Euler characteristic of the truncated order complex of generalized noncrossing partitions

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Dedicated to Anders Björner on the occasion of his 60th birthday

Submitted: May 29, 2009; Accepted: Nov 23, 2009; Published: Nov 30, 2009
2000 Mathematics Subject Classification: Primary 05E15; Secondary 05A10 05A15 05A18 06A07 20F55

Abstract

The purpose of this paper is to complete the study, begun in the first author's PhD thesis, of the topology of the poset of generalized noncrossing partitions associated to real reflection groups. In particular, we calculate the Euler characteristic of this poset with the maximal and minimal elements deleted. As we show, the result on the Euler characteristic extends to generalized noncrossing partitions associated to well-generated complex reflection groups.

1 Introduction

We say that a partition of the set $[n] := \{1, 2, \ldots, n\}$ is noncrossing if, whenever we have $\{a, c\}$ in block A and $\{b, d\}$ in block B of the partition with $a < b < c < d$, it follows that

*Research partially supported by NSF grant DMS-0603567.

†Research partially supported by the Austrian Science Foundation FWF, grants Z130-N13 and S9607-N13, the latter in the framework of the National Research Network "Analytic Combinatorics and Probabilistic Number Theory."

Key words and phrases. root systems, reflection groups, Coxeter groups, generalized non-crossing partitions, chain enumeration, Euler characteristics, Chu–Vandermonde summation.

$A = B$. For an introduction to the rich history of this subject, see [1, Chapter 4.1]. We say that a noncrossing partition of $[mn]$ is m-divisible if each of its blocks has cardinality divisible by m. The collection of m-divisible noncrossing partitions of $[mn]$ — which we will denote by $NC^{(m)}(n)$ — forms a join-semilattice under the refinement partial order. This structure was first studied by Edelman in his PhD thesis; see [7].

Twenty-six years later, in his own PhD thesis [1], the first author defined a generalization of Edelman's poset to all finite real reflection groups. (We refer the reader to [8] for all terminology related to real reflection groups.) Let W be a finite group generated by reflections in Euclidean space, and let $T \subseteq W$ denote the set of all reflections in the group. Let $\ell_T : W \to \mathbb{Z}$ denote the word length in terms of the generators T. Now fix a Coxeter element $c \in W$ and a positive integer m. We define the set of m-divisible noncrossing partitions as follows:

$$NC^{(m)}(W) = \Big\{ (w_0; w_1, \ldots, w_m) \in W^{m+1} : w_0 w_1 \cdots w_m = c \quad \text{and} \quad \sum_{i=0}^{m} \ell_T(w_i) = \ell_T(c) \Big\}. \tag{1.1}$$

That is, $NC^{(m)}(W)$ consists of the minimal factorizations of c into $m+1$ group elements. We define a partial order on $NC^{(m)}(W)$ by setting

$$(w_0; w_1, \ldots, w_m) \leqslant (u_0; u_1, \ldots, u_m) \quad \text{if and only if} \quad \ell_T(u_i) + \ell_T(u_i^{-1} w_i) = \ell_T(w_i) \quad \text{for } 1 \leqslant i \leqslant m. \tag{1.2}$$

In other words, we set $(w_0; w_1, \ldots, w_m) \leqslant (u_0; u_1, \ldots, u_m)$ if for each $1 \leqslant i \leqslant m$ the element u_i lies on a geodesic from the identity to w_i in the Cayley graph (W, T). We place no a priori restriction on the elements w_0, u_0, however it follows from the other conditions that $\ell_T(w_0) + \ell_T(w_0^{-1} u_0) = \ell_T(u_0)$. We note that the poset is graded with rank function

$$\mathrm{rk}(w_0; w_1, \ldots, w_m) = \ell_T(w_0), \tag{1.3}$$

hence the element $(c; \varepsilon, \ldots, \varepsilon) \in W^{m+1}$ — where $\varepsilon \in W$ is the identity — is the unique maximum element. When $m = 1$ there is a unique minimum element $(\varepsilon; c)$ but for $m > 1$ there are many minimal elements. It turns out that the isomorphism class of the poset $NC^{(m)}(W)$ is independent of the choice of Coxeter element c. Furthermore, when W is the symmetric group $\mathfrak{S}_n$ we recover Edelman's poset $NC^{(m)}(n)$.

In this note we are concerned with the topology of the order complex $\Delta(NC^{(m)}(W))$, which is the abstract simplicial complex whose d-dimensional faces are the chains $\pi_0 < \pi_1 < \cdots < \pi_d$ in the poset $NC^{(m)}(W)$. In particular, we would like to compute the homotopy type of this and some related complexes. The answers involve the following quantity, called the positive Fuß–Catalan number:

$$\mathrm{Cat}_+^{(m)}(W) := \prod_{i=1}^{n} \frac{mh + d_i - 2}{d_i}. \tag{1.4}$$

Here n is the rank of the group W (the number of simple reflections generating W), h is the Coxeter number (the order of a Coxeter element), and the integers $d_1, d_2, \ldots, d_n$ are the degrees of W (the degrees of the fundamental W-invariant polynomials). The prototypical theorem we wish to generalize is the following result Athanasiadis, Brady and Watt.

Theorem 1 ([2]). *The order complex of $NC^{(1)}(W)$ with its unique maximum and minimum elements deleted has reduced Euler characteristic $(-1)^n \operatorname{Cat}_+^{(1)}(W)$, and it is homotopy equivalent to a wedge of $\operatorname{Cat}_+^{(1)}(W)$ many $(n-2)$-dimensional spheres.*

The first author was able to prove the following theorem for general m.

Theorem 2 ([1], Theorem 3.7.7). *The order complex of $NC^{(m)}(W)$ with its unique maximum element deleted has reduced Euler characteristic $(-1)^{n-1} \operatorname{Cat}_+^{(m-1)}(W)$, and it is homotopy equivalent to a wedge of $\operatorname{Cat}_+^{(m-1)}(W)$ many $(n-1)$-dimensional spheres.*

However, if we set $m = 1$ in Theorem 2, we find that the reduced Euler characteristic of $NC^{(1)}(W)$ with its maximum element deleted is $(-1)^{n-1} \operatorname{Cat}_+^{(0)}(W) = 0$, which is not surprising because $NC^{(1)}(W)$ has a unique minimum element, which is a cone point for the order complex, and hence this complex is contractible. Thus, Theorem 2 is not a generalization of Theorem 1. To truly generalize Theorem 1, we must delete the maximum element and *all minimal elements* of $NC^{(m)}(W)$. The first author made a conjecture in this case [1, Conjecture 3.7.9], and our main result settles this conjecture.

Theorem 3. *Let W be a finite real reflection group of rank n and let m be a positive integer. The order complex of the poset $NC^{(m)}(W)$ with maximal and minimal elements deleted has reduced Euler characteristic*

$$(-1)^n \Big(\operatorname{Cat}_+^{(m)}(W) - \operatorname{Cat}_+^{(m-1)}(W) \Big), \tag{1.5}$$

and it is homotopy equivalent to a wedge of this many $(n-2)$-dimensional spheres.

A different, independent proof of this theorem was found simultaneously by Tomie in [13]. While our proof proceeds by explicitly enumerating the faces of the order complex involved in the above theorem, Tomie's proof is based on the EL-shellability of this order complex, a result due to Thomas and the first author [1, Cor. 3.7.3], which makes it possible to compute the Euler characteristic by enumerating certain chains in this order complex.

In Section 2 we will collect some auxiliary results and in Section 3 we will prove the main theorem.

In [3, 4], Bessis and Corran have shown that the notion of noncrossing partitions extends rather straightforwardly to well-generated complex reflection groups. It is not done explicitly in [1], but from [3, 4] it is obvious that the definition of generalized noncrossing partitions in [1] can be extended without any effort to well-generated complex reflection groups, the same being true for many (most?) of the results from [1] (cf. [1, Disclaimer 1.3.1]). In Section 4, we show that the assertion in Theorem 3 on the Euler characteristic of the truncated order complex of generalized noncrossing partitions continues to hold for well-generated complex reflection groups. We suspect that this is also true for the topology part of Theorem 3, but what is missing here is the extension to well-generated complex reflection groups of the result of Hugh Thomas and the first author [1, Cor. 3.7.3] that the poset of generalized noncrossing partitions associated to real reflection groups is shellable. This extension has so far not even been done for [2], the special case of the poset of noncrossing partitions.

2 Auxiliary results

In this section we record some results that are needed in the proof of the main theorem. The first result is Theorem 3.5.3 from [1].

Theorem 4. *The cardinality of $NC^{(m)}(W)$ is given by the* Fuß–Catalan number for reflection groups

$$\mathrm{Cat}^{(m)}(W) := \prod_{i=1}^{n} \frac{mh+d_i}{d_i}, \tag{2.1}$$

where, as before, n is the rank, h is the Coxeter number and the d_i are the degrees of W. Equivalently, given a Coxeter element c, the number of minimal decompositions

$$w_0 w_1 \cdots w_m = c \quad \textit{with} \quad \ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c)$$

is given by $\mathrm{Cat}^{(m)}(W)$.

Since the numbers $h - d_i + 2$ are a permutation of the degrees [8, Lemma 3.16], we have an alternate formula for the positive Fuß–Catalan number:

$$\mathrm{Cat}_{+}^{(m)}(W) = \prod_{i=1}^{n} \frac{mh+d_i-2}{d_i} = (-1)^n \,\mathrm{Cat}^{(-m-1)}(W).$$

Our next result is Theorem 3.6.9(1) from [1].

Theorem 5. *The total number of (multi-)chains*

$$\pi_1 \leqslant \pi_2 \leqslant \ldots \leqslant \pi_l$$

in $NC^{(m)}(W)$ is equal to $\mathrm{Cat}^{(ml)}(W)$.

And, moreover, we have the following.

Lemma 6. *The number of (multi-)chains $\pi_1 \leqslant \pi_2 \leqslant \ldots \leqslant \pi_l$ in $NC^{(m)}(W)$ with* $\mathrm{rk}(\pi_1) = 0$ *is equal to* $\mathrm{Cat}^{(ml-1)}(W)$.

Proof. If $\pi_1 = (w_0^{(1)}; w_1^{(1)}, \dots, w_m^{(1)})$ then the condition $\mathrm{rk}(\pi_1) = 0$ is equivalent to $w_0^{(1)} = \varepsilon$. We note that Theorem 3.6.7 of [1], together with the fundamental map between multichains and minimal factorizations [1, Definition 3.2.3], establishes a bijection between multichains $\pi_1 \leqslant \cdots \leqslant \pi_l$ in $NC^{(m)}(W)$ and elements $(u_0; u_1, \dots, u_{ml})$ of $NC^{(ml)}(W)$ for which $u_0 = w_0^{(1)}$. Since $\ell_T(\varepsilon) = 0$, we wish to count factorizations $u_1 u_2 \cdots u_{ml} = c$ in which $\ell_T(u_1) + \cdots + \ell_T(u_{ml}) = \ell_T(c)$. By the second part of Theorem 4, this number is equal to $\mathrm{Cat}^{(ml-1)}(W)$, as desired. $\square$

A stronger version of rank-selected chain enumeration will be important in the proof of our main theorem in Section 3. Given a finite reflection group W of rank n, let $R_W(s_1, s_2, \dots, s_l)$ denote the number of (multi-)chains

$$\pi_1 \leqslant \pi_2 \leqslant \ldots \leqslant \pi_{l-1}$$

in $NC^{(m)}(W)$, such that $\mathrm{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \dots, l-1$, and $s_1 + s_2 + \cdots + s_l = n$. The following lemma says that zeroes in the argument of $R_W(\,\cdot\,)$ can be suppressed except for a zero in the first argument.

Lemma 7. *Let W be a finite real reflection group of rank n and let $s_1, s_2, \dots, s_l$ be non-negative integers with $s_1 + s_2 + \cdots + s_l = n$. Then*

$$R_W(s_1, \dots, s_i, 0, s_{i+1}, \dots, s_l) = R_W(s_1, \dots, s_i, s_{i+1}, \dots, s_l) \tag{2.2}$$

for $i = 1, 2, \dots, l$. If $i = l$, equation (2.2) must be interpreted as

$$R_W(s_1, \dots, s_l, 0) = R_W(s_1, \dots, s_l).$$

Proof. This is obvious as long as $i < l$. If $i = l$, then, by definition, $R_W(s_1, \dots, s_l, 0)$ counts all multi-chains $\pi_1 \leqslant \pi_2 \leqslant \ldots \leqslant \pi_l$ with $\mathrm{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \dots, l$. In particular, $\mathrm{rk}(\pi_l) = s_1 + s_2 + \cdots + s_l = n$, so that π_l must be the unique maximal element $(c; \varepsilon, \dots, \varepsilon)$ of $NC^{(m)}(W)$. Thus we are counting multi-chains $\pi_1 \leqslant \pi_2 \leqslant \ldots \leqslant \pi_{l-1}$ with $\mathrm{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \dots, l-1$, and, again by definition, this number is given by $R_W(s_1, \dots, s_l)$. $\square$

Finally we quote the version of inclusion-exclusion given in [12, Sec. 2.1, Eq. (4)] that will be relevant to us.

Proposition 8. *Let A be a finite set and $w : A \to \mathbb{C}$ a weight function on A. Furthermore, let S be a set of properties an element of A may or may not have. Given a subset Y of S, we define the functions $f_=(Y)$ and $f_\geqslant(Y)$ by*

$$f_=(Y) := \sum_a{}' w(a),$$

where $\sum'$ is taken over all $a \in A$ which have exactly *the properties Y, and by*

$$f_\geqslant(Y) = \sum_{X \supseteq Y} f_=(X).$$

Then

$$f_{=}(\emptyset) = \sum_{Y \subseteq S} (-1)^{|Y|} f_{\geqslant}(Y). \tag{2.3}$$

3 Proof of Main Theorem

Let $\mathbf{c}$ denote the unique maximum element $(c; \varepsilon, \dots, \varepsilon)$ of $NC^{(m)}(W)$ and let mins denote its set of minimal elements, the cardinality of which is $\mathrm{Cat}^{(m-1)}(W)$. The truncated poset

$$NC^{(m)}(W) \backslash \big(\{\mathbf{c}\} \cup \mathsf{mins}\big)$$

is a rank-selected subposet of $NC^{(m)}(W)$, the latter being shellable due to [1, Cor. 3.7.3]. If we combine this observation with the fact (see [5, Theorem 4.1]) that rank-selected subposets of shellable posets are also shellable, we conclude that $NC^{(m)}(W) \backslash \big(\{\mathbf{c}\} \cup \mathsf{mins}\big)$ is shellable. Since it is known that a pure d-dimensional shellable simplicial complex Δ is homotopy equivalent to a wedge of $\tilde{\chi}(\Delta)$ d-dimensional spheres (this follows from Fact 9.19 in [6] and the fact that shellability implies the property of being homotopy-Cohen-Macaulay [6, Sections 11.2, 11.5]), it remains only to compute the reduced Euler characteristic $\tilde{\chi}(\,\cdot\,)$ of (the order complex of) $NC^{(m)}(W) \backslash \big(\{c\} \cup \mathsf{mins}\big)$.

For a finite real reflection group W of rank n, let us again write $R_W(s_1, s_2, \dots, s_l)$ for the number of (multi-)chains

$$\pi_1 \leqslant \pi_2 \leqslant \dots \leqslant \pi_{l-1}$$

in $NC^{(m)}(W)$ with $\mathrm{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \dots, l-1$, and $s_1 + s_2 + \cdots + s_l = n$. By definition, the reduced Euler characteristic is

$$-1 + \sum_{l=2}^{n} (-1)^l \sum_{\substack{s_1 + \cdots + s_l = n \\ s_1, \dots, s_l > 0}} R_W(s_1, s_2, \dots, s_l). \tag{3.1}$$

The sum over $s_1, s_2, \dots, s_l$ in (3.1) could be easily calculated from Theorem 5, if there were not the restriction $s_1, s_2, \dots, s_l > 0$. In order to overcome this difficulty, we appeal to the principle of inclusion-exclusion. More precisely, for a fixed l, in Proposition 8 choose $A = \{(s_1, s_2, \dots, s_l) : s_1 + s_2 + \cdots + s_l = n\}$, $w\big((s_1, s_2, \dots, s_l)\big) = R_W(s_1, s_2, \dots, s_l)$, and $S = \{S_i : i = 1, 2, \dots, l\}$, where S_i is the property of an element $(s_1, s_2, \dots, s_l) \in A$ to satisfy $s_i = 0$. Then (2.3) becomes

$$\sum_{\substack{s_1 + \cdots + s_l = n \\ s_1, \dots, s_l > 0}} R_W(s_1, s_2, \dots, s_l) = \sum_{I \subseteq \{1, \dots, l\}} (-1)^{|I|} \sum_{\substack{s_1 + \cdots + s_l = n \\ s_1, \dots, s_l \geqslant 0 \\ s_i = 0 \text{ for } i \in I}} R_W(s_1, s_2, \dots, s_l).$$

In view of Lemma 7, the right-hand side may be simplified, so that we obtain the equation

$$\begin{aligned}\sum_{\substack{s_1+\dots+s_l=n\\ s_1,\dots,s_l>0}} R_W(s_1,s_2,\dots,s_l) &= \sum_{\substack{I\subseteq\{1,\dots,l\}\\ 1\in I}} (-1)^{|I|} \sum_{\substack{s_2+\dots+s_{l-|I|+1}=n\\ s_2,\dots,s_{l-|I|+1}\geqslant 0}} R_W(0,s_2,\dots,s_{l-|I|+1})\\
&\qquad + \sum_{\substack{I\subseteq\{1,\dots,l\}\\ 1\notin I}} (-1)^{|I|} \sum_{\substack{s_1+\dots+s_{l-|I|}=n\\ s_1,\dots,s_{l-|I|}\geqslant 0}} R_W(s_1,s_2,\dots,s_{l-|I|})\\
&= \sum_{j=1}^{l}(-1)^j\binom{l-1}{j-1} \sum_{\substack{s_2+\dots+s_{l-j+1}=n\\ s_2,\dots,s_{l-j+1}\geqslant 0}} R_W(0,s_2,\dots,s_{l-j+1})\\
&\qquad + \sum_{j=0}^{l}(-1)^j\binom{l-1}{j} \sum_{\substack{s_1+\dots+s_{l-j}=n\\ s_1,\dots,s_{l-j}\geqslant 0}} R_W(s_1,s_2,\dots,s_{l-j}).\end{aligned}$$

By Lemma 6, the sum over $s_2,\dots,s_{l-j+1}$ on the right-hand side is equal to $\mathrm{Cat}^{((l-j)m-1)}(W)$, while by Theorem 5 the sum over $s_1,\dots,s_{l-j}$ is equal to $\mathrm{Cat}^{((l-j-1)m)}(W)$. If we substitute all this in (3.1), we arrive at the expression

$$-1+\sum_{l=2}^{n}(-1)^l\left(\sum_{j=1}^{l}(-1)^j\binom{l-1}{j-1}\mathrm{Cat}^{((l-j)m-1)}(W)\right.\\
\left.+\sum_{j=0}^{l-1}(-1)^j\binom{l-1}{j}\mathrm{Cat}^{((l-j-1)m)}(W)\right) \tag{3.2}$$

for the reduced Euler characteristic that we want to compute. We now perform the replacement $l=j+k$ in both sums. Thereby we obtain the expression

$$-1-\mathrm{Cat}^{(-1)}(W)-\mathrm{Cat}^{(-m)}(W)+\mathrm{Cat}^{(0)}(W)\\
+\sum_{k=0}^{n}(-1)^k\left(\sum_{j=1}^{n-k}\binom{j+k-1}{j-1}\mathrm{Cat}^{(km-1)}(W)+\sum_{j=0}^{n-k}\binom{j+k-1}{j}\mathrm{Cat}^{((k-1)m)}(W)\right), \tag{3.3}$$

the various terms in the first line being correction terms that cancel terms in the sums in the second line violating the condition $l=j+k\geqslant 2$, which is present in (3.2). Since we shall make use of it below, the reader should observe that, by the definition (2.1) of Fuß–Catalan numbers, both $\mathrm{Cat}^{(km-1)}(W)$ and $\mathrm{Cat}^{((k-1)m)}(W)$ are polynomials in k of degree n with leading coefficient $(mh)^n$.

Again by (2.1), we have $\mathrm{Cat}^{(-1)}(W)=0$ and $\mathrm{Cat}^{(0)}(W)=1$. Therefore, if we evaluate the sums over j in (3.3) (this is a special instance of the Chu–Vandermonde summation), then we obtain the expression

$$\begin{aligned}
-\operatorname{Cat}^{(-m)}(W) &+ \sum_{k=0}^{n}(-1)^k\left(\binom{n}{k+1}\operatorname{Cat}^{(km-1)}(W)+\binom{n}{k}\operatorname{Cat}^{((k-1)m)}(W)\right)\\
&= -\operatorname{Cat}^{(-m)}(W)+\operatorname{Cat}^{(-m-1)}(W)\\
&\qquad -\sum_{k=0}^{n}(-1)^k\binom{n}{k}\operatorname{Cat}^{((k-1)m-1)}(W)+\sum_{k=0}^{n}(-1)^k\binom{n}{k}\operatorname{Cat}^{((k-1)m)}(W)\\
&= -\operatorname{Cat}^{(-m)}(W)+\operatorname{Cat}^{(-m-1)}(W)-(-1)^n n!(mh)^n+(-1)^n n!(mh)^n\\
&= -\operatorname{Cat}^{(-m)}(W)+\operatorname{Cat}^{(-m-1)}(W)\\
&= -(-1)^n\operatorname{Cat}_+^{(m-1)}(W)+(-1)^n\operatorname{Cat}_+^{(m)}(W),
\end{aligned}$$

where, to go from the second to the third line, we used the well-known fact from finite difference calculus (cf. [12, Sec. 1.4, Eq. (26) and Prop. 1.4.2]), that, for any polynomial $p(k)$ in k of degree n and leading coefficient p_n, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}p(k)=(-1)^n n!p_n.$$

4 The case of well-generated complex reflection groups

We conclude the paper by pointing out that our result in Theorem 3 on the Euler characteristic of the truncated poset of generalized noncrossing partitions extends naturally to well-generated complex reflection groups. We refer the reader to [10, 11] for all terminology related to complex reflection groups.

Let W be a finite group generated by (complex) reflections in $\mathbb{C}^n$, and let $T \subseteq W$ denote the set of all reflections in the group. (Here, a reflection is a non-trivial element of $GL(\mathbb{C}^n)$ which fixes a hyperplane pointwise and which has finite order.) As in Section 1, let $\ell_T : W \to \mathbb{Z}$ denote the word length in terms of the generators T. Now fix a regular element $c \in W$ in the sense of Springer [11] and a positive integer m. (If W is a real reflection group, that is, if all generators in T have order 2, then the notion of "regular element" reduces to that of a "Coxeter element.") As in the case of Coxeter elements, it can be shown that any two regular elements are conjugate to each other. A further assumption that we need is that W is well-generated, that is, that it is generated by n reflections given that n is minimal such that W can be realized as reflection group on $\mathbb{C}^n$. A complex reflection group has two sets of distinguished integers $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ and $d_1^* \geqslant d_2^* \geqslant \cdots \geqslant d_n^*$, called its degrees and codegrees, respectively. If V is the geometric representation of W, the degrees arise from the W-invariant polynomials on V, and the codegrees arise from the W-invariant polynomials in the dual representation V^*. Orlik and Solomon [9] observed, using case-by-case checking, that W is well-generated if and only if its degrees and codegrees satisfy

$$d_i + d_i^* = d_n$$

for all $1 \leqslant i \leqslant n$. Together with the classification of Shephard and Todd [10], this constitutes a classification of well-generated complex reflection groups.

Given these extended definitions of ℓ_T and c, we define the set of m-divisible noncrossing partitions by (1.1), and its partial order by (1.2), as before. In the extension of Theorem 3 to well-generated complex reflection groups, we need the Fuß–Catalan number for W, which is again defined by (2.1), where the d_i's are the degrees of (homogeneous polynomial generators of the invariants of) W, and where h is the largest of the degrees.

Theorem 9. *Let W be a finite well-generated (complex) reflection group of rank n and let m be a positive integer. The order complex of the poset $NC^{(m)}(W)$ with maximal and minimal elements deleted has reduced Euler characteristic*

$$\operatorname{Cat}^{(-m-1)}(W) - \operatorname{Cat}^{(-m)}(W). \tag{4.1}$$

In order to prove this theorem, we may use the proof of Theorem 3 given in Sections 2 and 3 essentially verbatim. The only difference is that all notions (such as the reflections T or the order ℓ_T, for example), have to be interpreted in the extended sense explained above, and that "Coxeter element" has to be replaced by "regular element" everywhere. In particular, the extension of Theorem 4 to well-generated complex reflection groups is Proposition 13.1 in [3], and the proofs of Theorems 3.6.7 and Theorems 3.6.9(1) in [1] (which we used in order to establish Lemma 6 respectively Theorem 5) carry over essentially verbatim to the case of well-generated complex reflection groups.

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