moreover, since $(i,j) \notin D_2$, we have $2^i \leqslant \bar{d}_{\min}^{1/2} 2^j$. Hence, for any fixed $j \in \mathbf{Z}$ we have

$$\begin{aligned} c^{-1} \sum_{i:(i,j)\in D_5} e_{ij} 2^{i+j}/n &\leqslant \frac{2b_j 2^{2j}}{n} \ln\left(\frac{n}{b_j 2^{2j}}\right) \sum_{i:2^i \leqslant 2^j \bar{d}_{\min}^{1/2}} 2^{i-j} \\ &\leqslant 4\bar{d}_{\min}^{1/2} \cdot \frac{b_j 2^{2j}}{n} \ln\left(\frac{n}{b_j 2^{2j}}\right). \end{aligned} \tag{83}$$

Further, if $(i,j) \in D_5$, then $\sqrt{b_j/n} \leqslant 2^{-2j}$, whence for $j \geqslant 0$ we have

$$\begin{aligned} \frac{b_j 2^{2j}}{n} \ln\left(\frac{n}{b_j 2^{2j}}\right) &= \sqrt{\frac{b_j 2^{2j}}{n}} \left(-2\sqrt{\frac{b_j 2^{2j}}{n}} \ln \sqrt{\frac{b_j 2^{2j}}{n}}\right) \\ &\leqslant 2^{-j} \left(-2\sqrt{\frac{b_j 2^{2j}}{n}} \ln \sqrt{\frac{b_j 2^{2j}}{n}}\right). \end{aligned} \tag{84}$$

As the function $t \mapsto -t \ln t$ is $\leqslant 1$ for $t > 0$, we have

$$-2\sqrt{\frac{b_j 2^{2j}}{n}} \ln \sqrt{\frac{b_j 2^{2j}}{n}} \leqslant 2$$

for $j \geqslant 0$, and thus (84) yields

$$\frac{b_j 2^{2j}}{n} \ln\left(\frac{n}{b_j 2^{2j}}\right) \leqslant 2^{1-j} \qquad (j \geqslant 0). \tag{85}$$

Similarly, if $(i,j) \in D_5$ and $j < 0$, then

$$\frac{b_j 2^{2j}}{n} \ln\left(\frac{n}{b_j 2^{2j}}\right) \leqslant -2\sqrt{\frac{b_j 2^{2j}}{n}} \ln \sqrt{\frac{b_j 2^{2j}}{n}}. \tag{86}$$

Since $b_j \leqslant n/2$, we have

$$\sum_{j<0} -2\sqrt{\frac{b_j 2^{2j}}{n}} \ln \sqrt{\frac{b_j 2^{2j}}{n}} \leqslant -(2\ln 2) \sum_{j<0} j 2^j = 4 \ln 2. \tag{87}$$

Combining (83), (85), (86), and (87), we conclude

$$\sum_{(i,j)\in D_5} e_{ij} 2^{i+j}/n \leqslant 4c\bar{d}_{\min}^{1/2} \left(4\ln 2 + \sum_{j\geqslant 0} 2^{1-j}\right) \leqslant 32c\bar{d}_{\min}^{1/2}. \tag{88}$$

Finally, due to (73), (75), (77), (82), and (88), the assertion follows from (68). $\square$

Algorithm 6.1 `LowDisc`(G)
Input: A graph $G = (V, E)$.
Output: (α, β) such that G has (α, β)-low discrepancy.

1. Let $n = \#V$ and $\tilde{d} = 2\#E/n$.
   For $d = 1, \ldots, n$ do
2. Construct a subgraph $H(d)$ of G as follows.
   - Initially let $H(d) = G - \{v : d_G(v) \leqslant 0.01d\}$.
   - While there is a vertex $v \in H$ that has at least
   $$\max\{c_0, \exp(-d/c_0)\tilde{d}^{-\frac{1}{2}} d_G(v)\}$$
   neighbors in $G - H(d)$, remove v from $H(d)$.

   Here c_0 denotes a large enough constant. Then, compute the spectral gap $\alpha(d)$ of $\mathcal{L}(H(d))$, and set $\beta(d) = 2\sum_{v \in G - H(d)} d_G(v)$.
3. If there is some $1 \leqslant d \leqslant n$ such that $\alpha(d) \geqslant 1 - c_0 d^{-1/2}$ and $\beta(d) \leqslant 2\exp(-d/c_0)n$, then let d^* be the maximum such d and return $(\alpha(d^*), \beta(d^*))$. Otherwise just return $\alpha = 0$ and $\beta = 2\#E$.

Figure 1: the procedure `LowDisc`.

6 Algorithmic Results

In this section we present the algorithms for Corollaries 1.3 and 1.4. Let us start with the algorithm `LowDisc` for Corollary 1.3 (see Figure 1). If we assume that in addition to the input graph $G = G(n, \boldsymbol{d})$ we are given the minimum expected degree $\bar{d}_{\min}$, then we could just compute $H = \mathrm{core}(G)$ (cf. Section 3.2), determine the spectral gap α of $\mathcal{L}(H)$, and set $\beta = 2\sum_{v \in V - H} d_G(v)$. Then G has (α, β)-low discrepancy, and Theorem 1.2 ensures that w.h.p. α and β obey the bounds stated in the completeness condition of Corollary 1.3.

However, we of course desire an algorithm that just requires the graph G at the input. Therefore, the following procedure `LowDisc` basically tries all possible values for $\bar{d}_{\min}$ and outputs the best bound on the discrepancy discovered in the course of this process.

It is easily seen that the output (α, β) of `LowDisc` satisfies the correctness condition in Corollary 1.3. Further, the Chernoff bound (10) entails that w.h.p. $\tilde{d} \sim \bar{d}$, and Theorem 1.2 yields that w.h.p. $d^* \geqslant \bar{d}_{\min}$. Thus, w.h.p. we have $\alpha \geqslant 1 - c_0\bar{d}_{\min}^{-1/2}$ and $\beta \leqslant 2\exp(-\bar{d}_{\min}/c_0)n$, so that the completeness condition is satisfied as well.

The algorithm for Corollary 1.4 is as follows. At first the algorithm bounds the discrepancy of $G = G(n, \boldsymbol{d})$ using `LowDisc`. Let x be the number of vertices in $G - \mathrm{core}(G)$, and let (α, β) be the result of `LowDisc`. Then `BoundAlpha` outputs $400 \cdot (1 - \alpha) \cdot \#E\bar{d}_{\min}^{-1} + x$.

We claim that this is indeed an upper bound on $\alpha(G)$. To see this, let X be some independent set in $\mathrm{core}(G)$. By Step 2 of `LowDisc`, the core of G has $(\alpha, 0)$ discrepancy.

Using (4), we get $\sum_{v\in X} d_{\mathrm{core}(G)}(v) \leqslant 2(1-\alpha)\cdot \#E$. Since all vertices v in the core satisfy $d_{\mathrm{core}(G)}(v) \geqslant \bar{d}_{\min}/200$, we conclude that $\#X \leqslant 400\cdot(1-\alpha)\cdot \#E/\bar{d}_{\min}$. Therefore, the maximum independent set in G has at most $400\cdot(1-\alpha)\cdot \#E\bar{d}_{\min}^{-1} + x$ vertices. Hence, `BoundAlpha` satisfies the correctness statement in Corollary 1.4. Further, the completeness follows directly from Corollary 1.3.

7 Proofs of Auxiliary Lemmas

7.1 Proof of Lemma 2.2

The proof of Lemma 2.2 relies on the following general tail bound, which is a consequence of Azuma's inequality (cf. [17, p. 38] for a proof).

Lemma 7.1 *Let $\Omega = \prod_{i=1}^N \Omega_i$ be a product of finite probability spaces $\Omega_1,\ldots,\Omega_N$. Let $Y:\Omega\to\mathbf{R}$ be a random variable that satisfies the following condition for all $1\leqslant j\leqslant N$.*

> *If $\omega = (\omega_i)_{1\leqslant i\leqslant N}, \omega' = (\omega_i')_{1\leqslant i\leqslant N} \in \Omega$ differ only in the j'th component (i.e., $\omega_i = \omega_i'$ if $i\neq j$), then $|Y(\omega)-Y(\omega')|\leqslant \tau$.*

Then $\mathrm{P}\left[|Y-\mathrm{E}(Y)|\geqslant\lambda\right] \leqslant 2\exp\left(-\lambda^2/(2\tau^2N)\right)$ *for all $\lambda>0$.*

To derive Lemma 2.2 from Lemma 7.1, we let $\mathcal{E} = \{\{v,w\} : v,w\in V, v\neq w\}$ be the set of all $\binom{n}{2}$ possible edges. Further, for each $e=\{u,v\}\in\mathcal{E}$ we let Ω_e be a Bernoulli experiment with success probability $p_{uv} = \bar{d}(u)\bar{d}(v)(\bar{d}n)^{-1}$. Then the probability space $G(n,\boldsymbol{d})$ decomposes into a product $G(n,\boldsymbol{d}) = \prod_{e\in\mathcal{E}}\Omega_e$, because each edge $e=\{u,v\}\in\mathcal{E}$ is present in $G(n,\boldsymbol{d})$ with probability p_{uv} independently. However, we cannot apply Lemma 7.1 directly to this product decomposition, because the number $\binom{n}{2}$ of factors is too large. Therefore, we construct a different decomposition $G(n,\boldsymbol{d}) = \prod_{i=1}^K \Omega_i$, where each factor Ω_i is a combination of several factors Ω_e.

To this end, we partition $\mathcal{E}$ into $K\leqslant 2(\bar{d}n)^{1-\gamma}$ sets $\mathcal{E}_1,\ldots,\mathcal{E}_K$ such that $\frac{1}{2}(\bar{d}n)^\gamma \leqslant \sum_{e\in\mathcal{E}_i}\mathrm{P}\left[e\in G(n,\boldsymbol{d})\right] \leqslant (\bar{d}n)^\gamma$ for $i=1,\ldots,K$. Then we can represent $G(n,\boldsymbol{d})$ as a product space

$$G(n,\boldsymbol{d}) = \prod_{i=1}^K \Omega_i, \quad \text{where } \Omega_i = \prod_{e\in\mathcal{E}_i}\Omega_e. \tag{89}$$

We call $\mathcal{E}_i$ *critical* in $G=G(n,\boldsymbol{d})$ if $\#\mathcal{E}_i\cap E(G) > 2(\bar{d}n)^\gamma$. Then the generalized Chernoff bound (10) entails that $\mathrm{P}\left[\mathcal{E}_i \text{ is critical}\right] \leqslant \exp(-(\bar{d}n)^\gamma/3)$ for all i. Hence,

$$\mathrm{P}\left[\exists i : \mathcal{E}_i \text{ is critical}\right] \leqslant K\exp(-(\bar{d}n)^\gamma/3) \leqslant \exp(-(\bar{d}n)^\gamma/4). \tag{90}$$

For $G=G(n,\boldsymbol{d})$ we define $\tilde{G} = G - \bigcup_{i:\mathcal{E}_i \text{is critical}}\mathcal{E}_i$ and set $Y(G) = X(\tilde{G})$; we are going to apply Lemma 7.1 to the decomposition (89) and the random variable $Y(G(n,\boldsymbol{d}))$. To this end, we observe that (90) entails

$$\mathrm{P}\left[Y(G(n,\boldsymbol{d})) = X(G(n,\boldsymbol{d}))\right] \geqslant 1-\exp(-(\bar{d}n)^\gamma/4). \tag{91}$$

Moreover, since X satisfies the Lipschitz condition (11), we have $|X(G)-Y(G)| \leqslant n^2$ for *all* possible outcomes $G = G(n, \boldsymbol{d})$. In effect, our assumption $\bar{d} \geqslant 1$ yields

$$|\mathrm{E}(X) - \mathrm{E}(Y)| \overset{(91)}{\leqslant} n^2 \exp(-(\bar{d}n)^\gamma/4) < (\bar{d}n)^{\frac{1}{2}+\gamma}/2. \tag{92}$$

Furthermore, we claim that

> if G, G' are such that $G - \mathcal{E}_j = G' - \mathcal{E}_j$, i.e., G, G' differ only on edges corresponding to the factor Ω_j, then $|Y(G) - Y(G')| \leqslant 4(\bar{d}n)^\gamma$ (93)

for all $1 \leqslant j \leqslant K$. To prove (93), we consider four cases.

1st case: $\mathcal{E}_j$ is not critical in G and G'. Then $\tilde{G}'$ can be obtained from $\tilde{G}$ by removing all edges in $\mathcal{E}_j \cap E(G)$ and then adding all edges in $\mathcal{E}_j \cap E(G')$. Since in this process we delete/insert at most $4(\bar{d}n)^\gamma$ edges in total, (93) follows from the fact that X satisfies the Lipschitz condition (11).

2nd case: $\mathcal{E}_j$ is critical in both G and G'. Then $\tilde{G}' = \tilde{G}$, so that $Y(G) = Y(G')$.

3rd case: $\mathcal{E}_j$ is critical in G but not in G'. Then $\tilde{G}'$ is obtained from $\tilde{G}$ by adding the edges $\mathcal{E}_j \cap E(G')$; since $\#\mathcal{E}_j \cap E(G') \leqslant 2(\bar{d}n)^\gamma$, the Lipschitz condition (11) implies (93).

4th case: $\mathcal{E}_j$ is critical in G' but not in G. Analogous to the 3rd case.

Due to (93), we can apply Lemma 7.1 to $Y(G(n, \boldsymbol{d}))$ and obtain

$$\begin{aligned} \mathrm{P}\left[|Y(G(n,\boldsymbol{d})) - \mathrm{E}(Y(G(n,\boldsymbol{d})))| \geqslant \frac{(\bar{d}n)^{\frac{1}{2}+\gamma}}{2}\right] &\leqslant 2\exp\left[-\frac{(\bar{d}n)^{1+2\gamma}}{8(4(\bar{d}n)^\gamma)^2 K}\right] \\ &\leqslant 2\exp\left[-\frac{(\bar{d}n)^\gamma}{256}\right]. \end{aligned} \tag{94}$$

Finally, we obtain

$$\begin{aligned} &\mathrm{P}\left[|X(G(n,\boldsymbol{d})) - \mathrm{E}(X(G(n,\boldsymbol{d})))| \geqslant (\bar{d}n)^{\frac{1}{2}+\gamma}\right] \\ &\quad\leqslant \mathrm{P}\left[X(G(n,\boldsymbol{d})) \neq Y(G(n,\boldsymbol{d}))\right] + \\ &\qquad\qquad \mathrm{P}\left[|Y(G(n,\boldsymbol{d})) - \mathrm{E}(X(G(n,\boldsymbol{d})))| \geqslant (\bar{d}n)^{\frac{1}{2}+\gamma}\right] \\ &\quad\overset{(91),(92)}{\leqslant} \exp(-(\bar{d}n)^\gamma/4) + \mathrm{P}\left[|Y(G(n,\boldsymbol{d})) - \mathrm{E}(Y(G(n,\boldsymbol{d})))| \geqslant \frac{(\bar{d}n)^{\frac{1}{2}+\gamma}}{2}\right] \\ &\quad\overset{(94)}{\leqslant} \exp\left[-\frac{(\bar{d}n)^\gamma}{4}\right] + 2\exp\left[-\frac{(\bar{d}n)^\gamma}{256}\right] \leqslant \exp\left[-\frac{(\bar{d}n)^\gamma}{300}\right], \end{aligned}$$

as desired.

7.2 Proof of Corollary 2.3

Let ϕ denote the function (8). We assume throughout that $1 \leqslant \bar{d} \leqslant n^{0.99}$, and we let $G = G(n, \boldsymbol{d})$. Moreover, for each $t \geqslant 0$ we set $S_t = \{v \in V : 2^t \leqslant |d_G(v) - \bar{d}(v)| \cdot \bar{d}(v)^{-1/2} < 2^{t+1}\}$ and $\phi_t = \bar{d}_{\min} \cdot \phi(2^t (\bar{d}_{\min})^{-1/2})$. The following lemma is the main ingredient to the proof.

Lemma 7.2 *We have* $\mathrm{P}\left[\#S_t \leqslant 4\exp(-\phi_t/4)n\right] \geqslant 1 - n^{-\Omega(1)}$.

Further, to establish Lemma 7.2, we need the following estimate.

Lemma 7.3 *We have* $\bar{d}(v)\phi(2^t\bar{d}(v)^{-1/2}) \geqslant \phi_t = \bar{d}_{\min} \cdot \phi(2^t \bar{d}_{\min}^{-1/2})$ *for all* $v \in V$.

Proof. We will show that for all $\tau \geqslant 1$ the function

$$f(d, \tau) = d\phi(\tau d^{-1/2}) \quad \text{is monotonically increasing in } d. \tag{95}$$

Since $f(\bar{d}(v), 2^t) = \bar{d}(v)\phi(2^t\bar{d}(v)^{-1/2})$, $\bar{d}(v) \geqslant \bar{d}_{\min}$, and $f(\bar{d}_{\min}, 2^t) = \phi_t$, (95) implies the assertion.

In order to establish (95), we consider the function $\varphi(s) = \left(1 + \frac{s}{2}\right)\ln(1+s) - s$ $(s > 0)$. Then an easy computation shows that $\frac{\partial}{\partial d} f(d, \tau) = \varphi(\tau d^{-1/2})$. Thus, we just need to show that $\varphi(s) > 0$ for all $s > 0$. To this end, we observe that $\lim_{s \to 0} \varphi(s) = 0$. Furthermore, the derivative of φ is $\frac{d}{ds}\varphi(s) = \frac{1}{2(1+s)}\left[(1+s)\ln(1+s) - s\right]$. Finally, as $(1+s)\ln(1+s) > s$ for all $s > 0$, we conclude that $\varphi(s) > 0$ for all $s > 0$, thereby completing the proof. □

Proof of Lemma 7.2. Since $\phi(-x) > \phi(x)$ for all $0 < x < 1$, $(1 - o(1))\bar{d}(v) \leqslant \mathrm{Var}(d(v)) \leqslant \bar{d}(v)$, and because $\phi(y)$ is increasing for $y > 0$, the Chernoff bound (10) entails that

$$\begin{aligned} \mathrm{P}&\left[\left|d_{G(n,\boldsymbol{d})}(v) - \bar{d}(v)\right| \in \left[2^t, 2^{t+1}\right] \bar{d}(v)^{1/2}\right] \\ &\leqslant \quad 2\exp\left[-\mathrm{Var}(d(v))\phi(2^t\bar{d}(v)^{1/2}/\mathrm{Var}(d(v)))\right] \\ &\leqslant \quad 2\exp\left[-\frac{\bar{d}(v)}{2}\phi(2^t\bar{d}(v)^{-1/2})\right] \\ &\overset{\text{Lemma 7.3}}{\leqslant} \quad 2\exp(-\phi_t/2). \end{aligned}$$

Hence, $\mathrm{E}(\#S_t) \leqslant 2\exp(-\phi_t/2)n$. We consider two cases.

1st case: $\phi_t \geqslant 0.001 \ln n$**.** Then Markov's inequality implies that

$$\mathrm{P}\left[\#S_t \geqslant 2\exp(-\phi_t/4)n\right] \leqslant \exp(-\phi_t/4) \leqslant n^{-\Omega(1)}.$$

2nd case: $\phi_t < 0.001 \ln n$**.** Since adding or removing a single edge can change $\#S_t$ by at most 2, the random variable $\#S_t/2$ satisfies the Lipschitz condition (11). Therefore, Lemma 2.2 entails in combination with our assumption $1 \leqslant \bar{d} \leqslant n^{0.99}$ that

$$\begin{aligned} \mathrm{P}\left[\#S_t > 4\exp(-\phi_t/2)n\right] &\leqslant \mathrm{P}\left[\#S_t \geqslant \mathrm{E}(\#S_t) + (\bar{d}n)^{0.501}\right] \\ &\leqslant \exp(-(\bar{d}n)^{0.001}/300) \leqslant n^{-\Omega(1)}. \end{aligned}$$

Thus, $\mathrm{P}\left[\#S_t > 4\exp(-\phi_t/4)n\right] \leqslant n^{-\Omega(1)}$ in both cases. $\square$

Proof of Corollary 2.3. Let $S_{-1} = V - \bigcup_{j\geqslant 0} S_j$. Then Lemma 7.2 entails that w.h.p.

$$\begin{aligned}\sum_{v\in V}\frac{(d_G(v)-\bar{d}(v))^2}{\bar{d}(v)} &\leqslant 4\#S_{-1} + \sum_{j\geqslant 0} 2^{2j+2}\#S_j \leqslant 4n + 16n\sum_{j\geqslant 0} 2^{2j}\exp(-\phi_j/4)\\ &\overset{(9)}{\leqslant} 4n + 16n\sum_{j\geqslant 0} 2^{2j}\exp(-2^{j-4}) \leqslant 10^6 n,\end{aligned}$$

as desired. $\square$

7.3 Proof of Lemma 2.4

Let $0 < u \leqslant u' \leqslant \frac{n}{2}$. We first prove that for any two sets $U, U' \subset V$ of cardinality $1 \leqslant u = \#U \leqslant u' = \#U' \leqslant \frac{n}{2}$ we have

$$\begin{aligned}&\mathrm{P}[e(U,U') > 300\mu(U,U') \wedge \\ &\qquad e(U,U')\ln(e(U,U')/\mu(U,U')) > 300u'\ln(n/u')] \leqslant \binom{n}{u'}^{-5}.\end{aligned} \tag{96}$$

To show (96), let

$$x = \inf\left\{z > 0 : z \geqslant 100\mu(U,U') \text{ and } z\ln(z/\mu(U,U')) \geqslant 100u'\ln(n/u')\right\}. \tag{97}$$

Since $(1-o(1))\mu(U,U') \leqslant \mathrm{Var}(e(U,U')) \leqslant \mu(U,U')$ and because the function $z \mapsto \phi(z)$ is increasing for $z > 0$, the Chernoff bound (10) entails that

$$\begin{aligned}\mathrm{P}\left[e(U,U') \geqslant \mu(U,U') + x\right] &\leqslant \exp\left[-\mathrm{Var}(e(U,U'))\phi\left(\frac{x}{\mathrm{Var}(e(U,U'))}\right)\right]\\ &\leqslant \exp\left[-\frac{1}{2}\mu(U,U')\phi\left(\frac{x}{\mu(U,U')}\right)\right]\\ &\leqslant \exp\left[-\frac{x+\mu(U,U')}{2}\ln\left(\frac{x}{\mu(U,U')}\right) + \frac{x}{2}\right].\end{aligned} \tag{98}$$

Further, our choice (97) of x ensures that $\ln(x/\mu(U,U')) \geqslant 4$ and that $x\ln(x/\mu(U,U')) \geqslant 100u'\ln(n/u')$, whence (98) yields

$$\begin{aligned}\mathrm{P}\left[e(U,U') \geqslant \mu(U,U') + x\right] &\leqslant \exp\left[-\frac{x}{4}\ln\left(\frac{x}{\mu(U,U')}\right)\right]\\ &\leqslant \exp\left[-25u'\ln(n/u')\right] \leqslant \binom{n}{u'}^{-5},\end{aligned} \tag{99}$$

where the last step is due to our assumption $u' \leqslant n/2$. To complete the proof of (96), we claim that

$$\begin{aligned}\mu(U,U') + x &\leqslant 300\mu(U,U') \text{ or}\\ (\mu(U,U')+x)\ln((\mu(U,U')+x)/\mu(U,U')) &\leqslant 300u'\ln(n/u').\end{aligned} \tag{100}$$

In order to establish (100), we consider two cases.

1st case: $x \leqslant 100\mu(U,U')$**.** Then $\mu(U,U') + x \leqslant 101\mu(U,U')$.

2nd case: $x > 100\mu(U,U')$**.** Then $\mu(U,U') + x \leqslant 2x$, whence (97) yields

$$\begin{aligned}(\mu(U,U')+x)\ln\left(\frac{\mu(U,U')+x}{\mu(U,U')}\right) &\leqslant 2x\ln\left(\frac{2x}{\mu(U,U')}\right)\\ &\leqslant 3x\ln\left(\frac{x}{\mu(U,U')}\right) = 300u'\ln(n/u').\end{aligned}$$

Hence, combining (99) and (100), we obtain (96).

Let $1 \leqslant u \leqslant u' \leqslant n/2$. Due to (96) and the union bound, the probability that there *exist* sets U, U', $\#U = u$, $\#U' = u'$ such that $e(U,U') > 300\mu(U,U')$ and

$$e(U,U')\ln(e(U,U')/\mu(U,U')) > 300u'\ln(n/u')$$

is at most

$$\binom{n}{u}\binom{n}{u'}\binom{n}{u'}^{-5} \leqslant \binom{n}{u'}^{-3} \leqslant n^{-3}, \tag{101}$$

where we used our assumption $u' \leqslant n/2$. Finally, since there are at most n^2 ways to choose the numbers u, u', (101) implies the lemma.

7.4 Proof of Corollary 2.5

Since $\mu(U) \leqslant \mathrm{Vol}(U)^2/(\bar{d}n)$ for all $U \subset V$, Lemma 2.4 entails that w.h.p. for all $U \subset V$ of size $1 \leqslant \#U \leqslant \frac{n}{2}$ we have

$$e(U) \leqslant \frac{300\mathrm{Vol}(U)^2}{\bar{d}n} \vee e(U)\ln\left(\frac{e(U)\cdot\bar{d}n}{\mathrm{Vol}(U)^2}\right) \leqslant 300\#U\ln\left(\frac{n}{\#U}\right). \tag{102}$$

We shall prove that if (102) is true, then both properties stated in the corollary hold.

With respect to the first property, let us assume for contradiction that (102) is satisfied and that there is a set Q such that

$$\exp(2c'\bar{d}_{\min})\zeta\#Q \leqslant \mathrm{Vol}(Q) \leqslant \exp(-3c'\bar{d}_{\min})n, \tag{103}$$

$$e(Q) > \frac{1}{1000\zeta}\exp(-c'\bar{d}_{\min})\mathrm{Vol}(Q), \tag{104}$$

where $1 \leqslant \zeta \leqslant \bar{d}^{\frac{1}{2}}$. Observe that (103) implies that $\#Q \leqslant n/2$; for our assumption that $\min_{v\in V}\bar{d}(v) \geqslant \bar{d}_{\min} \geqslant d_0$ for a large enough d_0 entails that $\mathrm{Vol}(Q) \geqslant 2\#Q$. Moreover, (103) and (104) yield

$$\frac{300\mathrm{Vol}(Q)}{\bar{d}n} \leqslant 300\exp(-3c'\bar{d}_{\min})\bar{d}^{-1} \leqslant 0.001\zeta^{-1}\exp(-c'\bar{d}_{\min}) < \frac{e(Q)}{\mathrm{Vol}(Q)},$$

whence $e(Q) > 300\mathrm{Vol}(Q)^2/(\bar{d}n)$ (provided that $\bar{d}_{\min} \geqslant d_0$ for a sufficiently large d_0). Hence, if (102) holds, then

$$\begin{aligned} 300\#Q\ln\left(\frac{n}{\#Q}\right) &\geqslant e(Q)\ln\left(\frac{e(Q)\cdot\bar{d}n}{\mathrm{Vol}(Q)^2}\right) \\ &\geqslant \frac{\mathrm{Vol}(Q)}{1000\zeta\exp(c'\bar{d}_{\min})}\ln\left(\frac{\exp(-c'\bar{d}_{\min})\bar{d}n}{1000\zeta\mathrm{Vol}(Q)}\right). \end{aligned} \tag{105}$$

Furthermore, (103) yields

$$\begin{aligned} \frac{\exp(-c'\bar{d}_{\min})\bar{d}n}{1000\zeta\mathrm{Vol}(Q)} &\geqslant \left(\frac{\exp(-c'\bar{d}_{\min})\bar{d}n}{1000\zeta\mathrm{Vol}(Q)}\right)^{\frac{1}{2}}\left(\frac{\exp(-c'\bar{d}_{\min})\bar{d}n}{1000\zeta n\exp(-3c'\bar{d}_{\min})}\right)^{\frac{1}{2}} \\ &\geqslant \left(\frac{10^6\zeta\exp(c'\bar{d}_{\min})n}{\mathrm{Vol}(Q)}\right)^{\frac{1}{2}}. \end{aligned} \tag{106}$$

Let $t = \frac{\mathrm{Vol}(Q)}{10^6\zeta\exp(c'\bar{d}_{\min})n}$ and $t' = \frac{\#Q}{n}$. Combining (105) and (106), we conclude that

$$-t\ln t \leqslant -t'\ln t'. \tag{107}$$

Invoking (103) once more and recalling that $\bar{d}_{\min} \geqslant d_0$ for a large d_0, we obtain

$$t' = \frac{\#Q}{n} < \frac{\exp(c'\bar{d}_{\min})\#Q}{10^6 n} \leqslant t \leqslant \frac{1}{100}. \tag{108}$$

However, the function $x \mapsto -x\ln x$ is strictly increasing for $0 < x < 1/100$, so that (107) contradicts (108). Consequently, if (102) is satisfied, then (104) will be false.

In order to show that (102) implies the second part of the corollary, we assume for contradiction that (102) holds and that there is a set $Q \subset V$ such that

$$\mathrm{Vol}(Q) \leqslant \bar{d}^{\,\frac{1}{2}}\#Q^{5/8}n^{3/8} \text{ and } e(Q) > 3000\#Q. \tag{109}$$

Remember that we are assuming $\#Q \leqslant n/2$. Moreover, (102) and (109) entail that

$$\begin{aligned} 300\#Q\ln(n/\#Q) &\geqslant e(Q)\ln\left(\frac{e(Q)\bar{d}n}{\mathrm{Vol}(Q)^2}\right) \\ &\geqslant 3000\#Q\ln\left(\frac{\#Q\bar{d}n}{\mathrm{Vol}(Q)^2}\right) \\ &\geqslant 750\#Q\ln(n/\#Q), \end{aligned}$$

which is a contradiction. Thus, if (102) holds, then $e(Q) \leqslant 3000\#Q$.

7.5 Proof of Lemma 2.6

Let $1 \leqslant q \leqslant n/2$, and let $Q \subset V$, $\#Q = q$ be such that $\mathrm{Vol}(Q) > 1000\#Q^{5/8}n^{3/8}$. We are going to prove

$$\mathrm{P}\left[\sum_{v\in Q} d(v) < \frac{1}{4}\mathrm{Vol}(Q)\right] \leqslant \binom{n}{q}^{-2}. \tag{110}$$

Then the union bound implies that the property stated in Lemma 2.6 holds w.h.p.

To establish (110), we consider the random variable $e(Q,V)$, whose expectation satisfies $\mathrm{E}(e(Q,V)) \geqslant (\frac{1}{2}-o(1))\mathrm{Vol}(Q)$. As $e(Q,V)$ is a sum of mutually independent Bernoulli variables, the Chernoff bound (10) yields

$$\mathrm{P}\left[e(Q,V) \leqslant 0.51\mathrm{E}(e(Q,V))\right] \leqslant \exp\left[-\mathrm{E}(e(Q,V))/10\right] \leqslant \exp\left[-\mathrm{Vol}(Q)/24\right]. \tag{111}$$

Furthermore, if $\mathrm{Vol}(Q) > 1000\#Q^{5/8}n^{3/8}$, then

$$\frac{q\ln(n/q)}{\mathrm{Vol}(Q)/24} \leqslant \frac{3}{125}\cdot\frac{q^{3/8}}{n^{3/8}}\ln(n/q) = -\frac{8}{125}\cdot\left(\frac{q}{n}\right)^{3/8}\ln\left[\left(\frac{q}{n}\right)^{3/8}\right] \leqslant \frac{8}{125},$$

because the function $x \mapsto -x\ln x$ is $\leqslant 1$. Consequently, $\mathrm{Vol}(Q)/24 > 10q\ln(n/q)$, so that (111) gives

$$\mathrm{P}\left[e(Q,V) \leqslant 0.51\mathrm{E}(e(Q,V))\right] \leqslant \exp\left[-10q\ln(n/q)\right] \leqslant \binom{n}{q}^{-2}, \tag{112}$$

because $q \leqslant n/2$. Finally, since

$$\frac{1}{2}\sum_{v\in Q} d_G(v) \leqslant e(Q,V) \leqslant \sum_{v\in Q} d_G(v),$$

(112) entails (110).

7.6 Proof of Lemma 2.7

We shall prove that w.h.p. for *all* sets $X \subset V$ such that $\mathrm{Vol}(X) \leqslant n\exp(-\bar{d}_{\min}/C)$ the bounds

$$\begin{aligned} e(X) &\leqslant \exp(-\bar{d}_{\min}/(2C))n, & (113)\\ e(X, V-X) &\leqslant \exp(-\bar{d}_{\min}/(2C))n & (114)\end{aligned}$$

hold. Since $\sum_{v\in X} d_G(v) \leqslant 2e(X) + e(V-X)$, (113) and (114) imply the assertion.

To establish (113), we prove that if

$$\mathrm{Vol}(X) \leqslant n\exp(-\bar{d}_{\min}/C) \text{ but } e(X) > n\exp(-\bar{d}_{\min}/(2C)),$$

then the condition (12) is violated. As Lemma 2.4 shows that (12) is true w.h.p., this implies $e(X) \leqslant n\exp(-\bar{d}_{\min}/(2C))$ w.h.p. Thus, assume that $e(X) > n\exp(-\bar{d}_{\min}/(2C))$. Then

$$e(X)\ln\left(\frac{e(X)}{\mu(X)}\right) \geqslant n\exp\left(-\bar{d}_{\min}/(2C)\right)\ln\left(\frac{\bar{d}n^2\exp(-\bar{d}_{\min}/(2C))}{\mathrm{Vol}(X)^2}\right) \geqslant n\exp\left(-\bar{d}_{\min}/(2C)\right). \tag{115}$$

Moreover, as $\#X \leqslant \mathrm{Vol}(X)$ by (7), our assumption $\mathrm{Vol}(X) \leqslant n\exp(-\bar{d}_{\min}/C)$ entails

$$\#X\ln(n/\#X) \leqslant n\bar{d}_{\min}\exp(-\bar{d}_{\min}/C)/C \leqslant n\exp(-2\bar{d}_{\min}/(3C)), \tag{116}$$

prodived that $\bar{d}_{\min} \geqslant d_0$ for a large enough $d_0 > 0$. Combining (115) and (116), we conclude that

$$e(X)\ln\left(\frac{e(X)}{\mu(X)}\right) > \exp(\bar{d}_{\min}/(6C))\#X\ln(n/\#X) > 300\#X\ln(n/\#X),$$

and thus indeed (12) is violated.

In order to prove (114), we set $Z = e(X, V-X)$. Clearly, $\mathrm{E}(Z) \leqslant \mathrm{Vol}(X)$, and Z is a sum of mutually independent Bernoulli random variables. Therefore, letting $t = \exp\left(-\bar{d}_{\min}/(4C)\right)n > \mathrm{E}(Z)$ and applying the Chernoff bound (10), we obtain

$$\mathrm{P}\left[Z \geqslant 2t\right] \leqslant \mathrm{P}\left[Z \geqslant \mathrm{E}(Z)+t\right] \leqslant \exp\left(-\frac{t^2}{2(\mathrm{E}(Z)+t/3)}\right) \leqslant \exp\left(-t/6\right). \tag{117}$$

Thus,

$$\begin{aligned}
&\mathrm{P}\left[\exists X \subset V : \mathrm{Vol}(X) \leqslant \exp(-\bar{d}_{\min}/C)n \wedge e(X, V-X) \geqslant 2t\right] \\
&\leqslant \sum_{x \leqslant \exp(-\bar{d}_{\min}/C)n} \mathrm{P}\left[\exists X \subset V : \#X = x \wedge \mathrm{Vol}(X) \leqslant \exp(-\bar{d}_{\min}/C)n \wedge e(X, V-X) \geqslant 2t\right] \\
&\overset{(117)}{\leqslant} \sum_{x \leqslant \exp(-\bar{d}_{\min}/C)n} \binom{n}{x}\exp\left(-t/6\right) \leqslant \sum_{x \leqslant \exp(-\bar{d}_{\min}/C)n} \exp\left(2x\ln(n/x) - t/6\right) \\
&\leqslant n\exp\left(2n\exp(-\bar{d}_{\min}/C)\ln\left(\frac{n}{\exp(-\bar{d}_{\min}/C)n}\right) - t/6\right) \\
&\leqslant n\exp\left(2\bar{d}_{\min}\exp(-\bar{d}_{\min}/C)n/C - t/6\right).
\end{aligned} \tag{118}$$

Now, if $\bar{d}_{\min} \geqslant d_0$ for a large enough $d_0 > 0$, then the last term in (118) is $\leqslant n\exp(-t/12)$. Hence, if $t \geqslant \sqrt{n}$, then (118) yields

$$\mathrm{P}\left[\exists X \subset V : \mathrm{Vol}(X) \leqslant \exp(-\bar{d}_{\min}/C)n \wedge e(X, V-X) \geqslant 2t\right] \leqslant n\exp(-\sqrt{n}/12) = o(1). \tag{119}$$

If, on the other hand, $t = \exp\left(-\bar{d}_{\min}/(4C)\right)n < \sqrt{n}$, then $\bar{d}_{\min}/C \geqslant 2\ln n$, so that

$$\#X \leqslant \mathrm{Vol}(X) \leqslant \exp(-\bar{d}_{\min}/C)n < n^{-1} < 1,$$

whence $X = \emptyset$; thus, in the case $t \geqslant \sqrt{n}$ we simply know $e(X, V-X) = 0$. Therefore, (119) implies that (114) holds for all X w.h.p.

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