Spectral saturation: inverting the spectral Turán theorem

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Abstract

Let $\mu(G)$ be the largest eigenvalue of a graph G and $T_r(n)$ be the r-partite Turán graph of order n.

We prove that if G is a graph of order n with $\mu(G) > \mu(T_r(n))$, then G contains various large supergraphs of the complete graph of order $r+1$, e.g., the complete r-partite graph with all parts of size $\log n$ with an edge added to the first part.

We also give corresponding stability results.

Keywords: *complete r-partite graph; stability, spectral Turán's theorem; largest eigenvalue of a graph.*

1 Introduction

This note is part of an ongoing project aiming to build extremal graph theory on spectral basis, see, e.g., [3], [13, 18].

Let $\mu(G)$ be the largest adjacency eigenvalue of a graph G and $T_r(n)$ be the r-partite Turán graph of order n. The spectral Turán theorem [15] implies that if G is a graph of order n with $\mu(G) > \mu(T_r(n))$, then G contains a K_{r+1}, the complete graph of order $r+1$.

On the other hand, it is known (e.g., [2], [4], [9], [12]) that if $e(G) > e(T_r(n))$, then G contains large supergraphs of K_{r+1}. It turns out that essentially the same results also follow from the inequality $\mu(G) > \mu(T_r(n))$.

Recall first a family of graphs, studied initially by Erdős [7] and recently in [2]: an *r-joint* of size t is the union of t distinct r-cliques sharing an edge. Write $js_r(G)$ for the maximum size of an r-joint in a graph G. Erdős [7], Theorem 3′, showed that *if G is a graph of sufficiently large order n satisfying $e(G) > e(T_r(n))$, then $js_{r+1}(G) > n^{r-1}/(10(r+1))^{6(r+1)}$.*

Here is a explicit spectral analogue of this result.

Theorem 1 *Let $r \geq 2$, $n > r^{15}$, and G be a graph of order n. If $\mu(G) > \mu(T_r(n))$, then $js_{r+1}(G) > n^{r-1}/r^{2r+4}$.*

Erdős [4] introduced yet another graph related to Turán's theorem: let $K_r^+(s_1, \ldots, s_r)$ be the complete r-partite graph with parts of sizes $s_1 \geq 2, s_2, \ldots, s_r$, with an edge added to the first part. The extremal results about this graph given in [4] and [9] were recently extended in [12] to:

Let $r \geq 2$, $2/\ln n \leq c \leq r^{-(r+7)(r+1)}$, and G be a graph of order n. If G has $t_r(n)+1$ edges, then G contains a $K_r^+\left(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lceil n^{1-\sqrt{c}}\rceil\right)$.

Here we give a similar spectral extremal result.

Theorem 2 *Let $r \geq 2$, $2/\ln n \leq c \leq r^{-(2r+9)(r+1)}$, and G be a graph of order n. If $\mu(G) > \mu(T_r(n))$, then G contains a $K_r^+\left(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lceil n^{1-\sqrt{c}}\rceil\right)$.*

As an easy consequence of Theorem 2 we obtain

Theorem 3 *Let $r \geq 2$, $c = r^{-(2r+9)(r+1)}$, $n \geq e^{2/c}$, and G be a graph of order n. If $\mu(G) > \mu(T_r(n))$, then G contains a $K_r^+(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor)$.*

Theorems 1, 2, and 3 have corresponding stability results.

Theorem 4 *Let $r \geq 2$, $0 < b < 2^{-10}r^{-6}$, $n \geq r^{20}$, and G be a graph of order n. If $\mu(G) > (1 - 1/r - b)n$, then G satsisfies one of the conditions:*

(a) $js_{r+1}(G) > n^{r-1}/r^{2r+5}$;

(b) G contains an induced r-partite subgraph G_0 of order at least $\left(1 - 4b^{1/3}\right)n$ with minimum degree $\delta(G_0) > \left(1 - 1/r - 7b^{1/3}\right)n$.

Theorem 5 *Let $r \geq 2$, $2/\ln n \leq c \leq r^{-(2r+9)(r+1)}/2$, $0 < b < 2^{-10}r^{-6}$, and G be a graph of order n. If $\mu(G) > (1 - 1/r - b)n$, then G satsisfies one of the conditions:*

(a) G contains a $K_r^+\left(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lceil n^{1-2\sqrt{c}}\rceil\right)$;

(b) G contains an induced r-partite subgraph G_0 of order at least $\left(1 - 4b^{1/3}\right)n$ with minimum degree $\delta(G_0) > \left(1 - 1/r - 7b^{1/3}\right)n$.

Theorem 6 *Let $r \geq 2$, $c = r^{-(2r+9)(r+1)}/2$, $0 < b < 2^{-10}r^{-6}$, $n \geq e^{2/c}$, and G be a graph of order n. If $\mu(G) > (1 - 1/r - b)n$, then one of the following conditions holds:*

(a) G contains a $K_r^+(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor)$;

(b) G contains an induced r-partite subgraph G_0 of order at least $\left(1 - 4b^{1/3}\right)n$ with minimum degree $\delta(G_0) > \left(1 - 1/r - 7b^{1/3}\right)n$.

Remarks

- Obviously Theorems 1, 2, and 3 are tight since $T_r(n)$ contains no $(r+1)$-cliques.

- Theorems 2, 3, 5, and 6 are essentially best possible since for every $\varepsilon > 0$, choosing randomly a graph G of order n with $e(G) = \lceil (1-\varepsilon) n^2/2 \rceil$ edges we see that $\mu(G) > (1-\varepsilon)n$, but G contains no $K_2(c \ln n, c \ln n)$ for some $c > 0$, independent of n.

- In Theorem 1, it is not known what is the best possible value of $js_{r+1}(G)$, given G is a graph of order n and $\mu(G) > \mu(T_r(n))$.

- Theorem 1 implies in turn spectral versions of other known results, like Theorem 3.8 in [8]:

  Every graph G of order n with $\mu(G) > \mu(T_r(n))$ contains cn distinct $(r+1)$-cliques sharing an r-clique, where $c > 0$ is independent of n.

- It is not difficult to show that if G is a graph of order n, then the inequality $e(G) > e(T_r(n))$ implies the inequality $\mu(G) > \mu(T_r(n))$. Therefore, Theorems 1-6 imply the corresponding nonspectral extremal results of [12] with narrower ranges of the parameters.

- The relations between c and n in Theorems 2 and 5 need some explanation. First, for fixed c, they show how large must be n so that the vertex classes of the required $K_r^+(s, \ldots s, t)$ are nonempty. But also c may depend on n, e.g., letting $c = 1/\ln \ln n$, the conclusion is meaningful for sufficiently large n.

- Note that, in Theorems 2 and 5, if the conclusion holds for some c, it holds also for $0 < c' < c$, provided n is sufficiently large, i.e., as n grows, we can find a larger and more lopsided $K_r^+(s, \ldots s, t)$;

- The stability conditions *(b)* in Theorems 4, 5, and 6 are stronger than the conditions in the stability theorems of [6], [19] and [11]. Indeed, in all these theorems, condition *(b)* implies that G_0 is an induced, almost balanced, and almost complete r-partite graph containing almost all the vertices of G;

- The exponents $1 - \sqrt{c}$ and $1 - 2\sqrt{c}$ in Theorems 2 and 5 are far from the best ones, but are simple.

The next section contains notation and results needed to prove the theorems. The proofs are presented in Section 3.

2 Preliminary results

Our notation follows [1]. Given a graph G, we write:

- $V(G)$ for the vertex set of G and $|G|$ for $|V(G)|$;
- $E(G)$ for the edge set of G and $e(G)$ for $|E(G)|$;
- $d(u)$ for the degree of a vertex u;
- $\delta(G)$ for the minimum degree of G;
- $k_r(G)$ for the number of r-cliques of G;
- $K_r(s_1, \ldots, s_r)$ for the complete r-partite graph with parts of sizes $s_1, \ldots, s_r$.

The following facts play crucial roles in our proofs.

Fact 7 ([15], Theorem 1) *Every graph G of order n with $\mu(G) > \mu(T_r(n))$ contains a K_{r+1}.* □

Fact 8 ([16], Theorem 5) *Let $0 < \alpha \leq 1/4$, $0 < \beta \leq 1/2$, $1/2 - \alpha/4 \leq \gamma < 1$, $K \geq 0$, $n \geq (42K + 4)/\alpha^2\beta$, and G be a graph of order n. If*

$$\mu(G) > \gamma n - K/n \quad \text{and} \quad \delta(G) \leq (\gamma - \alpha) n,$$

then G contains an induced subgraph H satisfying $|H| \geq (1 - \beta) n$ and one of the conditions:
(a) $\mu(H) > \gamma(1 + \beta\alpha/2)|H|$;
(b) $\mu(H) > \gamma|H|$ and $\delta(H) > (\gamma - \alpha)|H|$. □

Fact 9 ([2], Lemma 6) *Let $r \geq 2$ and G be graph a of order n. If G contains a K_{r+1} and $\delta(G) > (1 - 1/r - 1/r^4) n$, then $js_{r+1}(G) > n^{r-1}/r^{r+3}$.* □

Fact 10 ([3], Theorem 2) *If $r \geq 2$ and G is a graph of order n, then*

$$k_r(G) \geq \left(\frac{\mu(G)}{n} - 1 + \frac{1}{r}\right) \frac{r(r-1)}{r+1} \left(\frac{n}{r}\right)^{r+1}.$$

□

Fact 11 ([3], Theorem 4) *Let $r \geq 2$, $0 \leq b \leq 2^{-10}r^{-6}$, and G be a graph of order n. If G contains no K_{r+1} and $\mu(G) \geq (1 - 1/r - b) n$, then G contains an induced r-partite graph G_0 satisfying $|G_0| \geq \left(1 - 3b^{1/3}\right) n$ and $\delta(G_0) > \left(1 - 1/r - 6b^{1/3}\right) n$.* □

Fact 12 ([12], Theorem 6) *Let $r \geq 2$, $2/\ln n \leq c \leq r^{-(r+8)r}$, and G be a graph of order n. If G contains a K_{r+1} and $\delta(G) > (1 - 1/r - 1/r^4) n$, then G contains a $K_r^+\left(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \left\lceil n^{1-cr^3} \right\rceil\right)$.* □

Fact 13 ([10], Theorem 1) *Let $r \geq 2$, $c^r \ln n \geq 1$, and G be a graph of order n. If $k_r(G) \geq cn^r$, then G contains a $K_r(s, \ldots s, t)$ with $s = \lfloor c^r \ln n \rfloor$ and $t > n^{1-c^{r-1}}$.* □

Fact 14 *The number of edges of $T_r(n)$ satisfies $2e(T_r(n)) \geq (1 - 1/r) n^2 - r/4$.* □

3 Proofs

Below we prove Theorems 1, 2, 4, and 5. We omit the proofs of Theorems 3 and 6 since they are easy consequences of Theorems 2 and 5.

All proofs have similar simple structure and follow from the facts listed above.

Proof of Theorem 1

Let G be a graph of order n with $\mu(G) > \mu(T_r(n))$; thus, by Fact 7, G contains a K_{r+1}. If

$$\delta(G) > \left(1 - r^{-1} - r^{-4}\right) n, \tag{1}$$

then, by Fact 9, $js_{r+1}(G) > n^{r-1}/r^{r+3}$, completing the proof.

Thus, we shall assume that (1) fails. Then, letting

$$\alpha = 1/r^4, \quad \beta = 1/2, \quad \gamma = 1 - 1/r, \quad K = r/4, \tag{2}$$

we see that

$$\delta(G) \leq (\gamma - \alpha) n \tag{3}$$

and also, in view of Fact 14,

$$\mu(G) > \mu(T_r(n)) \geq 2e(T_r(n))/n \geq (1 - 1/r) n - r/4n = \gamma n - K/n. \tag{4}$$

Given (2), (3) and (4), Fact 8 implies that, for $n \geq r^{15}$, G contains an induced subgraph H satisfying $|H| \geq n/2$ and one of the conditions:

(i) $\mu(H) > (1 - 1/r + 1/(4r^4))|H|$;

(ii) $\mu(H) > (1 - 1/r)|H|$ and $\delta(H) > (1 - 1/r - 1/r^4)|H|$.

If condition *(i)* holds, Fact 10 gives

$$k_{r+1}(H) > \left(\frac{\mu(H)}{|H|} - 1 - \frac{1}{r}\right) \frac{r(r-1)}{r+1} \left(\frac{|H|}{r}\right)^{r+1} > \frac{r(r-1)}{4r^4(r+1)} \left(\frac{|H|}{r}\right)^{r+1},$$

and so,

$$\begin{aligned} js_{r+1}(G) &\geq js_{r+1}(H) \geq \binom{r+1}{2} \frac{k_{r+1}(H)}{e(H)} > r(r+1) \frac{k_{r+1}(H)}{|H|^2} \\ &> \frac{r(r+1) r(r-1)}{4r^4(r+1) r^{r+1}} |H|^{r-1} \geq \frac{1}{4r^{r+3}} |H|^{r-1} \geq \frac{1}{2^{r+1} r^{r+3}} n^{r-1} \geq \frac{1}{r^{2r+4}} n^{r-1}, \end{aligned}$$

completing the proof.

If condition *(ii)* holds, then H contains a K_{r+1}; thus, $js_{r+1}(H) > |H|^{r-1}/r^{r+3}$ by Fact 9. To complete the proof, notice that

$$js_{r+1}(G) > js_{r+1}(H) > \frac{|H|^{r-1}}{r^{r+3}} \geq \frac{1}{2^{r-1} r^{r+3}} n^{r-1} > \frac{1}{r^{2r+4}} n^{r-1}.$$

□

Proof of Theorem 2

Let G be a graph of order n with $\mu(G) > \mu(T_r(n))$; thus, by Fact 7, G contains a K_{r+1}. If

$$\delta(G) > \left(1 - 1/r - 1/r^4\right) n, \tag{5}$$

then, by Fact 12, G contains a $K_r^+\left(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \left\lceil n^{1-cr^3} \right\rceil\right)$, completing the proof, in view of $cr^3 < \sqrt{c}$.

Thus, we shall assume that (5) fails. Then, letting

$$\alpha = 1/r^4, \quad \beta = 1/2, \quad \gamma = 1 - 1/r, \quad K = r/4, \tag{6}$$

we see that

$$\delta(G) \leq (\gamma - \alpha) n \tag{7}$$

and also, in view of Fact 14,

$$\mu(G) > \mu(T_r(n)) \geq 2e(T_r(n))/n \geq (1 - 1/r) n - r/4n = \gamma n - K/n. \tag{8}$$

Given (6), (7) and (8), Fact 8 implies that, for $n > r^{15}$, G contains an induced subgraph H satisfying $|H| \geq n/2$ and one of the conditions:

(i) $\mu(H) > (1 - 1/r + 1/(4r^4))|H|$;
(ii) $\mu(H) > (1 - 1/r)|H|$ and $\delta(H) > (1 - 1/r - 1/r^4)|H|$.

If condition *(i)* holds, Fact 10 gives

$$\begin{aligned} k_{r+1}(H) &> \left(\frac{\mu(H)}{|H|} - 1 - \frac{1}{r}\right) \frac{r(r-1)}{r+1} \left(\frac{|H|}{r}\right)^{r+1} > \frac{r(r-1)}{4r^4(r+1)} \left(\frac{|H|}{r}\right)^{r+1} \\ &> \frac{1}{2^{r+3} r^{r+4}(r+1)} n^{r+1} > \frac{1}{r^{2r+9}} n^{r+1} \geq c^{1/(r+1)} n^{r+1}. \end{aligned}$$

Thus, by Fact 13, G contains a $K_{r+1}(s, \ldots, s, t)$ with $s = \lfloor c \ln n \rfloor$ and $t > n^{1-c^{r/(r+1)}} > n^{1-\sqrt{c}}$. Then, obviously, G contains a $K_r^+\left(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \left\lceil n^{1-\sqrt{c}} \right\rceil\right)$, completing the proof.

If condition *(ii)* holds, then H contains a K_{r+1}; thus, by Fact 12, H contains a

$$K_r^+\left(\lfloor 2c \ln |H| \rfloor, \ldots, \lfloor 2c \ln |H| \rfloor, \left\lceil |H|^{1-2cr^3} \right\rceil\right).$$

To complete the proof, note that $2c \ln |H| \geq 2c \ln \frac{n}{2} > c \ln n$ and

$$|H|^{1-2cr^3} \geq \left(\frac{n}{2}\right)^{1-2cr^3} \geq \frac{1}{2} n^{1-2cr^3} > n^{1-\sqrt{c}}.$$

□

Proof of Theorem 4 Let G be a graph of order n with $\mu(G) > (1 - 1/r - b) n$. If G contains no K_{r+1}, then condition *(b)* follows from Fact 11; thus we assume that G contains a K_{r+1}. If

$$\delta(G) > \left(1 - 1/r - 1/r^4\right) n, \tag{9}$$

then Fact 9 implies condition *(a)*.

Thus, we shall assume that (9) fails. Then, letting

$$\alpha = 1/r^4 - b, \quad \beta = 4b/\alpha, \quad \gamma = 1 - 1/r - b, \quad K = 0, \tag{10}$$

we easily see that

$$\beta = \frac{4b}{1/r^4 - b} \leq \frac{1}{2}, \qquad \delta(G) \leq (\gamma - \alpha) n, \tag{11}$$

and

$$\mu(G) > (1 - 1/r - b) n = \gamma n. \tag{12}$$

Given (10), (11) and (12), Theorem 8 implies that, for $n \geq r^{20}$, G contains an induced subgraph H satisfying $|H| \geq (1-\beta) n$ and one of the conditions:

(i) $\mu(H) > (1 - 1/r) |H|$;

(ii) $\mu(H) > (1 - 1/r - b) |H|$ and $\delta(H) > (1 - 1/r - 1/r^4) |H|$.

If condition *(i)* holds, by Theorem 1 we have

$$\begin{aligned} js_{r+1}(G) &\geq js_{r+1}(H) \geq \frac{|H|^{r-1}}{r^{2r+4}} \geq (1-\beta)^{r-1} \frac{n^{r-1}}{r^{2r+4}} = \left(1 - \frac{4b}{1/r^4 - b}\right)^{r-1} \frac{n^{r-1}}{r^{2r+4}} \\ &> \left(1 - \frac{1}{r^2}\right)^{r-1} \frac{n^{r-1}}{r^{2r+4}} \geq \left(1 - \frac{r-1}{r^2}\right) \frac{n^{r-1}}{r^{2r+4}} > \frac{n^{r-1}}{r^{2r+5}}, \end{aligned}$$

implying condition *(a)* and completing the proof.

Suppose now that condition *(ii)* holds. If H contains a K_{r+1}, by Fact 9, we see that

$$js_{r+1}(G) \geq js_{r+1}(H) \geq \frac{|H|^{r-1}}{r^{r+3}} \geq (1-\beta)^{r-1} \frac{n^{r-1}}{r^{r+3}} > \frac{n^{r-1}}{2^{r-1} r^{r+3}} > \frac{n^{r-1}}{r^{2r+5}},$$

implying condition *(a)*.

If H contains no K_{r+1}, by Fact 11, H contains an induced r-partite subgraph H_0 satisfying $|H_0| > \left(1 - 3b^{1/3}\right) |H|$ and $\delta(H_0) > \left(1 - 6b^{1/3}\right) |H|$. Now from

$$\beta = \frac{4b}{1/r^4 - b} \leq \frac{4b}{1/r^4 - 1/\left(2^{10} r^6\right)} \leq 8r^4 b < b^{1/3},$$

we deduce that

$$|H_0| \geq \left(1 - 3b^{1/3}\right) |H| \geq \left(1 - 3b^{1/3}\right) (1-\beta) n > \left(1 - 4b^{1/3}\right) n$$

and

$$\delta(H_0) \geq \left(1 - 6b^{1/3}\right) |H| \geq \left(1 - 6b^{1/3}\right) (1-\beta) n > \left(1 - 7b^{1/3}\right) n.$$

Thus condition *(b)* holds, completing the proof. □

Proof of Theorem 5 Let G be a graph of order n with $\mu(G) > (1 - 1/r - b) n$. If G contains no K_{r+1}, then condition *(b)* follows from Fact 11; thus we assume that G contains a K_{r+1}. If

$$\delta(G) > \left(1 - 1/r - 1/r^4\right) n, \tag{13}$$

then Fact 12 implies condition *(a)*.

Thus, we shall assume that (13) fails. Then, letting

$$\alpha = 1/r^4 - b, \quad \beta = 4b/\alpha, \quad \gamma = 1 - 1/r - b, \quad K = 0, \tag{14}$$

we easily see that

$$\beta = \frac{4b}{1/r^4 - b} \leq \frac{1}{2}, \qquad \delta(G) \leq (\gamma - \alpha) n, \tag{15}$$

and

$$\mu(G) > (1 - 1/r - b) n = \gamma n. \tag{16}$$

Given (14), (15) and (16), Theorem 8 implies that, for $n \geq r^{20}$, G contains an induced subgraph H satisfying $|H| \geq (1-\beta) n$ and one of the conditions:

(i) $\mu(H) > (1 - 1/r) |H|$;

(ii) $\mu(H) > (1 - 1/r - b) |H|$ and $\delta(H) > (1 - 1/r - 1/r^4) |H|$.

If condition *(i)* holds, Theorem 2 implies that H contains a

$$K_r^+\left(\lfloor 2c \ln |H| \rfloor, \ldots, \lfloor 2c \ln |H| \rfloor, \left\lceil |H|^{1-2cr^3} \right\rceil\right).$$

Now condition *(a)* follows in view of $2c \ln |H| \geq 2c \ln \frac{n}{2} > c \ln n$ and

$$|H|^{1-2cr^3} \geq \left(\frac{n}{2}\right)^{1-2cr^3} \geq \frac{1}{2} n^{1-2cr^3} > n^{1-\sqrt{c}},$$

completing the proof.

Suppose now that condition *(ii)* holds. If H contains a K_{r+1}, by Fact 12, H contains a

$$K_r^+\left(\lfloor 2c \ln |H| \rfloor, \ldots, \lfloor 2c \ln |H| \rfloor, \left\lceil |H|^{1-2cr^3} \right\rceil\right).$$

This implies condition *(a)* in view of $2c \ln |H| \geq 2c \ln \frac{n}{2} > c \ln n$ and

$$|H|^{1-2cr^3} \geq \left(\frac{n}{2}\right)^{1-2cr^3} \geq \frac{1}{2} n^{1-2cr^3} > n^{1-\sqrt{c}}.$$

If H contains no K_{r+1}, the proof is completed as the proof of Theorem 4. $\square$

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