Certificates of factorisation for chromatic polynomials

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Submitted: Sep 15, 2008; Accepted: Jun 11, 2009; Published: Jun 19, 2009
Mathematics Subject Classification: 05C15, 05C75, 68R10

Abstract

The chromatic polynomial gives the number of proper λ-colourings of a graph G. This paper considers factorisation of the chromatic polynomial as a first step in an algebraic study of the roots of this polynomial. The chromatic polynomial of a graph is said to have a chromatic factorisation if $P(G,\lambda) = P(H_1,\lambda)P(H_2,\lambda)/P(K_r,\lambda)$ for some graphs H_1 and H_2 and clique K_r. It is known that the chromatic polynomial of any clique-separable graph, that is, a graph containing a separating r-clique, has a chromatic factorisation. We show that there exist other chromatic polynomials that have chromatic factorisations but are not the chromatic polynomial of any clique-separable graph and identify all such chromatic polynomials of degree at most 10. We introduce the notion of a certificate of factorisation, that is, a sequence of algebraic transformations based on identities for the chromatic polynomial that explains the factorisations for a graph. We find an upper bound of $n^2 2^{n^2/2}$ for the lengths of these certificates, and find much smaller certificates for all chromatic factorisations of graphs of order ≤ 9.

1 Introduction

The number of proper λ-colourings of a graph G is given by the chromatic polynomial $P(G,\lambda) \in \mathbb{Z}[\lambda]$. This polynomial was introduced by Birkhoff [5, 6] in an attempt to prove the four colour theorem by algebraic means. Read and Tutte [17] comment that calculating the chromatic polynomial of a graph is at least as difficult as determining the chromatic number of the graph which is known to be NP-complete [10].

The study of *chromatic roots*, the roots of chromatic polynomials, may be divided into three areas: integer chromatic roots, real chromatic roots and complex chromatic roots. Surveys of results on this topic have been given by Woodall [26] and Jackson [9].

The integer roots have provided information on some properties of graphs including the chromatic number and connectivity [23, 26, 24]. Studies of the real roots include the identification of intervals that are zero-free in $\mathbb{R}$ [23, 26, 8, 22, 27, 9]. Studies of complex roots have emphasised the limits of zeros of chromatic polynomials of families of graphs in the complex plane [4, 2, 3, 17, 14, 19, 20].

The chromatic polynomial also has applications in statistical mechanics where the partition function generalises this polynomial. The limit points of the complex zeros of this function are of particular interest, as they correspond to possible locations of physical phase transitions. Furthermore, no phase transitions are located in any zero-free region of the complex plane [11]. Sokal gives a good overview of the applications to statistical mechanics in [21].

Although there has been considerable work on the location of chromatic roots, there has been little work on the algebraic properties of these roots. The main exception to this is the the exclusion of the Beraha numbers $B_i = 2\cos 2\pi/i$, $i \geq 5$, as possible roots (except possibly B_{10}), proved algebraically by Salas and Sokal [18] and in the case of B_5 by Tutte [23].

Our motivation is to begin the study of the algebraic structure of chromatic polynomials and their roots. A first step is understanding factorisations of the chromatic polynomial, and this is the subject of this paper.

We say the chromatic polynomial of a graph G has a *chromatic factorisation* if there exist graphs H_1 and H_2 with fewer vertices than G such that

$$P(G,\lambda) = \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)} \tag{1}$$

for some $r \geq 0$, where by convention $P(K_0,\lambda) := 1$. The graph G is said to have a *chromatic factorisation*, if $P(G,\lambda)$ has a chromatic factorisation. The graph G is said to be *clique-separable* if G is disconnected or is isomorphic to the graph obtained by identifying graphs H_1 and H_2 at some clique. It is well-known that the chromatic polynomial of any clique-separable graph has a chromatic factorisation [28, 16]. A graph G' is *chromatically equivalent* to G if $P(G,\lambda) = P(G',\lambda)$. We denote this by $G \sim H$. A graph is said to be *quasi-clique-separable* if it is chromatically equivalent to a clique-separable graph. Any quasi-clique-separable graph has a chromatic factorisation.

Clique-separability is the most obvious way to determine some information about the factorisation of $P(G,\lambda)$ just from the structure of G itself. It is therefore natural to begin investigation of factorisation of $P(G,\lambda)$ by looking at situations where it factorises like the case of a clique-separable graph.

A search of all chromatic polynomials of degree at most 10 was undertaken to identify which of these polynomials had chromatic factorisations. This demonstrated that there exist chromatic polynomials that have chromatic factorisations but which are not the chromatic polynomial of any clique-separable graph. We identified 512 such factorisations.

In order to provide an explanation of these factorisations, we introduce the notion of a *certificate of factorisation*. This certificate is a sequence of steps using various identities for the chromatic polynomial that explains the chromatic factorisation of a given chromatic polynomial. The certificate starts with the chromatic polynomial $P(G, \lambda)$ and by applying steps using known properties of the chromatic polynomial and basic algebraic operations expresses $P(G, \lambda)$ as $P(H_1, \lambda)P(H_2, \lambda)/P(K_r, \lambda)$. In such cases a certificate of factorisation can always be found, in principle. However, naive approaches to finding certificates may not be feasible, as they may produce certificates of exponential length. We establish an upper bound on certificate length of $n^2 2^{n^2/2}$. Furthermore, as calculating the chromatic polynomial is NP-hard, it is not surprising that finding a certificate appears to be difficult.

In the light of these remarks short certificates of factorisation might be expected to be rare, and significant when they occur. Most of the certificates we give are in fact reasonably short. Furthermore, the two shortest certificates we found appear to be the shortest possible, when the graph is not quasi-clique-separable.

We find it helpful to group some certificates of factorisation together into *schemas*. A *schema* is, in effect, a template for a certificate of factorisation. Although the schema may include some of the actual certification steps, the schema also includes gaps, where each gap must be replaced by a sequence of certification steps to form an actual certificate. So a schema represents a class of certificates that all share certain designated subsequences of steps. These certificates may be said to *belong* to the schema.

We give a useful schema for certificates of factorisation and a number of classes of certificates belonging to this schema. Certificates from this schema can explain most chromatic factorisations of graphs of order at most 9. We give some other certificates, not from this schema, which explain the remaining cases.

If a graph is clique-separable, then (1) is a certificate of factorisation. Graphs that have a chromatic factorisation that satisfies this simplest of certificates have a common structural property, namely clique-separability. The graphs that have chromatic factorisations that satisfy the schema presented in this paper also have a common structural property. Although these graphs are not clique-separable, they can be obtained by adding, or removing, an edge from some clique-separable graph. Graphs that have chromatic factorisations satisfying some particular certificate belonging to this schema have additional common structure. In [13] we give an infinite family of graphs that have chromatic factorisations satisfying a certificate belonging to this schema. In addition to the common properties of all graphs with chromatic factorisations satisfying the schema, these graphs are triangle-free K_4-homeomorphs.

The paper is organised as follows. Section 2 provides definitions and some properties of chromatic polynomials. Section 3 then presents the results of our search for previously unexplained chromatic factorisations in graphs of order at most 10. In Section 4 certificates of factorisation are defined and an upper bound on the length of these certificates is proved. A schema for certificates of factorisation is then introduced and a number of certificates produced from this schema.

2 Preliminaries

2.1 Definitions

Standard definitions are used. We refer the reader to [7] for more information. As the presence of multiple edges does not affect the number of colourings, we will assume graphs have no multiple edges. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum number of colours required to colour the vertices of the graph so that no adjacent vertices are assigned the same colour.

If disjoint graphs, H_1 and H_2, each contain a clique of size at least r, let G be the graph formed by identifying an r-clique in H_1 with an r-clique in H_2. We say G is an r*-gluing*, or *clique-gluing*, of H_1 and H_2. If G can be obtained by a sequence of clique-gluings, we say G is an $(r_1, \ldots, r_t)$*-gluing* where:

- An (r_1)-gluing is an r_1-gluing of graphs H_1 and H_2
- An $(r_1, \ldots, r_t)$*-gluing* of graphs $H_1, \ldots, H_{t+1}$ is an r_t-gluing of H_{t+1} and a graph obtained by an $(r_1, \ldots, r_{t-1})$-gluing of graphs $H_1, \ldots, H_t$.

If G is a graph formed by an r-gluing of graphs H_1 and H_2, and a graph G' is the graph formed by identifying a different pair of r-cliques in H_1 and H_2 (if a different pair exists), then G' is a *re-gluing* of G. Although the graphs G and G' may not be isomorphic, they are chromatically equivalent.

Let G be the graph obtained from graphs G_1 and G_2 by identifying vertices a_1 and b_1 in G_1 with vertices a_2 and b_2 in G_2 respectively. Then the graph obtained by identifying vertices a_1 and b_1 in G_1 with vertices b_2 and a_2 in G_2 respectively is said to be 2*-isomorphic* to G.

2.2 Basic Properties

Some basic properties of the chromatic polynomial are listed in this section. Further details can be found in [15, 16, 17, 23, 28].

The deletion-contraction relation states that for any $e \in E$,

$$P(G, \lambda) = P(G \setminus e, \lambda) - P(G/e, \lambda).$$

The addition-identification relation states that for any $u, v \in V$, $uv \notin E$,

$$P(G, \lambda) = P(G + uv, \lambda) + P(G/uv, \lambda),$$

where we write G/uv for the graph obtained from G by identifying u and v and deleting any multiple edges so formed.

2.3 Computations

The chromatic polynomial can be calculated in terms of the complete graph basis, that is as a sum of chromatic polynomials of complete graphs, or in terms of the null graph basis, that is as a sum of chromatic polynomials of null graphs. The chromatic polynomials of all non-isomorphic connected graphs of order at most 10 were calculated in the null graph basis by the repeated application of the deletion-contraction relation.[1] Each chromatic polynomial was then factorised in $\mathbb{Z}[\lambda]$ using Pari [1]. We identified all non-clique-separable graphs using the algorithm in [25]. Any quasi-clique-separable graphs were then removed from this list. All possible chromatic factorisations of the chromatic polynomials of the remaining non-clique-separable graphs were constructed and basic search techniques used to determine if there exist graphs H_1 and H_2 satisfying such a factorisation.

3 Chromatic Factorisation

If the chromatic polynomial of a graph G has a chromatic factorisation then

$$\boxed{P(G,\lambda) = \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)}} \tag{2}$$

where H_1 and H_2 are graphs of lower order than G and $0 \leq r \leq \min\{\chi(H_1), \chi(H_2)\}$, and neither H_1 nor H_2 are isomorphic to K_r. The *chromatic factors* of $P(G,\lambda)$ are H_1 and H_2.

Any quasi-clique-separable graph has a chromatic factorisation. We say that a graph is *strongly non-clique-separable* if it is not quasi-clique-separable. We found that a number of chromatic polynomials of strongly non-clique-separable graphs have chromatic factorisations, by undertaking a search of all chromatic polynomials of strongly non-clique-separable graphs of at most 10. In all such cases, the graphs have at least 8 vertices. There are 512 such polynomials corresponding to 3118 non-isomorphic graphs and 4705 non-isomorphic pairs (G, g), where g is the unordered pair $\{H_1, H_2\}$, satisfying (2). (The pairs $(G, \{H_1, H_2\})$ and $(G', \{H_1', H_2'\})$ are *isomorphic* if $G \cong G'$ and either $H_1 \cong H_1'$ and $H_2 \cong H_2'$, or $H_1 \cong H_2'$ and $H_2 \cong H_1'$.) Details are given in Tables 1 and 2.

These 512 chromatic polynomials have chromatic factorisations that cannot be explained by the graph being quasi-clique-separable. In order to provide an explanation for these factorisations, we introduce the concept of a certificate of factorisation in Section 4. Certificates are then presented to explain the chromatic factorisations of some of these polynomials.

[1]These graphs are provided by B. McKay at http://cs.anu.edu.au/people/bdm/data/graphs.html. Code for calculating chromatic polynomials was provided by J. Reicher. Chromatic polynomials calculated by this code agreed with the author's own code that produced chromatic polynomials in the complete graph basis and hand calculations.

n	A	B	C
8	1,650	663	2
9	21,121	5319	25
10	584,432	74,016	485
$8 \leq n \leq 10$	607,203	79,998	512

Table 1: Numbers of chromatic polynomials of degree at most 10. (A) Total number of chromatic polynomials, (B) number of chromatic polynomials of clique-separable graphs and (C) number of chromatic polynomials of strongly non-clique-separable graphs with chromatic factorisations.

n	# chromatic polys.	# graphs	# pairs $(G, \{H_1, H_2\})$
8	2	3	3
9	25	97	114
10	485	3018	4588
$8 \leq n \leq 10$	512	3118	4705

Table 2: Chromatic factorisations of chromatic polynomials of degree $n \leq 10$ of strongly non-clique-separable graph.

4 Certificates of Factorisation

Definition A *certificate of factorisation* of $P(G, \lambda)$ with chromatic factors H_1 and H_2 is a sequence $P_0, P_1, \ldots, P_i$ where each P_j is an expression formed from chromatic polynomials $P(\_, \lambda)$ as follows. Each chromatic polynomial $P(\_, \lambda)$ is treated as a formal symbol and not an actual polynomial. Let $\{p_0, p_1, \ldots\}$ be the set of formal symbols representing chromatic polynomials $P(\_, \lambda)$. Let $\mathbb{Q}(p_0, p_1, \ldots)$ be the field of rational functions in indeterminates $p_1, p_2, \ldots$. The sequence $P_0, P_1, \ldots, P_i$ starts and ends with $P_0 = P(G, \lambda)$ and $P_i = P(H_1, \lambda)P(H_2, \lambda)/P(K_r, \lambda)$ respectively. Each P_j, $1 \leq j \leq i$, in the sequence is obtained from P_{j-1} by one of the following *certification steps*:

(CS1) $P(G', \lambda)$ becomes $P(G' \setminus e, \lambda) - P(G'/e, \lambda)$ for some $e \in E(G')$

(CS2) $P(G_1, \lambda) - P(G_2, \lambda)$ becomes $P(G', \lambda)$ where G' is isomorphic to $G_1 + uv$, $uv \notin E(G_1)$, and G_1/uv is isomorphic to G_2

(CS3) $P(G', \lambda)$ becomes $P(G' + uv, \lambda) + P(G'/uv, \lambda)$ for some $uv \notin E(G')$

(CS4) $P(G_1, \lambda) + P(G_2, \lambda)$ becomes $P(G', \lambda)$ where G' is isomorphic to $G_1 \setminus e$, $e \in E(G_1)$, and G_1/e is isomorphic to G_2

(CS5) $P(G_1, \lambda) - P(G_2, \lambda)$ becomes $P(G', \lambda)$ where G' is isomorphic to G_2/e, $e \in E(G_2)$, and G_1 is isomorphic to $G_2 \setminus e$

(CS6) $P(G', \lambda)$ becomes $P(G_1, \lambda)P(G_2, \lambda)/P(K_r, \lambda)$ where G' is isomorphic to the graph obtained by an r-gluing of G_1 and G_2

(CS7) $P(G_1, \lambda)P(G_2, \lambda)/P(K_r, \lambda)$ becomes $P(G', \lambda)$ where G' is isomorphic to the graph obtained by an r-gluing of G_1 and G_2

(CS8) By applying the field axioms, for $\mathbb{Q}(p_0, p_1, \ldots)$, a finite number of times, so as to produce a different expression for the same field element

(CS9) $P(G', \lambda)$ becomes $P(G'', \lambda)$ where $G' \sim G''$

Each P_j is a formal expression. If these expressions were evaluated to actual polynomials, all these polynomials would be equal. Thus, the certificate of factorisation fully explains the chromatic factorisation of $P(G, \lambda)$.

We say that $P(G, \lambda)$ (and by overloading the terminology its chromatic factorisation, and also G itself) *satisfies* its certificate of factorisation.

Step (CS9) requires that $G' \sim G''$. In order to be able to show that two graphs are chromatically equivalent, we define a *certificate of equivalence*. A certificate of equivalence is similar to a certificate of factorisation. It is a sequence of steps $P_0, P_1, \ldots, P_i$ where the steps are the same certification steps (excluding the step of interchanging $P(G', \lambda)$ and $P(G'', \lambda)$ when $G' \sim G''$), and $P_0 = P(G, \lambda)$ and $P_i = P(H, \lambda)$ where $G \sim H$.

An additional certification step of interchanging graphs that are 2-isomorphic could be added to the certification steps. As 2-isomorphic graphs are chromatically equivalent (since their cycle matroids are isomorphic), the certificate of factorisation can use (CS9) to interchange 2-isomorphic graphs. In the case of certificates of equivalence, showing G and G' are 2-isomorphic can be achieved using a sequence of the existing steps, as follows.

In the case where G' is a re-gluing of G, the steps are

$$P(G, \lambda) = \frac{P(H_1, \lambda)P(H_2, \lambda)}{P(K_2, \lambda)} = P(G', \lambda).$$

In the case where G' is not a re-gluing of G, as the graphs are 2-isomorphic there exists $uv \notin E(G)$ and $wx \notin E(G')$ such that $G + uv$ is a re-gluing of $G + wx$ and G/uv is isomorphic to G'/wx. Thus the steps are

$$\begin{aligned} P(G, \lambda) &= P(G + uv, \lambda) + P(G/uv, \lambda) \\ &= \frac{P(H_1, \lambda)P(H_2, \lambda)}{P(K_2, \lambda)} + P(G'/wx, \lambda) \\ &= P(G' + wx, \lambda) + P(G'/wx, \lambda) \\ &= P(G', \lambda). \end{aligned}$$

An *extended certificate of factorisation* is a certificate of factorisation which only uses certification steps (CS1–CS8). Thus, an extended certificate of factorisation can be obtained from a certificate of factorisation by replacing any step of type (CS9) (if such exists) by the sequence of steps in a certificate of equivalence showing $G' \sim G''$.

The average length of the certificates of factorisation we found for all strongly non-clique-separable graphs of order 9 was 16.88 steps (and an average length of 19.2 steps for the extended certificate of factorisation).

Two certificates of factorisation, $C = (P_0, P_1, \ldots, P_i)$ and $C' = (P'_0, P'_1, \ldots, P'_i)$, are *equivalent* if there is a bijection f from the symbols $P(\_, \lambda)$ appearing in C to those appearing in C' such that the replacement of all symbols in C by their images under f transforms C into C', with all certification steps still being valid. A *CF-class* (Certificate of Factorisation class) of graphs is a maximal set of graphs with equivalent certificates of factorisation. Note that these classes may overlap, as a graph may have different, inequivalent certificates of factorisation. Informally, a CF-class is a maximal set of all graphs having "essentially" the same certificate of factorisation. Later (in Section 4.3) we will see that a graph's CF-class can be related to its structure.

4.1 Simple Certificates

If G is a clique-separable graph, then (2) is a certificate of factorisation. If G is chromatically equivalent to a clique-separable graph G', then $P(G, \lambda)$ has the following certificate:

$$\begin{aligned} P(G, \lambda) &= P(G', \lambda) \\ &= \frac{P(H_1, \lambda) P(H_2, \lambda)}{P(K_r, \lambda)} \end{aligned}$$

Certificate 1.
Graph G is chromatically equivalent to Graph G'.

However, these simple certificates cannot explain all chromatic factorisations. In Section 4.3 more complex certificates for chromatic factorisations are presented.

4.2 Construction of Certificates of Factorisation

It would appear that finding certificates of factorisation for strongly non-clique-separable graphs is hard. The length of the certificate for a graph of n vertices is $\leq n^2 2^{n^2/2}$. We establish this bound below, using a naive approach to constructing a certificate of factorisation for any chromatic factorisation. Certificates of this form are exponential both in length and in time taken to compute them. In Section 4.3 we present a schema for certificates of factorisation that produces much shorter certificates than this approach, in cases to which it applies.

Any chromatic polynomial can be expressed as the sum of chromatic polynomials of complete graphs by repeated application of the addition-identification relation [16].

Proposition 1 *The chromatic polynomial of a graph G can be expressed as the sum of chromatic polynomials of complete graphs in at most $2^{\overline{m}} - 1$ applications of the addition-identification relation where $\overline{m}$ is the number of edges in the complement $\overline{G}$.*

Theorem 2 *If G is a strongly non-clique-separable graph having chromatic factorisation $P(G,\lambda) = P(H_1,\lambda)P(H_2,\lambda)/P(K_r,\lambda)$, then there exists an extended certificate of factorisation for $P(G,\lambda)$ of length $\le n^2 2^{n^2/2}$.*

Proof Let n, n_1, n_2 be the number of vertices in G, H_1 and H_2 respectively, and let $\overline{m}, \overline{m_1}$ and $\overline{m_2}$ be the number of edges in $\overline{G}, \overline{H_1}$ and $\overline{H_2}$ respectively.

A certificate can be obtained as follows. Firstly, express both $P(H_1,\lambda)$ and $P(H_2,\lambda)$ as sums of chromatic polynomials of complete graphs. By Proposition 1 this gives a sequence of at most $2^{\overline{m_1}} + 2^{\overline{m_2}} - 2$ steps showing

$$\frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)} = \frac{(\sum_{i=\chi(H_1)}^{n_1} a_i P(K_i,\lambda))(\sum_{j=\chi(H_2)}^{n_2} b_j P(K_j,\lambda))}{P(K_r,\lambda)} \tag{3}$$

where the a_i and b_j are positive integers and $a_{n_1} = b_{n_2} = 1$.

Applying Step (CS8) to the product in (3),

$$\frac{(\sum_{i=\chi(H_1)}^{n_1} a_i P(K_i,\lambda))(\sum_{j=\chi(H_2)}^{n_2} b_j P(K_j,\lambda))}{P(K_r,\lambda)} = \sum_{i,j} \frac{a_i b_j P(K_i,\lambda)P(K_j,\lambda)}{P(K_r,\lambda)}. \tag{4}$$

For each i, j, let G_{ij} be the graph formed by an r-gluing of K_i and K_j. (This is always possible as $\chi(H_1) \ge r$ and $\chi(H_2) \ge r$.) Then by performing a sequence of $(n_1 - \chi(H_1) + 1)(n_2 - \chi(H_2) + 1) \le (n_1 - 2)(n_2 - 2)$ clique-gluings, we obtain

$$\sum_{i,j} \frac{a_i b_j P(K_i,\lambda)P(K_j,\lambda)}{P(K_r,\lambda)} = \sum_{i,j} a_i b_j P(G_{ij},\lambda). \tag{5}$$

Now each $P(G_{ij},\lambda)$ in (5) can be expressed as the sum of chromatic polynomials of complete graphs. There are at most $(n_1 - \chi(H_1) + 1)(n_2 - \chi(H_2) + 1) \le (n_1 - 2)(n_2 - 2)$ of these graphs G_{ij}. Each of the G_{ij} has at most n vertices and at least $\binom{r}{2}$ edges. So, each $\overline{G_{ij}}$ must have at most $\binom{n}{2} - \binom{r}{2} < n(n-1)/2$ edges. Thus, by Proposition 1, in $< (n_1 - 2)(n_2 - 2)(2^{n(n-1)/2} - 1)$ steps we obtain

$$\sum_{i,j} a_i b_j P(G_{ij},\lambda) = \sum_{k=\chi(G)}^{n} c_k P(K_k,\lambda) \tag{6}$$

where each c_k is a positive integer and $c_n = 1$. But the right hand sum in (6) must also be the expression for $P(G,\lambda)$ as the sum of chromatic polynomials of complete graphs, since this expression is unique. Thus reversing this sequence of steps we have the desired certificate, namely

$$
\begin{aligned}
&P(G,\lambda)\\
&= \sum_{k=\chi(G)}^{n} c_k P(K_k,\lambda) && \text{in} \le 2^{\overline{m}}-1 \text{ steps by Proposition 1}\\
&= \sum_{i,j} a_i b_j P(G_{ij},\lambda) && \text{in} \le (n_1-2)(n_2-2)(2^{n(n-1)/2}-1) \text{ steps by (6)}\\
&= \sum_{i,j} \frac{a_i b_j P(K_i,\lambda)P(K_j,\lambda)}{P(K_r,\lambda)} && \text{in} \le (n_1-2)(n_2-2) \text{ steps by (5)}\\
&= \frac{(\sum_{i=\chi(H_1)}^{n_1} a_i P(K_i,\lambda))(\sum_{j=\chi(H_2)}^{n_2} b_j P(K_j,\lambda))}{P(K_r,\lambda)} && \text{in a single application of (CS8)}\\
&= \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)} && \text{in} \le 2^{\overline{m_1}}+2^{\overline{m_2}}-2 \text{ steps by (3).} \qquad (7)
\end{aligned}
$$

This certificate has at most $2^{\overline{m}}-1+(n_1-2)(n_2-2)(2^{n(n-1)/2}-1)+(n_1-2)(n_2-2)+1+2^{\overline{m_1}}+2^{\overline{m_2}}-2$ steps. Now as $(n_1-2)(n_2-2) \le (n-3)^2$ and $2^{\overline{m_1}}+2^{\overline{m_2}}-2 < 2^{(n-2)(n-3)/2}$, the total number of steps in the certificate is

$$< (n-3)^2 2^{n(n-1)/2} + 2^{n(n-3)/2} + 2^{(n-2)(n-3)/2} \tag{8}$$

which is $\le n^2 2^{n^2/2}$. $\square$

The proof in Theorem 2 gives us the means to find a certificate of factorisation, albeit a very long one, whenever a graph has a chromatic factorisation.

Although a certificate of factorisation can always be found by this simple approach, the length of certificate means that this method is infeasible for all but very small graphs. The upper bound in (8) shows that this approach produces certificates for strongly non-clique-separable graphs of order 8 and 9 with $< 6{,}711{,}967{,}744$ and $< 2{,}474{,}037{,}477{,}376$ steps respectively. Our certificates for graphs of order 9 were < 57 steps and on average 16.88 steps. This approach also does not provide any insight into any link between the structure of a strongly non-clique-separable graph and its chromatic factorisation.

In Section 4.3 a more efficient schema for some certificates of factorisation is presented. These certificates are much more concise than those produced by (7). The lengths of these certificates (which we call A–E) are given in Table 3 with the certificates A–E themselves given in Appendix A.1. The schema can be used to form certificates for most of the chromatic factorisations of the strongly non-clique-separable graphs of degree at most 9. The average length of certificates of factorisation using this schema for strongly non-clique-separable graphs of order 9 was 13.0625 steps (and an average length of 15.6875 steps for the extended certificate of factorisation). Both certificates A and B have constant length of 8 and 7 steps respectively, which makes them the shortest known certificates for strongly non-clique-separable graphs. Certificates for the chromatic factorisations of all strongly non-clique-separable graphs of degree 9 not explained by this schema (which we call F–K) are given in Appendix A.2. The lengths of these certificates were at most 57 steps with an average length of 23.67 steps.

Certificate	n	# Chromatic polynomials	s	$\overline{s}$
D	8	2	$10 \leq s \leq 11$	$10 \leq \overline{s} \leq 11$
A	9	2	8	8
B	9	1	7	7
C	9	2	$10 \leq s \leq 11$	$10 \leq \overline{s} \leq 11$
D	9	9	$10 \leq s \leq 23$	$12 \leq \overline{s} \leq 24$
E	9	2	$18 \leq s \leq 21$	$21 \leq \overline{s} \leq 34$
F	9	1	18	18
G	9	3	$12 \leq s \leq 18$	$16 \leq \overline{s} \leq 18$
H	9	1	26	26
I	9	1	39	39
J	9	1	57	66
K	9	2	$12 \leq s \leq 15$	$12 \leq \overline{s} \leq 16$

Table 3: Number of steps s ($\overline{s}$) in certificates of (extended) factorisation for chromatic polynomials of 8- and 9-vertex strongly non-clique-separable graphs. For each certificate the number of chromatic polynomials with this certificate is given.

Theorem 3 *If $G \sim G'$, then there exists a certificate of equivalence of length $< 2^{n^2/2}$.*

Proof By Proposition (1) the chromatic polynomials of G and G' can each be expressed as a sum of complete graphs in at most $2^{\overline{m}} - 1$ applications of the addition-identification relation. Thus, in at most $2(2^{\overline{m}} - 1) < 2^{n^2/2}$ steps it can be shown that both G and G' can be expressed as the same sum of complete graphs. □

4.3 New Chromatic Factorisations

Strongly non-clique-separable graphs are precisely those to which Certificate 1 does not apply. So, if such a graph has a chromatic factorisation, a more complex certificate will be needed to explain it. This section considers such certificates. We identify some useful classes of certificates and give numbers of chromatic factorisations that are explained by various types of certificate.

These classes of certificates are remarkably short in comparison to the upper bound of $n^2 2^{n^2/2}$ given in Section 4.2, and are the shortest known certificates of factorisation for strongly non-clique-separable graphs.

In this section we consider strongly non-clique-separable graphs that are *almost clique-separable*, that is graphs that can obtained by adding a single edge to, or removing a single edge from, a clique-separable graph. We present a schema for certificates of factorisation for these graphs. This allows us to link the structure of these graphs to their CF-class.

4.3.1 Graphs that are almost clique-separable

In most cases of strongly non-clique-separable graph with chromatic factorisations we examined ($n \leq 10$), there either exists an edge $e \in E(G)$ such that both $G \setminus e$ and G/e are clique-separable, or there exists $uv \notin E(G)$ such that both $G + uv$ and G/uv are clique-separable. In these cases, the chromatic polynomial of G can be expressed as the sum (or difference) of two clique-separable chromatic polynomials by the use of a single addition-identification or deletion-contraction relation. The majority of certificates presented in this section use this technique as their starting point.

Now, if G is a strongly non-clique-separable graph with the chromatic factorisation $P(G, \lambda) = P(H_1, \lambda)P(H_2, \lambda)/P(K_r, \lambda)$, we say that $P(H_1, \lambda)$ can be *isolated by a single application of the addition-identification relation* if $G + uv$, $uv \notin E(G)$, is an s-gluing of H_1 and some graph H_3, $r \geq s$, and G/uv is a t-gluing of H_1 and some graph H_4, $r \geq t$. If $G + uv$ is isomorphic to an s-gluing of H_1 and some graph H_3, we say $P(H_1, \lambda)$ can be *partially isolated by a single application of the addition-identification relation.*

Similarly, if there exists $e \in E(G)$ such that $G \setminus e$ is an s-gluing of H_1 and some graph H_3, $r \geq s$, and G/e is a t-gluing of H_1 and some graph H_4, $r \geq t$, we say that the chromatic factor $P(H_1, \lambda)$ can be *isolated by a single application of the deletion-contraction relation.* If $G \setminus e$ is isomorphic to an s-gluing of H_1 and some graph H_3, we say $P(H_1, \lambda)$ can be *partially isolated by a single application of the deletion-contraction relation.*

	Degree of $P(G, \lambda)$: 8	9	Certificates
$P(H_1, \lambda)$ can be isolated by single deletion-contraction	2	12	B, D, E
$P(H_1, \lambda)$ can be isolated by single deletion-contraction, but the certificate uses partial isolation.	0	2	G
$P(H_1, \lambda)$ can be isolated by single addition-identification	0	4	A, C
$P(H_1, \lambda)$ cannot be isolated but can be partially isolated by single deletion-contraction	0	3	G, K
$P(H_1, \lambda)$ cannot be isolated or partially isolated by single addition-identification or deletion-contraction	0	3	H, I, J
$P(G, \lambda)$ has 3 chromatic factors	0	1	F
TOTAL:	2	25	

Table 4: Number of chromatic factorisations where chromatic factor H_1 can be isolated by a single operation, and $P(G, \lambda)$ is the chromatic polynomial of a strongly non-clique-separable graph.

Table 4 lists the number of instances where one of the chromatic factors could be isolated, or partially isolated, in one of the above ways in all chromatic polynomials of strongly non-clique-separable graphs of at most 9 vertices. A chromatic factor could be isolated by a single application of either the addition-identification or the deletion-contraction relation in all of the chromatic polynomials of degree 8 and most of the chromatic polynomials of degree 9. Thus, the initial step in most of the certificates is to isolate a chromatic factor.

4.3.2 A Schema for Certificates of Factorisation

The schema for certificates of factorisation presented in this section has isolation of the chromatic factor H_1 as the initial step, that is

$$\begin{aligned} P(G,\lambda) =& P(G',\lambda) \pm P(G/uv,\lambda) \\ =& \frac{P(H_1,\lambda)P(H_3,\lambda)}{P(K_s,\lambda)} \pm \frac{P(H_1,\lambda)P(H_4,\lambda)}{P(K_t,\lambda)} \\ =& \frac{P(H_1,\lambda)}{P(K_r,\lambda)}\left(\frac{P(K_r,\lambda)P(H_3,\lambda)}{P(K_s,\lambda)} \pm \frac{P(K_r,\lambda)P(H_4,\lambda)}{P(K_t,\lambda)}\right) \end{aligned} \tag{9}$$

where $G' \cong G + uv$ if $uv \notin E(G)$, otherwise $G' \cong G \setminus uv$.

Suppose the initial steps in the certificate are those in (9). Suppose also that there exist graphs H_5 and H_6 and sequences of certification steps showing:

$$P(H_5,\lambda) = \frac{P(K_r,\lambda)P(H_3,\lambda)}{P(K_s,\lambda)}, \tag{10}$$

$$P(H_6,\lambda) = \frac{P(K_r,\lambda)P(H_4,\lambda)}{P(K_t,\lambda)} \qquad \text{and} \tag{11}$$

$$P(H_2,\lambda) = P(H_5,\lambda) \pm P(H_6,\lambda). \tag{12}$$

Then the following, Schema 1, is a schema for a class of certificate:

$$\begin{aligned}
P(G,\lambda) =& P(G',\lambda) \pm P(G/uv,\lambda)\\
=& \frac{P(H_1,\lambda)P(H_3,\lambda)}{P(K_s,\lambda)} \pm \frac{P(H_1,\lambda)P(H_4,\lambda)}{P(K_t,\lambda)}\\
=& \frac{P(H_1,\lambda)}{P(K_r,\lambda)}\left(\frac{P(K_r,\lambda)P(H_3,\lambda)}{P(K_s,\lambda)} \pm \frac{P(K_r,\lambda)P(H_4,\lambda)}{P(K_t,\lambda)}\right)
\end{aligned}$$

Insert certification steps showing (10) and (11)

$$= \frac{P(H_1,\lambda)}{P(K_r,\lambda)}\left(P(H_5,\lambda) \pm P(H_6,\lambda)\right)$$

Insert certification steps showing (12)

$$= \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)}$$

where $G' \cong G + uv$ if $uv \notin E(G)$, otherwise $G' \cong G \setminus uv$.

Schema 1 for Certificates of Factorisation

Appendix A.1 lists some certificates (A–E) that satisfy Schema 1. Most chromatic factorisations of strongly non-clique-separable graphs of degree at most 9 (in fact all but 9) satisfied this schema. Certificates for the remaining nine chromatic polynomials (F–K) are given in Appendix A.2. Three of these certificates, F, G and K (corresponding to six of the nine cases), contain some of the elements of Schema 1.

4.3.3 Some Schema 1 Certificates of Factorisation

In this section we will consider certificates that satisfy Schema 1. There are many different sequences of steps that can be used in the certification steps to show (10) and (11) in Schema 1. We present two possible sequences for (10) and three possible sequences for (11).

Certification steps to show (10)**.**
Now, if (10) holds then one of the following applies:
Case 1. $r = s$ and $H_5 \cong H_3$.
In this case the numerator and denominator have a common factor, $P(K_r,\lambda)$. Thus, the certification step is to replace $P(H_3,\lambda)P(K_r,\lambda)/P(K_s,\lambda)$ by $P(H_3,\lambda)$. This step is used in Certificate C step (27), in Certificate D step (29), in Certificate E step (30) and in Certificate K step (32).
Case 2. $r > s$ and H_5 is isomorphic to an s-gluing of H_3 and K_r.
In this case the certification step is to replace $P(H_3,\lambda)P(K_r,\lambda)/P(K_s,\lambda)$ by $P(H_5,\lambda)$. This step is used in Certificate A step (23) where $H_5 \cong H_2 + wx$, and in Certificate B step (25) where $H_5 \cong H_2 \setminus f$.

Certification steps to show (11).
If (11) holds then one of the following applies:
Case 1. $r = t$ and $H_6 \cong H_4$.
In this case the numerator and denominator have a common factor, $P(K_r, \lambda)$. Thus, the certification step is to replace $P(H_4, \lambda)P(K_r, \lambda)/P(K_t, \lambda)$ by $P(H_4, \lambda)$. This step is used in Certificate D step (29), in Certificate E step (30) and in Certificate K step (32).
Case 2. $r > t$ and H_6 is isomorphic to a t-gluing of H_4 and K_r.
In this case the certification step is to replace $P(H_4, \lambda)P(K_r, \lambda)/P(K_t, \lambda)$ by $P(H_6, \lambda)$. This step is used in Certificate B step (25) where $H_6 \cong H_2/f$ and in Certificate C step (28).
Case 3. $r > t + 1$ and H_6 is not isomorphic to a t-gluing of H_4 and K_r, but H_6 is isomorphic to the graph obtained by an $(r-1, t)$-gluing of graphs H_4, K_r and K_{r-1}.
In this case there are two certification steps. The first step replaces the expression $P(H_4, \lambda)P(K_r, \lambda)/P(K_t, \lambda)$ by $P(H_4, \lambda)P(K_r, \lambda)P(K_{r-1}, \lambda)/\left(P(K_{r-1}, \lambda)P(K_t, \lambda)\right)$. The second step replaces the latter expression by $P(H_6, \lambda)$ where H_6 is the graph obtained by an $(r-1, t)$-gluing of graphs H_4, K_r and K_{r-1}. These steps are used in Certificate A steps (22) and (23) where H_2/wx is isomorphic to a $(2, 1)$-gluing of graphs H_4, K_3 and K_2

Certification steps to show (12).
Schema 1 also requires certification steps to show (12). We will consider the case where $|V(H_5)| = |V(H_6)| + 1$. In this case, it is clear that either

Case 1

$$P(H_2, \lambda) = P(H_5, \lambda) + P(H_6, \lambda) \text{ and} \tag{13}$$
$$|E(H_2)| = |E(H_5)| - 1, \tag{14}$$

or

Case 2

$$P(H_2, \lambda) = P(H_5, \lambda) - P(H_6, \lambda) \text{ and} \tag{15}$$
$$|E(H_2)| = |E(H_5)| + 1. \tag{16}$$

Case 1 When (14) holds, there exist $e_0, \ldots, e_p \in E(H_5)$ and $f_1, \ldots, f_p \notin E(H_5)$ such that
$$H_5 \setminus \{e_0, \ldots, e_p\} + \{f_1, \ldots, f_p\} \cong H_2, \qquad p \geq 0. \tag{17}$$
When $p = 0$,
$$H_5 \setminus e_0 \cong H_2,$$
so
$$H_5 \cong H_2 + e_0.$$

For (13) to hold we must then have

$$H_6 \sim H_2/e_0,$$

which would certainly be satisfied if

$$H_6 \cong H_2/e_0.$$

The addition-identification relation is used to replace $P(H_5, \lambda) + P(H_6, \lambda)$ with $P(H_2, \lambda)$ in this certification step. This is used in Certificate A step (24).

Case 2 Similarly, when (16) holds, there exist $e_1, \ldots, e_p \in E(H_5)$ and $f_0, \ldots, f_p \notin E(H_5)$ such that

$$H_5 + \{f_0, \ldots, f_p\} \setminus \{e_1, \ldots, e_p\} \cong H_2, \qquad p \geq 0. \tag{18}$$

When $p = 0$,

$$H_5 + f_0 \cong H_2,$$

so

$$H_5 \cong H_2 \setminus f_0.$$

For (15) to hold we must then have

$$H_6 \sim H_2/f_0,$$

which would certainly be satisfied if

$$H_6 \cong H_2/f_0.$$

The certification step uses the deletion-contraction relation to replace $P(H_5, \lambda) - P(H_6, \lambda)$ by $P(H_2, \lambda)$. This is used in Certificate B step (26).

Case 1 and Case 2 when $p > 0$. We have seen that Certificate A and Certificate B, our shortest certificates, include the steps in Case 1 and Case 2 when $p = 0$. In the case where either (17) or (18) holds and $p > 0$, a sequence of addition-identification and deletion-contraction relations can be applied to show

$$P(H_5, \lambda) = P(H_2, \lambda) + \sum_{i=0}^{2p+1} c_i P(D_i, \lambda), \qquad c_i \in \{1, -1\} \tag{19}$$

for some graphs D_i. If a sequence of certification steps can be found that show

$$\sum_{i=0}^{2p+1} c_i P(D_i, \lambda) \pm P(H_6, \lambda) = 0 \tag{20}$$

then these steps can be combined with those used to show (19) to show

$$\begin{aligned} P(H_5,\lambda) \pm P(H_6,\lambda) =& P(H_2,\lambda) + \sum_{i=0}^{2p+1} c_i P(D_i,\lambda) \pm P(H_6,\lambda) \\ =& P(H_2,\lambda). \end{aligned} \tag{21}$$

Thus a sequence of addition-identification and deletion-contraction steps to show (19), combined with the sequence of certification steps to show (20), shows that $P(H_2,\lambda) = P(H_5,\lambda) \pm P(H_6,\lambda)$ as required in Schema 1.

Tables 5 and 6 list the numbers of chromatic polynomials of degree at most 9 with certificates that use sequences of steps of the kind we have been discussing, with $p \geq 0$. Examples of certificates of factorisation using this type of sequence of steps are provided in Figure 1 ($p = 0$) and Figure 2 ($p = 1$) (these figures represent the chromatic polynomial of a graph by the graph itself). Both these certificates satisfy Schema 1. The certificate of factorisation in Figure 1 has the form of Certificate B, the shortest certificate we found for strongly non-clique-separable graphs; and the certificate of factorisation in Figure 2 has the form of Certificate C.

	Certificate	$P(G,\lambda)$ with degree 9
$H_2 + e \cong H_5$, $e \notin E(H_2)$, where H_5 is an s-gluing of H_3 and K_r	A	2
$H_2 + e + f - g \cong H_5$, $e, f \notin E(H_2)$ and $g \in E(H_2)$, where H_5 is an s-gluing of H_3 and K_r	C	2
TOTAL:		4

Table 5: Relationship between graphs H_2 and H_5 in Certificate of Factorisation Schema 1 when graph H_1 is isolated by a single addition-identification .

5 Conclusion

In order to explain the chromatic factorisation of strongly non-clique-separable graphs, the concept of a certificate of factorisation was developed. A series of these certificates were presented that provide explanations of all chromatic factorisations of graphs of order at most 9. Most of these certificates were found to satisfy Schema 1. These certificates were much shorter than those that could be obtained by a naive approach. It seems likely that these certificates are the shortest possible certificates for strongly non-clique-separable graphs that have chromatic factorisations. It would be interesting to find a better upper bound on the lengths of certificates of factorisation than that presented in Theorem 2.

We have demonstrated that there exist strongly non-clique-separable graphs that have chromatic factorisations. In [13] we demonstrate that there exist infinitely many strongly

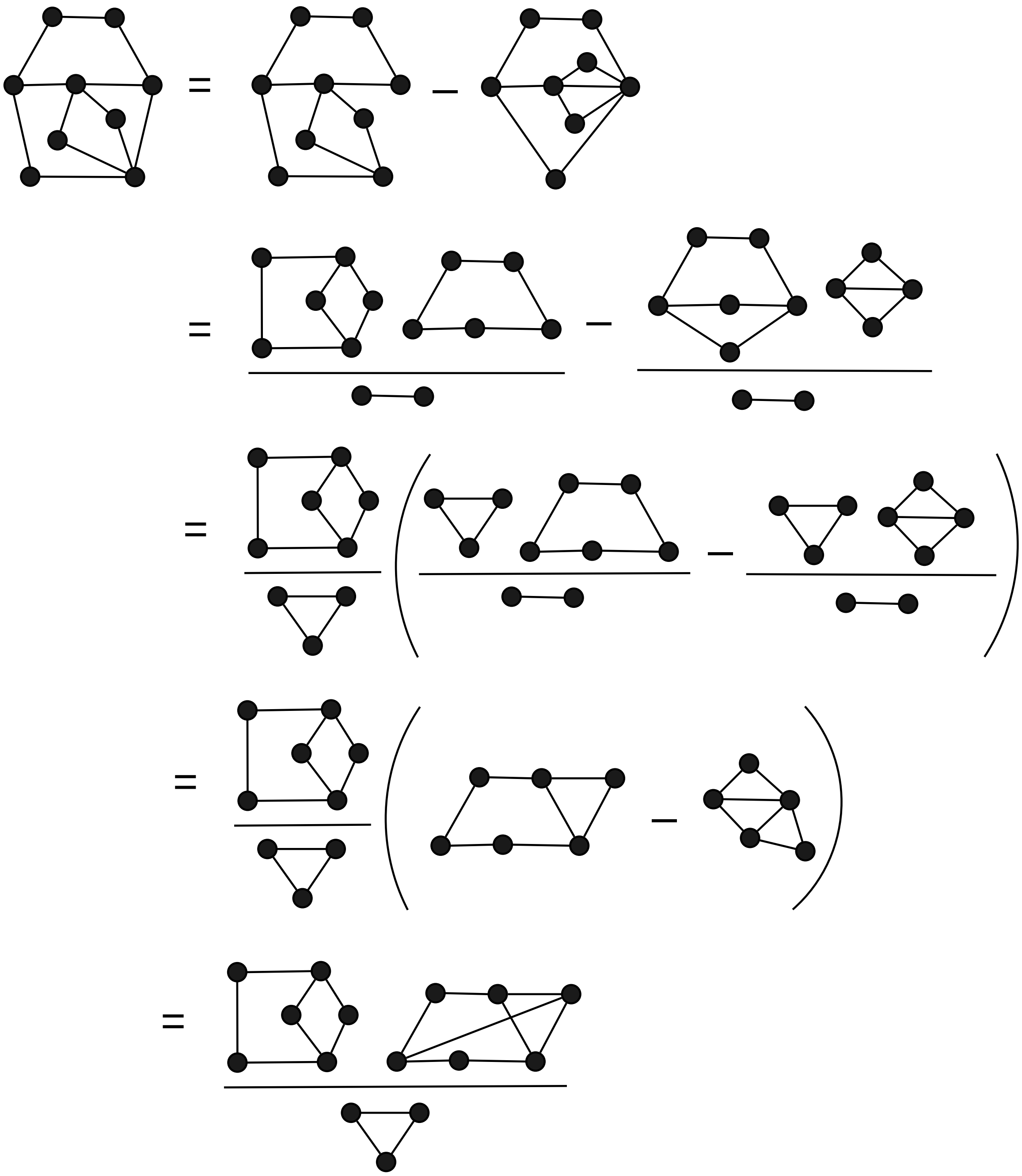

Figure 1: Example of chromatic factorisation satisfying Certificate B

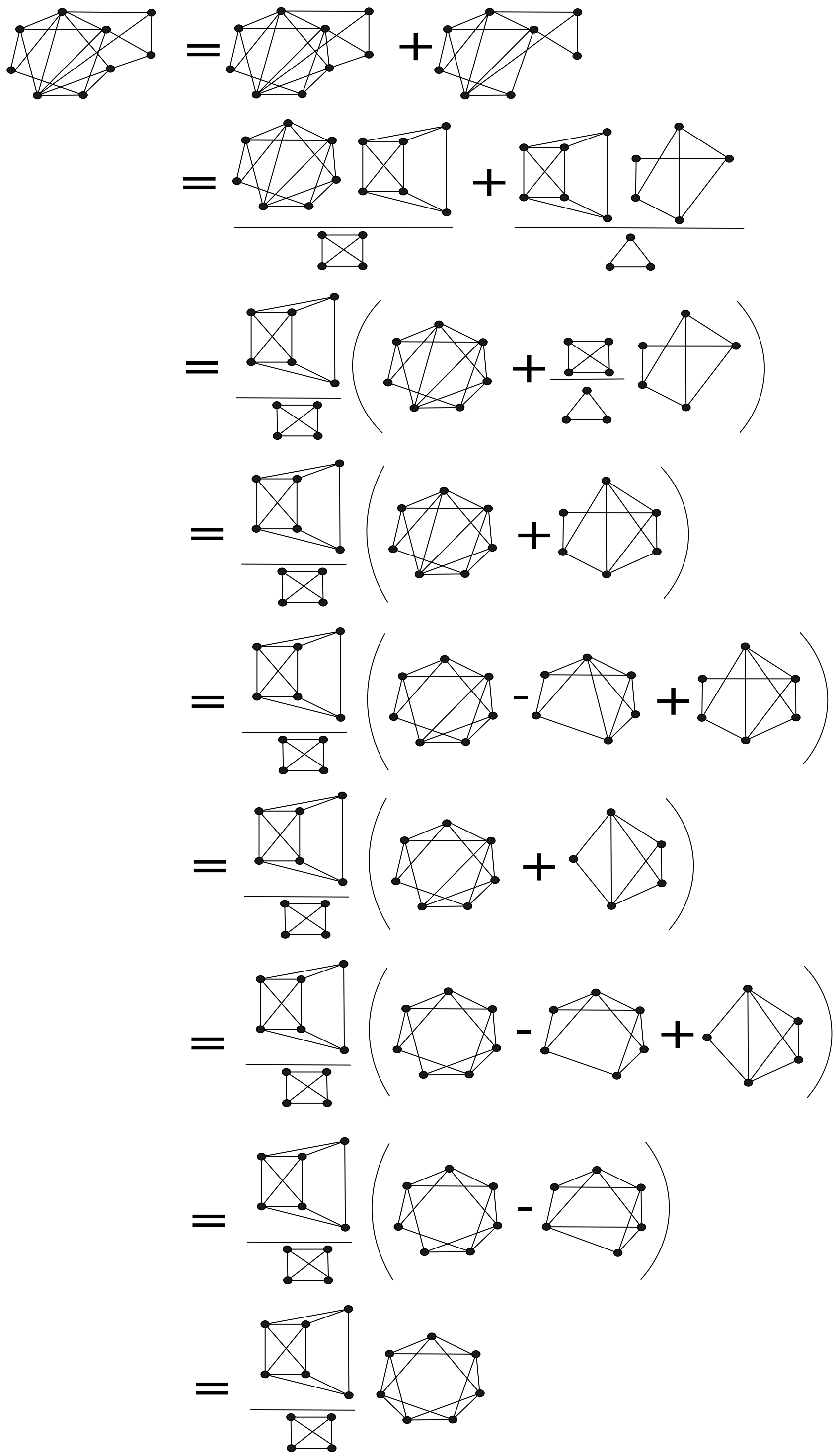

Figure 2: Example of chromatic factorisation satisfying Certificate C

	Certificate	# $P(G,\lambda)$ with degree 8	degree 9
$H_2 - e \cong H_5$, $e \in E(H_2)$ where H_5 is an s-gluing of H_3 and K_r	B	0	1
$H_2 - e - f + g \cong H_5$, $e, f \in E(H_2)$ and $g \notin E(H_2)$ where H_5 is an s-gluing of H_5 and K_r	D	2	9
$H_2 - e - f - g + h + i \cong H_5$, $e, f, g \in E(H_2)$ and $h, i \notin E(H_2)$ where H_5 is an s-gluing of H_3 and K_r	E	0	2
TOTAL:		2	12

Table 6: Relationship between graphs H_2 and H_5 in Certificate of Factorisation Schema 1 when graph H_1 is isolated by a single deletion-contraction .

non-clique-separable graphs that have chromatic factorisations, and provide a certificate of factorisation satisfying Schema 1 for these graphs. The length of this certificate is $O(1)$, which is a large improvement on the general upper bound of $n^2 2^{n^2/2}$ obtained by the more naive approach.

The shortest certificates we found for chromatic factorisations of strongly non-clique-separable graphs had less than 10 steps. However, it is not known if these are the shortest certificates for these graphs. Finding shortest certificates, in general, is likely to be difficult.

An open problem is the characterisation of graphs belonging to the same CF-class. Many of the certificates given in this article, particular those belonging to Schema 1, explain chromatic factorisations of graphs that are almost clique-separable. In [13] we give an infinite family of graphs that have a chromatic factorisation explained by Certificate B. These graphs are graphs that can be obtained by replacing two non-adjacent edges in K_4 with paths of length $2n-1$ and $2n$, $n \geq 2$. As there are infinitely many graphs in this family, we know that there exist infinitely many strongly non-clique-separable graphs that have chromatic factorisations. However, the proportion of graphs that are strongly non-clique-separable is unknown.

Another open question is which graphs can be chromatic factors. When is it possible to find a graph G that has a chromatic factorisation with chromatic factors, H_1 and H_2, where H_1 and H_2 are an arbitrary pair of r-colourable graphs? In [12] we show that any triangle-free graph H_1 with $\chi(H_1) \geq 3$ is a chromatic factor of some chromatic factorisation $P(H_1,\lambda)P(H_2,\lambda)/P(K_3,\lambda)$ explained by Certificate B. However, in this case the second chromatic factor H_2 must contain H_1 as a subgraph, and must contain a triangle.

Acknowledgement

We thank Alan Sokal and the referee for their suggestions and comments.

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Appendices

A Some Certificates of Factorisation

In this appendix a number of certificates of factorisation are presented. These certificates explain the factorisation of all chromatic polynomials of strongly non-clique-separable graphs of order at most 9. The certificates in Appendix A.1 are Schema 1 certificates. Some further certificates are presented in Appendix A.2.

A.1 Schema 1 certificates

The certificates in this section provide explanations for the factorisations of all the degree 8 and 16 of the degree 9 chromatic polynomials of strongly non-clique-separable graphs. Tables 5 and 6 provide a breakdown of the numbers of these polynomials that satisfy each certificate.

$$\begin{aligned}
P(G,\lambda) =& P(G+uv,\lambda) + P(G/uv,\lambda) \\
=& \frac{P(H_1,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_1,\lambda)P(H_4,\lambda)}{P(K_1,\lambda)} \\
=& \frac{P(H_1,\lambda)}{P(K_3,\lambda)}\left(\frac{P(K_3,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(K_3,\lambda)P(H_4,\lambda)}{P(K_1,\lambda)}\right) \\
=& \frac{P(H_1,\lambda)}{P(K_3,\lambda)}\left(\frac{P(K_3,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(K_2,\lambda)P(K_3,\lambda)P(H_4,\lambda)}{P(K_2,\lambda)P(K_1,\lambda)}\right) \qquad (22) \\
=& \frac{P(H_1,\lambda)}{P(K_3,\lambda)}\left(P(H_2+wx,\lambda) + P(H_2/wx,\lambda)\right) \qquad (23) \\
=& \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_3,\lambda)}. \qquad (24)
\end{aligned}$$

Certificate A. (Schema 1)

$$
\begin{aligned}
P(G,\lambda) =& P(G \setminus e,\lambda) - P(G/e,\lambda)\\
=& \frac{P(H_1,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} - \frac{P(H_1,\lambda)P(H_4,\lambda)}{P(K_2,\lambda)}\\
=& \frac{P(H_1,\lambda)}{P(K_3,\lambda)}\left(\frac{P(K_3,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} - \frac{P(K_3,\lambda)P(H_4,\lambda)}{P(K_2,\lambda)}\right)\\
=& \frac{P(H_1,\lambda)}{P(K_3,\lambda)}\left(P(H_2 \setminus f,\lambda) - P(H_2/f,\lambda)\right) \qquad (25)\\
=& \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_3,\lambda)}. \qquad (26)
\end{aligned}
$$

Certificate B. (Schema 1)

$$
\begin{aligned}
P(G,\lambda) =& P(G+uv,\lambda) + P(G/uv,\lambda)\\
=& \frac{P(H_1,\lambda)P(H_3,\lambda)}{P(K_4,\lambda)} + \frac{P(H_1,\lambda)P(H_4,\lambda)}{P(K_3,\lambda)}\\
=& \frac{P(H_1,\lambda)}{P(K_4,\lambda)}\left(\frac{P(K_4,\lambda)P(H_3,\lambda)}{P(K_4,\lambda)} + \frac{P(K_4,\lambda)P(H_4,\lambda)}{P(K_3,\lambda)}\right)\\
=& \frac{P(H_1,\lambda)}{P(K_4,\lambda)}\left(P(H_3,\lambda) + \frac{P(K_4,\lambda)P(H_4,\lambda)}{P(K_3,\lambda)}\right) \qquad (27)\\
=& \frac{P(H_1,\lambda)}{P(K_4,\lambda)}\left(P(H_2+e+f-g,\lambda) + P(H_6,\lambda)\right) \qquad (28)\\
=& \dots\\
=& \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_4,\lambda)}.
\end{aligned}
$$

Certificate C. (Schema 1)

$$
\begin{aligned}
P(G,\lambda) =& P(G\setminus e,\lambda) - P(G/e,\lambda)\\
=& \frac{P(K_5,\lambda)P(H_3,\lambda)}{P(K_4,\lambda)} - \frac{P(K_5,\lambda)P(H_4,\lambda)}{P(K_4,\lambda)}\\
=& \frac{P(K_5,\lambda)}{P(K_4,\lambda)}\left(\frac{P(K_4,\lambda)P(H_3,\lambda)}{P(K_4,\lambda)} - \frac{P(K_4,\lambda)P(H_4,\lambda)}{P(K_4,\lambda)}\right)\\
=& \frac{P(K_5,\lambda)}{P(K_4,\lambda)}\left(P(H_3,\lambda) - P(H_4,\lambda)\right) \qquad (29)\\
=& \frac{P(K_5,\lambda)}{P(K_4,\lambda)}\left(P(H_2 - e - f + g,\lambda) - P(H_4,\lambda)\right)\\
=& \dots\\
=& \frac{P(K_5,\lambda)P(H_2,\lambda)}{P(K_4,\lambda)}.
\end{aligned}
$$

Certificate D. (Schema 1)

$$
\begin{aligned}
P(G,\lambda) =& P(G\setminus e,\lambda) - P(G/e,\lambda)\\
=& \frac{P(K_5,\lambda)P(H_3,\lambda)}{P(K_4,\lambda)} - \frac{P(K_5,\lambda)P(H_4,\lambda)}{P(K_4,\lambda)}\\
=& \frac{P(K_5,\lambda)}{P(K_4,\lambda)}\left(\frac{P(K_4,\lambda)P(H_3,\lambda)}{P(K_4,\lambda)} - \frac{P(K_4,\lambda)P(H_4,\lambda)}{P(K_4,\lambda)}\right)\\
=& \frac{P(K_5,\lambda)}{P(K_4,\lambda)}\left(P(H_3,\lambda) - P(H_4,\lambda)\right) \qquad (30)\\
=& \frac{P(K_5,\lambda)}{P(K_4,\lambda)}\left(P(H_2 - e - f - g + h + i,\lambda) - P(H_4,\lambda)\right)\\
=& \dots\\
=& \frac{P(K_5,\lambda)P(H_2,\lambda)}{P(K_4,\lambda)}.
\end{aligned}
$$

Certificate E. (Schema 1)

A.2 Other Certificates of Factorisation

The certificates in the previous section did not explain all chromatic factorisations of chromatic polynomials of degree 9. Table 7 lists the numbers of chromatic polynomials of strongly non-clique-separable graphs of order nine with certificates not following Schema 1. This section presents the certificates used to explain these cases. Certificate F explains a chromatic factorisation with three chromatic factors. It uses similar techniques to those used in Schema 1. Certificates G and I express $P(G,\lambda)$ as the sum of at least three terms with each term having $P(H_1,\lambda)$ as a chromatic factor. In the case of Certificate G, the only difference from Schema 1 is that it requires both an addition-identification and a deletion-contraction operation to isolate $P(H_1,\lambda)$ rather than a single operation. Certificates H and J both use the chromatic factorisation of another strongly non-clique-separable graph. Certificate K only differs from Certificate D in the second step. In Certificate D, the graph G/e is isomorphic to the graph obtained by an r-gluing of K_5 and H_4. In Certificate K, the graph G/e is not isomorphic to the graph obtained by an r-gluing of K_5 and H_4, but is isomorphic to the graph obtained by a $(4,3)$-gluing of K_5, K_4 and H_4' where H_4 is isomorphic to a 3-gluing of K_4 and H_4'.

Certificate	# Chromatic Polynomials
F	1
G	3
H	1
I	1
J	1
K	2

Table 7: Number of chromatic polynomials of degree 9 satisfying non-Schema 1 Certificates.

$$
\begin{aligned}
P(G,\lambda) =& P(G\setminus e,\lambda) - P(G/e,\lambda)\\
=& P(G\setminus e+f,\lambda) + P(G\setminus e/f,\lambda) - P(G/e+g,\lambda) - P(G/e/g,\lambda)\\
=& \frac{P(K_4,\lambda)P(K_4,\lambda)P(H_3,\lambda)}{P(K_3,\lambda)P(K_3,\lambda)} + \frac{P(K_4,\lambda)P(H_4,\lambda)}{P(K_3,\lambda)}\\
& - \frac{P(K_4,\lambda)P(K_4,\lambda)P(H_5,\lambda)}{P(K_3,\lambda)P(K_3,\lambda)} - \frac{P(K_4,\lambda)P(H_6,\lambda)}{P(K_3,\lambda)}\\
=& \frac{P(K_4,\lambda)P(K_4,\lambda)}{P(K_3,\lambda)P(K_3,\lambda)}\left(P(H_3,\lambda) - P(H_5,\lambda)\right) + \frac{P(K_4,\lambda)}{P(K_3,\lambda)}\left(P(H_4,\lambda) - P(H_6,\lambda)\right)\\
=& \dots\\
=& \frac{P(K_4,\lambda)P(K_4,\lambda)}{P(K_3,\lambda)P(K_3,\lambda)}\left(P(H_3,\lambda) - P(H_5,\lambda)\right) + \frac{P(K_4,\lambda)P(K_4,\lambda)P(H_7,\lambda)}{P(K_3,\lambda)P(K_3,\lambda)}\\
=& \frac{P(K_4,\lambda)P(K_4,\lambda)}{P(K_3,\lambda)P(K_3,\lambda)}\left(P(H_3,\lambda) - P(H_5,\lambda) + P(H_7,\lambda)\right)\\
=& \frac{P(K_4,\lambda)P(K_4,\lambda)}{P(K_3,\lambda)P(K_3,\lambda)}\left(P(H_2+h-i,\lambda) - P(H_5,\lambda) + P(H_7,\lambda)\right)\\
=& \dots\\
=& \frac{P(K_4,\lambda)P(K_4,\lambda)P(H_2,\lambda)}{P(K_3,\lambda)P(K_3,\lambda)}
\end{aligned}
$$

Certificate F (3 chromatic factors).

$$
\begin{aligned}
P(G,\lambda) =& P(G\setminus e,\lambda) - P(G/e,\lambda)\\
=& P(G\setminus e,\lambda) - P(G/e\setminus f,\lambda) + P(G/e/f,\lambda)\\
=& \frac{P(K_5,\lambda)P(H_3,\lambda)}{P(K_4,\lambda)} - \frac{P(K_5,\lambda)P(H_4,\lambda)}{P(K_4,\lambda)} + \frac{P(K_5,\lambda)P(H_5,\lambda)}{P(K_4,\lambda)}\\
=& \frac{P(K_5,\lambda)}{P(K_4,\lambda)}\left(P(H_3,\lambda) - P(H_4,\lambda) + P(H_5,\lambda)\right)\\
=& \frac{P(K_5,\lambda)}{P(K_4,\lambda)}\left(P(H_2-g-h+i,\lambda) - P(H_4,\lambda) + P(H_5,\lambda)\right)\\
=& \dots\\
=& \frac{P(K_5,\lambda)P(H_2,\lambda)}{P(K_4,\lambda)}
\end{aligned}
$$

Certificate G.

$$
\begin{aligned}
P(G,\lambda) =& P(G \setminus e,\lambda) - P(G/e,\lambda) \\
=& P(G \setminus e \setminus f,\lambda) - P(G \setminus e/f,\lambda) - P(G/e,\lambda) \\
=& P(G \setminus e \setminus f+g,\lambda) + P(G \setminus e \setminus f/g,\lambda) - P(G \setminus e/f,\lambda) - P(G/e,\lambda) \\
& \text{But } G \setminus e \setminus f+g \cong G' \text{and the factorisation of } P(G',\lambda) \text{ is known} \\
=& P(G',\lambda) + P(G \setminus e \setminus f/g,\lambda) - P(G \setminus e/f,\lambda) - P(G/e,\lambda) \\
=& \frac{P(H_1,\lambda)P(H_2 \setminus h,\lambda)}{P(K_3,\lambda)} + P(G \setminus e \setminus f/g,\lambda) - P(G \setminus e/f,\lambda) - P(G/e,\lambda) \\
=& \ldots \\
=& \frac{P(H_1,\lambda)P(H_2 \setminus h,\lambda)}{P(K_3,\lambda)} - \frac{P(H_1,\lambda)P(H_2/h,\lambda)}{P(K_3,\lambda)} \\
=& \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_3,\lambda)}
\end{aligned} \tag{31}
$$

Certificate H.

$$
\begin{aligned}
P(G,\lambda) =& P(G \setminus e,\lambda) - P(G/e,\lambda) \\
=& P(G \setminus e \setminus f,\lambda) - P(G \setminus e/f,\lambda) - P(G/e,\lambda) \\
=& P(G \setminus e \setminus f+g,\lambda) + P(G \setminus e \setminus f/g,\lambda) - P(G \setminus e/f,\lambda) - P(G/e,\lambda) \\
=& \frac{P(H_1,\lambda)P(H_3,\lambda)}{P(K_4,\lambda)} + P(G \setminus e \setminus f/g,\lambda) - P(G \setminus e/f,\lambda) - P(G/e,\lambda) \\
& \text{Further certification steps to get each term having chromatic} \\
& \text{factor } P(H_1,\lambda) \\
=& \frac{P(H_1,\lambda)}{P(K_3,\lambda)} \left(P(H_3,\lambda) - P(H_4,\lambda) + P(H_5,\lambda) + P(H_6,\lambda)\right) \\
=& \frac{P(H_1,\lambda)}{P(K_3,\lambda)} \left(P(H_2-h-i+j,\lambda) - P(H_4,\lambda) + P(H_5,\lambda) + P(H_6,\lambda)\right) \\
=& \ldots \\
=& \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_3,\lambda)}
\end{aligned}
$$

Certificate I.

$$
\begin{aligned}
P(G,\lambda) =& P(G\setminus e,\lambda) - P(G/e,\lambda)\\
=& P(G\setminus e\setminus f,\lambda) - P(G\setminus e/f,\lambda) - P(G/e,\lambda)\\
=& P(G\setminus e\setminus f+g,\lambda) + P(G\setminus e\setminus f/g,\lambda) - P(G\setminus e/f,\lambda) - P(G/e,\lambda)\\
& \text{But } G\setminus e\setminus f+g \cong G'\text{and the factorisation of } P(G',\lambda) \text{ is known}\\
=& P(G',\lambda) + P(G\setminus e\setminus f/g,\lambda) - P(G\setminus e/f,\lambda) - P(G/e,\lambda)\\
=& \frac{P(H_1,\lambda)P(H_2\setminus h\setminus i+j,\lambda)}{P(K_4,\lambda)}\\
& + P(G\setminus e\setminus f/g,\lambda) - P(G\setminus e/f,\lambda) - P(G/e,\lambda)\\
=& \ldots\\
=& \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_4,\lambda)}
\end{aligned}
$$

Certificate J.

$$
\begin{aligned}
P(G,\lambda) =& P(G\setminus e,\lambda) - P(G/e,\lambda)\\
=& \frac{P(K_5,\lambda)P(H_3,\lambda)}{P(K_4,\lambda)} - \frac{P(K_4,\lambda)P(K_5,\lambda)P(H_4',\lambda)}{P(K_4,\lambda)P(K_3,\lambda)}\\
=& \frac{P(K_5,\lambda)}{P(K_4,\lambda)}\left(P(H_3,\lambda) - \frac{P(K_4,\lambda)P(H_4',\lambda)}{P(K_3,\lambda)}\right)\\
=& \frac{P(K_5,\lambda)}{P(K_4,\lambda)}\left(P(H_3,\lambda) - P(H_4,\lambda)\right) \qquad (32)\\
=& \frac{P(K_5,\lambda)}{P(K_4,\lambda)}\left(P(H_2-e-f+g,\lambda) - P(H_4,\lambda)\right)\\
=& \ldots\\
=& \frac{P(K_5,\lambda)P(H_2,\lambda)}{P(K_4,\lambda)}.
\end{aligned}
$$

Certificate K.