Distinguishing Chromatic Numbers of Bipartite Graphs

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Abstract

Extending the work of K. L. Collins and A. N. Trenk, we characterize connected bipartite graphs with large distinguishing chromatic number. In particular, if G is a connected bipartite graph with maximum degree $\Delta \geq 3$, then $\chi_D(G) \leq 2\Delta - 2$ whenever $G \not\cong K_{\Delta-1,\Delta}$, $K_{\Delta,\Delta}$.

1 Introduction

A *colouring* of a graph G is an assignment of labels (colours) to the vertices of G; the colouring is *proper* if and only if adjacent vertices receive different labels, and the colouring is *distinguishing* provided that no automorphism of G, other than the identity, preserves the labels. The *distinguishing chromatic number* of G, written $\chi_D(G)$, is the minimum number of labels required to produce a colouring that is both proper and distinguishing. The distinguishing chromatic number is introduced by K.L. Collins and A.N. Trenk [3], as a natural extension of the *distinguishing number* of a graph, defined by M.O. Albertson and K.L. Collins [1].

In [3], Collins and Trenk compute the distinguishing chromatic number for various classes of graphs. In particular, they characterize the connected graphs with maximum possible distinguishing chromatic number, showing that $\chi_D(G) = |V(G)|$ if and only if G is a complete multipartite graph [3, Theorem 2.3]. Further, they show that for a connected graph G with maximum degree Δ, $\chi_D(G) \leq 2\Delta$ with equality if and only if $G \cong K_{\Delta,\Delta}$ or $G \cong C_6$, a cycle of length six [3, Theorem 4.5]. For connected graphs with $\Delta \leq 2$, they also completely determine the distinguishing chromatic number. Note that if G is

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connected and has $\Delta \leq 2$, then G is either a path or a cycle. Let P_k denote a path with k vertices, and C_k a cycle with k vertices. Then

$$\chi_D(P_k) = \begin{cases} 2 & \text{for } k \geq 2 \text{ and even,} \\ 3 & \text{for } k \geq 3 \text{ and odd,} \end{cases}$$

and

$$\chi_D(C_k) = \begin{cases} 3 & \text{for } k = 3, 5, k \geq 7, \\ 4 & \text{for } k = 4, 6. \end{cases}$$

Since, for $\Delta \geq 3$, $\chi_D(G) = 2\Delta$ if and only if $G \cong K_{\Delta,\Delta}$, it is natural to consider the distinguishing chromatic number for the class of connected bipartite graphs. In this paper, we further characterize connected bipartite graphs with large distinguishing chromatic number, proving that, for $\Delta \geq 3$, $\chi_D(G) \leq 2\Delta - 2$ whenever $G \ncong K_{\Delta-1,\Delta}, K_{\Delta,\Delta}$. This solves Conjecture 5.1 of [3]. We also compute the distinguishing chromatic number of the complete bipartite graph minus a perfect matching; this provides an interesting example of a graph that is "close" to $K_{\Delta,\Delta}$, but whose distinguishing chromatic number is generally much less than 2Δ.

Unless otherwise specified, we us the notation and terminology of [2]. If G is a graph, and v is a vertex of G, we denote by $d(v)$ the degree of v in G. For any connected graph G, and any vertex $u \in V(G)$, one can easily construct a *breadth-first search* spanning tree rooted at u. In order to facilitate the proofs in Sections 3 and 4, we visualize such a spanning tree as a plane graph in which the children of a vertex in the tree are added in order from left to right (so the leftmost child is the first child added to the tree, and the rightmost child is the last child added to the tree).

The remainder of this paper is structured as follows. In Section 2 we provide an exact value for the distinguishing chromatic number of the complete bipartite graph minus a perfect matching. Section 3 contains a modification of the basic algorithm developed by Collins and Trenk [3] for giving G a distinguishing colouring with $2\Delta - 1$ colours. This is necessary preparation for the proof of the main result, presented in Section 4, where a colouring with $2\Delta - 1$ colours is modified to produce a colouring with $2\Delta - 2$ colours. Finally, Section 5 contains some discussion and open problems.

2 The distinguishing chromatic number of $K_{n,n} - M$

We denote by $K_{n,n} - M$ the graph obtained from the complete bipartite graph $K_{n,n}$ by deleting the edges of a perfect matching. This graph arises in the proof of our main result, but is also of interest on its own. Other than $K_{\Delta,\Delta}$, C_6 is the only graph with $\chi_D(G) = 2\Delta$ [3, Theorem 2.3]. But $K_{3,3} - M \cong C_6$, providing an alternate context for C_6; i.e., from Theorem 1, $\chi_D(C_6) = \chi_D(K_{3,3} - M) = \lceil 2\sqrt{3}\,\rceil = 4$, which happens to equal 2Δ. Also, $K_{n,n} - M$ is a regular bipartite graph with a high degree of symmetry, much like $K_{n,n}$, yet $\chi_D(K_{n,n} - M)$ is generally much less than 2Δ.

Definition 1. *Let G be a connected graph and c a colouring of G. An automorphism σ of G is called* **colour preserving** *if, for every $u \in V(G)$, $c(\sigma(u)) = c(u)$. A vertex u in*

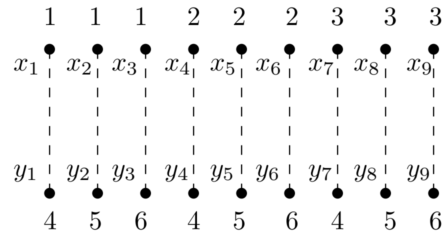

Figure 1: A distinguishing colouring of $K_{9,9} - M$.

$V(G)$ is **pinned** *(under the colouring c) if, for any colour preserving automorphism σ of G, $\sigma(u) = u$.*

Theorem 1. *Let $n \geq 3$, and let $G \cong K_{n,n} - M$, where M is a perfect matching in $K_{n,n}$. Then $\chi_D(G) = \lceil 2\sqrt{n}\,\rceil = \left\lceil 2\sqrt{\Delta + 1}\,\right\rceil$.*

Proof. Let G have bipartition (X, Y) where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, and $E(G) = \{x_i y_j \mid 1 \leq i \neq j \leq n\}$. Vertices x_i and y_i are said to be *corresponding*. We first describe a distinguishing colouring of G with $\lceil 2\sqrt{n}\,\rceil$ colours.

The case when n is a perfect square provides a general idea of how the colouring is defined, but is much simpler to describe than the general case, so we begin with this case. If n is a perfect square, then the vertices of both X and Y consist of $\sqrt{n}$ subsets of $\sqrt{n}$ vertices (the vertices of subset 1 having subscripts 1 through $\sqrt{n}$, those of subset 2 having subscripts $\sqrt{n}+1$ through $2\sqrt{n}$, etc.). In X, colour the vertices of the first subset with colour 1, the vertices of the second subset with colour 2, and so on, colouring the vertices of the last subset with colour $\sqrt{n}$. In Y, colour the vertices of each subset, in order from smallest to largest, with colours $\sqrt{n}+1, \sqrt{n}+2, \ldots, 2\sqrt{n}$. The case $n = 3^2$ is illustrated in Figure 1, where the dashed lines indicate the (missing) edges of the perfect matching.

More generally, let k be the positive integer so that $k^2 < n \leq (k+1)^2$; i.e., $k = \sqrt{n} - 1$ if n is a perfect square, and $k = \lfloor \sqrt{n} \rfloor$ otherwise.

If $k^2 < n \leq k(k+1)$, write $n = k^2 + r$ where $1 \leq r \leq k$. Partition X into sets as follows: for $1 \leq i \leq k$, define

$$X_i = \{x_{(i-1)k+1}, x_{(i-1)k+2}, \ldots, x_{ik}\},$$

and

$$X_{k+1} = \{x_{k^2+1}, x_{k^2+2}, \ldots, x_{k^2+r}\},$$

and assign colours to the vertices of X so that $c(x) = i$ if and only if $x \in X_i$, $1 \leq i \leq k+1$. Partition Y into sets as follows: for $1 \leq j \leq k$, define

$$Y_j = \{y_i \mid i \equiv j \pmod{k}\},$$

and assign colours to the vertices of Y so that $c(y) = k + j + 1$ if and only if $y \in Y_j$, $1 \leq j \leq k$. This results in a colouring of G with $2k+1$ colours. Since $k^2 < n \leq k(k+1)$, $k < \sqrt{n} \leq k + \frac{1}{2}$, and thus $2k < 2\sqrt{n} \leq 2k+1$. From this we conclude that the number of colours used is exactly $\lceil 2\sqrt{n}\,\rceil = 2k+1$.

If $k(k+1) < n \leq (k+1)^2$, write $n = k(k+1) + r$ where $1 \leq r \leq k+1$. Partition X into sets as follows: for $1 \leq i \leq r$, define

$$X_i = \{x_{(i-1)(k+1)+1}, x_{(i-1)(k+1)+2}, \dots, x_{i(k+1)}\},$$

and for $r+1 \leq i \leq k+1$, define

$$X_i = \{x_{(i-1)k+r+1}, x_{(i-1)k+r+2}, \dots, x_{ik+r}\}.$$

Then assign colours to the vertices of X so that $c(x) = i$ if and only if $x \in X_i$, $1 \leq i \leq k+1$. Partition Y into sets as follows: for $1 \leq j \leq k+1$, define

$$Y_j = \{y_i \mid i \equiv j \pmod{(k+1)}\},$$

and assign colours to the vertices of Y so that $c(y) = k + j + 1$ if and only if $y \in Y_j$, $1 \leq j \leq k+1$. This results in a colouring of G with $2k+2$ colours. Since $k^2 + k < n \leq (k+1)^2$ and n, k are integers, $k^2 + k + 1 \leq n \leq (k+1)^2$; also, $(k + \frac{1}{2})^2 < k^2 + k + 1$, and so $k + \frac{1}{2} < \sqrt{n} \leq k+1$. It follows that $2k + 1 < 2\sqrt{n} \leq 2k+2$. From this we conclude that the number of colours used is exactly $\lceil 2\sqrt{n}\, \rceil = 2k+2$.

This colouring is certainly proper: no colour occurs in both X and Y. To see that this colouring is distinguishing, observe that $x_i \in X$ is adjacent to every vertex of Y except its corresponding vertex, y_i ($1 \leq i \leq n$). The colouring has been defined so that if $c(x_i) = c(x_j)$ for some $1 \leq i \neq j \leq n$, then $c(y_i) \neq c(y_j)$; therefore, G has no colour preserving automorphism that maps x_i to x_j. Similarly, if $c(y_i) = c(y_j)$ for some $1 \leq i \neq j \leq n$, then $c(x_i) \neq c(x_j)$, so again there is no colour preserving automorphism of G that maps y_i to y_j. Thus this colouring pins all vertices of G.

We must now prove that any colouring of G with fewer than $\lceil 2\sqrt{n}\, \rceil$ colours results in G having a colour preserving automorphism. We say that a colouring c of G contains a **bad** configuration if, for some $i \neq j$, $c(x_i) = c(x_j)$ and $c(y_i) = c(y_j)$.

Suppose that G has a proper colouring that contains a bad configuration. Define the automorphism σ so that

$$\sigma(x_i) = x_j, \sigma(x_j) = x_i, \sigma(y_i) = y_j, \sigma(y_j) = y_i,$$

and $\sigma(u) = u$ otherwise. Then σ is a colour preserving automorphism of G.

To prove that $\chi_D(G) = \lceil 2\sqrt{n}\, \rceil$, it suffices to prove that any proper colouring of G with fewer than $\lceil 2\sqrt{n}\, \rceil$ colours contains a bad configuration. In order to do so, we require the following.

Claim. G has a distinguishing colouring with $\chi_D(G)$ colours so that no colour appears in both X and Y.

Proof. Let c be a distinguishing colouring of G with $\chi_D(G)$ colours. Suppose, without loss of generality, that colour 1 appears in both X and Y. We may assume that $c(x_1) = 1$; since x_1 is adjacent to every vertex of Y except y_1, it must be that $c(y_1) = 1$, and no other vertices in X or Y can be assigned colour 1. A colouring of G in which $c(x_i) = c(y_i)$ for each i is certainly not distinguishing, and thus for some i, $2 \leq i \leq n$, $c(x_i) \neq c(y_i)$. Without loss of generality, assume that $c(x_2) = 2$ and $c(y_2) = 3$; then no vertex in Y has colour 2. We recolour x_1 by setting $c(x_1) = 2$ (the same colour as x_2). The resulting colouring is still proper. If the colouring is not distinguishing, then there is a colour preserving automorphism, σ, with $\sigma(x_1) = u$ for some $u \in X$, $u \neq x_1$ ($u \notin Y$ because no vertex in Y has colour 2). But this is impossible because u must be adjacent to y_1, which has colour 1, but x_1 has no neighbour of colour 1. □

Suppose that c is a proper distinguishing colouring of G with fewer than $\lceil 2\sqrt{n}\ \rceil$ colours. By the preceding Claim, we may assume that no colour appears in both X and Y; let p denote the number of colours used to colour X, and let q denote the number of colours used to colour Y. By the Pigeonhole Principle, some colour in Y occurs on at least $\lceil n/q \rceil$ vertices; since G contains no bad configurations, the corresponding vertices must have distinct colours, and thus the number of colours used in X must be at least $\lceil n/q \rceil$; i.e., $p \geq \lceil n/q \rceil$. Therefore $pq \geq n$. Since $p + q \leq \lceil 2\sqrt{n}\ \rceil - 1$ and $pq \leq \left(\frac{p+q}{2}\right)^2$, it is routine to verify that $pq < n$, a contradiction. □

3 General machinery: proper distinguishing colouring with $2\Delta - 1$ colours

Let G be a connected bipartite graph with $\Delta \geq 3$, and assume $G \ncong K_{\Delta,\Delta}$. In this section, we describe the general method for producing a proper distinguishing colouring of G with at most $2\Delta - 1$ colours. This is a modification of the algorithm of Collins and Trenk [3], and is the basis for our later result.

Suppose G has bipartition (X, Y). Choose $r \in V(G)$ to be a vertex of minimum degree in G; without loss of generality, we may assume $r \in Y$. Construct T, a breadth-first search spanning tree rooted at r; a vertex at distance i from r is in *level* i of T, and a level is *even (odd)* if the level number is even (odd). Observe that any vertex of X is therefore in an odd level, and any vertex of Y is in an even level.

We construct a proper colouring $c : V(G) \to \{1, 2, \ldots, 2\Delta - 1\}$ using a modification of the algorithm of [3]. Begin by setting $c(r) = 2\Delta - 1$, and then discard colour $2\Delta - 1$; this clearly pins vertex r. The vertices of X will be coloured with colours from the set $\{1, 2, \cdots \Delta - 1\}$, and the vertices of $Y \backslash \{r\}$ will be coloured with colours from the set $\{\Delta, \Delta + 1, \ldots, 2\Delta - 2\}$. If h denotes the height of the tree, then for each level i, $i = h, h-1, \ldots, 2, 1$, colour vertices in level i from left to right in the order they are added to T so that each vertex is assigned the smallest available colour (from its corresponding set) that does not appear among its siblings.

If $d(r) \leq \Delta - 1$, this results in a proper distinguishing colouring of G: it is proper because any edge of G joins vertices in adjacent levels of T, and the colours on adjacent levels of T form disjoint sets; it is distinguishing because for each $u \in V(G)$, the children of u in T have distinct colours, so once r is pinned, the remaining vertices of T, and hence of G, are pinned.

If $d(r) = \Delta$, it is not possible to complete the colouring of G as prescribed, since there are Δ vertices in level 1 that should receive distinct colours, but there are only $\Delta - 1$ available colours. However, since $d(r) = \Delta$, the choice of r in this case implies that G is Δ-regular. Also, since $G \ncong K_{\Delta,\Delta}$, there must exist $x, y \in N(r)$ so that $N(x) \neq N(y)$. Let $N(r) = \{a_1, a_2, \ldots, a_{\Delta-2}, x, y\}$, and assume without loss of generality that the vertices in $N(r)$ are added to T in the order $a_1, a_2, \ldots, a_{\Delta-2}, x, y$ (so that x and y are the last two children of r). Now recolour the vertices of G in level 1 so that $c(a_i) = i$ and $c(x) = c(y) = \Delta - 1$. The result is a proper colouring of G. If this colouring is also distinguishing, there is nothing to do. Otherwise, define G' to be the subgraph of G induced by r and $a_1, a_2, \ldots, a_{\Delta-2}$, along with the descendants of $a_1, a_2, \ldots, a_{\Delta-2}$ in T. Suppose that σ is a nontrivial colour preserving automorphism of G. Then the vertices of G' are pinned, and σ must exchange x and y; i.e., $\sigma(x) = y$ and $\sigma(y) = x$.

Define S_x to be the set of children of x in T that are not adjacent to y, and similarly define S_y to be the set of children of y in T. Because T is a breadth-first search spanning tree, no vertex of S_y is adjacent to x. Since σ exchanges x and y, σ must also exchange S_x and S_y. We claim that $S_y \neq \emptyset$. To justify this, recall that $N(x) \neq N(y)$ but $d(x) = d(y) = \Delta$. Thus, there exists some vertex $z \in N(y) \backslash N(x)$. If $z \in V(G')$, then z is pinned; but $zy \in E(G)$ and $zx \notin E(G)$ implies that σ can not exchange x and y, a contradiction. Therefore $z \in S_y$, so $S_y \neq \emptyset$. (In fact, this proves $N(y) \setminus N(x) \subseteq S_y$.)

Suppose that $|S_y| < \Delta - 1$; then no child of y receives colour $2\Delta - 2$ (the algorithm dictates that the children of y be coloured with colours $\Delta, \Delta + 1, \ldots, \Delta - 1 + |S_y|$). Since σ exchanges S_x with S_y, no vertex in S_x is coloured $2\Delta - 2$. Now recolour the rightmost child of y with $2\Delta - 2$. This results in a proper colouring, and because colour $2\Delta - 2$ appears in S_y but not S_x, σ is no longer a colour preserving automorphism of G. This implies that x and y are pinned, and since the children of x and y in T each have distinct colours, the colouring is distinguishing as desired.

Finally, if $|S_y| = \Delta - 1$, then $|S_x| = \Delta - 1$ as well, and the children of each of x and y are coloured $\Delta, \Delta + 1, \ldots, 2\Delta - 2$ in order from left to right. Choose x' and y' as the two rightmost children of y in T, and recolour y' so that $c(x') = c(y') = 2\Delta - 3$. This is still a proper colouring, and because $2\Delta - 2$ appears as a colour in S_x but is no longer a colour in S_y, σ is no longer a colour preserving automorphism of G. Therefore x and y are pinned under c, and since the children of x are coloured distinctly, all descendants of x are pinned under any colour preserving automorphism. Similarly, the children of y coloured $\Delta, \Delta+1, \ldots, 2\Delta-4$, along with their descendants, are pinned. If x' and y' are also pinned, then the colouring is distinguishing. Otherwise, there is a colour preserving automorphism σ' that exchanges x' and y'. When constructing T, if it is not possible to choose the children of y so that $N(x') \neq N(y')$, then for any two vertices $u, v \in S_y$, $N(u) = N(v)$ (see Figure 2[‡]). This implies that the subgraph of G induced by $S_y \cup \{N(y')\backslash\{y\}\}$ is isomorphic to $K_{n-1,n-1}$. We now construct a different breadth-first search spanning tree

[‡]In all figures, solid lines denote edges of the spanning tree; dotted lines are edges of G that may or may not be in the spanning tree.

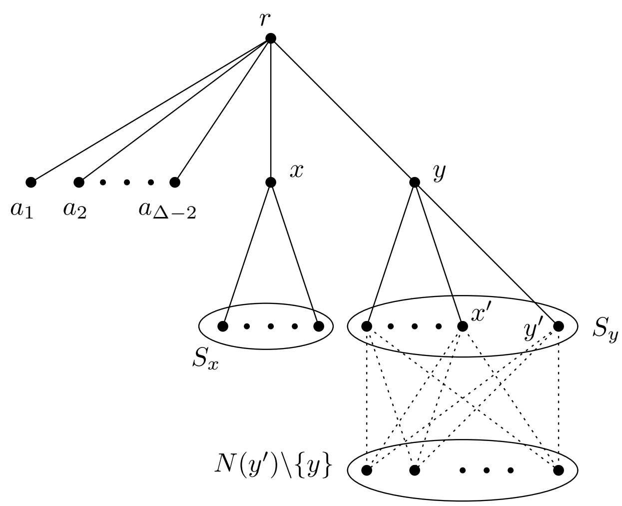

Figure 2: $N(u) = N(v)$ for all $u, v \in S_y$.

rooted at x', and proceed as before, with the two rightmost children of x', say u and v, chosen so that $N(u) \neq N(v)$. Notice that in this case, $y' \in N(u) \cap N(v)$, implying that $|S_v| < \Delta - 1$. Therefore we obtain the desired proper distinguishing colouring of G as described.

We may now assume that $N(x') \neq N(y')$, and proceed by induction on the number of levels of T, with x replaced by x', and y replaced by y'. Because T has finite height, this process eventually ends and results in a proper distinguishing colouring of G.

It is important to note that after any recolouring, the vertices of X are still coloured with colours from the set $\{1, 2, \ldots, \Delta - 1\}$, while the vertices of $Y \backslash \{r\}$ are coloured with colours from the set $\{\Delta, \Delta + 1, \ldots, 2\Delta - 2\}$. The vertex r is the only vertex of G to receive colour $2\Delta - 1$. In the next section, we describe ways to eliminate colour $2\Delta - 1$, the precise details of which depend on the structure of G.

4 Proper distinguishing colouring with $2\Delta - 2$ colours

Theorem 2. *If G is a connected bipartite graph with maximum degree $\Delta \geq 3$, and $G \ncong K_{\Delta-1,\Delta}, K_{\Delta,\Delta}$, then $\chi_D(G) \leq 2\Delta - 2$.*

Before proceeding with the proof, we observe that $\chi_D(K_{\Delta-1,\Delta}) = 2\Delta - 1$ [3, Theorem 2.3], and so it follows from Theorem 2 that:

Corollary 3. *If G is a connected bipartite graph with maximum degree $\Delta \geq 3$, then $\chi_D(G) = 2\Delta - 1$ if and only if $G \cong K_{\Delta-1,\Delta}$.*

The proof of Theorem 2 uses, as an initial colouring, the proper distinguishing colouring with $2\Delta - 1$ colours that is described in Section 3. This colouring is subsequently modified to eliminate colour $2\Delta-1$; the modification requires several cases, based on the structure of the graph.

Proof. Let G be a connected bipartite graph with maximum degree $\Delta \geq 3$, and assume $G \ncong K_{\Delta,\Delta}, G \ncong K_{\Delta-1,\Delta}$. We assume throughout that G has bipartition (X, Y), and choose $r \in V(G)$ to be a vertex of minimum degree in G; without loss of generality, we assume $r \in Y$. Following the method described in Section 3, construct a breadth-first search spanning tree T rooted at r, and give G the corresponding proper distinguishing colouring with $2\Delta-1$ colours as specified. The proof proceeds with three main cases, depending on δ, the minimum degree of G; in each case, the initial colouring is appropriately modified.

Case 1. $\delta \leq \Delta - 2$.

Since $d(r) = \delta \leq \Delta - 2$, colour $\Delta - 1$ does not occur on any vertex of $N(r)$, so modify the colouring by setting $c(r) = \Delta - 1$. This results in a proper colouring with $2\Delta-2$ colours. To see that this colouring is distinguishing, note that r is the only vertex with colour $\Delta - 1$ that is adjacent to a vertex of colour 1 (the first child of r in T); any other vertex coloured $\Delta - 1$ is in X, and all its neighbours are in Y (coloured from $\{\Delta, \Delta+1, \ldots, 2\Delta-2\}$). Therefore r is pinned, and it follows that the remaining vertices of G are pinned.

Case 2. $\delta = \Delta - 1$.

In this case, every vertex of G has degree Δ or $\Delta - 1$. Note that $\delta = \Delta - 1$ implies that G contains at least one other vertex $p \neq r$ with $d(p) = \Delta - 1$. To see this, assume $r \in Y$ is the unique vertex in G with degree $\Delta-1$. Since every edge of G is incident with one vertex of X and one vertex of Y, we have $|E(G)| = \sum_{u \in X} d(u) \equiv 0 \pmod{\Delta}$, and $|E(G)| = \sum_{v \in Y} d(v) \equiv -1 \pmod{\Delta}$, a contradiction.

Choose $p \neq r$ to be a vertex with degree $\Delta-1$ whose distance from r in G is minimum.

Case 2a. $d(r, p) = 1$.

We may assume that p is the first vertex of $N(r)$ to be added to T, so $c(p) = 1$. The $\Delta - 2$ vertices of $N(p)\backslash\{r\}$ are children of p in T, and so are coloured using colours $\{\Delta, \Delta + 1, \ldots, 2\Delta - 3\}$; i.e., no vertex in $N(p)$ has colour $2\Delta - 2$. We now recolour p and r: set $c(p) = 2\Delta - 2$ and $c(r) = 1$. This colouring is still proper. Furthermore, the colouring is distinguishing because r is pinned (r is the only vertex in Y with colour 1), and the children of every vertex in T are distinctly coloured. Thus, colour $2\Delta - 1$ has been eliminated, completing Case 2a.

In what follows, $d(r, p) \geq 2$, implying that each vertex in $N(r)$ has degree Δ. Let $P = ra_1a_2 \ldots a_kp$ be a shortest path between r and p in G. Construct the breadth-first search spanning tree T so that a_1 is the first child of r, a_j is the leftmost descendant of a_1 on level j ($2 \leq j \leq k$), and p is the first child of a_k. The colouring scheme described in Section 3 results in a proper distinguishing colouring of G in which $c(a_j) = 1$ if j is odd, $c(a_j) = \Delta$ if j is even, and $c(p) = 1$ or $c(p) = \Delta$ according as p's level is odd or even.

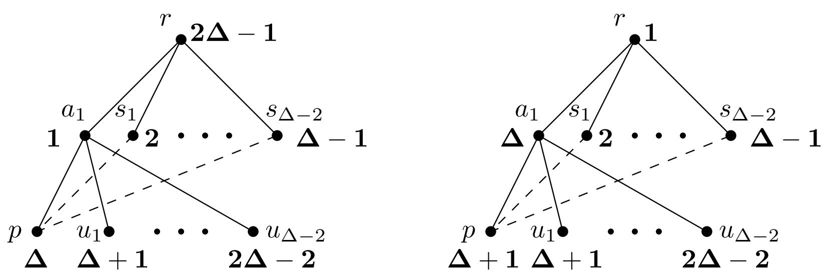

Figure 3: Case 2b, $N(p) = N(r)$. Initial and modified colouring.

Case 2b. $d(r, p) = 2$.

Let $N(r) = \{a_1, s_1, s_2, \ldots, s_{\Delta-2}\}$ and $N(a_1) = \{r, p, u_1, u_2, \ldots, u_{\Delta-2}\}$. If $N(p) = N(r)$, then we have the situation depicted in Figure 3.[§] Since $G \not\cong K_{\Delta-1,\Delta}$, there is some i $(1 \leq i \leq \Delta - 2)$ for which $N(u_i) \neq N(p)$. When constructing T, we may choose u_1 so that $N(u_1) \neq N(p)$, and $c(u_1) = \Delta + 1$. We now recolour p, a_1 and r: set $c(p) = \Delta + 1 (= c(u_1))$, $c(a_1) = \Delta$ and $c(r) = 1$. This modified colouring is still proper. In addition, it is distinguishing: r is pinned (r is the only vertex of colour 1 in Y), so $N(r)$ is pinned (vertices in $N(r)$ are distinctly coloured). Vertices $p, u_1, u_2, \ldots, u_{\Delta-2}$ are pinned because $N(p) \neq N(u_1)$ and $u_1, \ldots, u_{\Delta-2}$ are coloured distinctly. The other vertices of G remain pinned as before.

If $N(p) \neq N(r)$, then p has at least one child in T, and the first child of p receives colour 1. Thus p has two neighbours coloured 1 (its first child and a_1), implying that some colour, t, in $\{2, \ldots, \Delta - 1\}$ does not occur on any vertex of $N(p)$. We now recolour p, a_1, and r: set $c(p) = t$, $c(a_1) = \Delta$, and $c(r) = 1$. This modified colouring is still proper. In addition, it is distinguishing: r is pinned (r is the only vertex of colour 1 in Y), and the children of every vertex in T are distinctly coloured.

Case 2c. $d(r, p) \geq 3$.

First assume that $d(r, p)$ is odd, i.e., $p \in X$ and $c(p) = 1$. For notational convenience, let $a_0 = r$.

If $N(p) = N(a_{k-1}) \backslash \{a_{k-2}\}$, then we have the situation depicted in Figure 4. In this case, choose a different breadth-first search spanning tree, T', this one with root p, and add vertices to T' so that a_k is the first child of p, and a_{k-1} is the first child of a_k (see Figure 5). Colour G using the scheme from Section 3. Then $c(p) = 2\Delta - 1$, $c(a_k) = 1$, and $c(a_{k-1}) = \Delta$. Also, since $d(a_{k-1})$ is one more than $d(p)$, a_{k-1} has a unique child, q, in level 3 of T', and $c(q) = 1$. We now modify the colouring as follows: set $c(q) = 2$, $c(a_{k-1}) = 1$, $c(a_k) = \Delta$ and $c(p) = 1$. This colouring is still proper: no neighbour of q has colour 2. To see that this colouring is distinguishing, observe that p is the only vertex of degree $\Delta - 1$ in Y that has colour 1, and so p is pinned. The remaining vertices of G are pinned because, in T', the children of any vertex are coloured with distinct colours.

[§]In all figures, dashed lines are edges of G that are not in the spanning tree; boldface labels such as $\mathbf{1}, \mathbf{2}, \boldsymbol{\Delta}$, etc. denote colours.

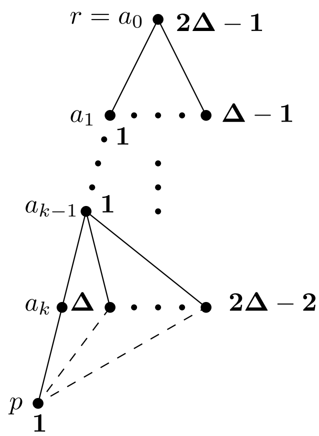

Figure 4: Case 2c, $N(p) = N(a_{k-1}) \backslash \{a_{k-2}\}$.

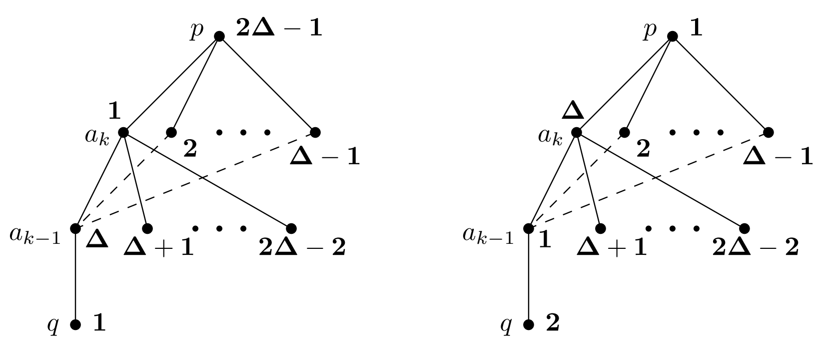

Figure 5: Case 2c, $N(p) = N(a_{k-1}) \backslash \{a_{k-2}\}$. Initial and modified colouring.

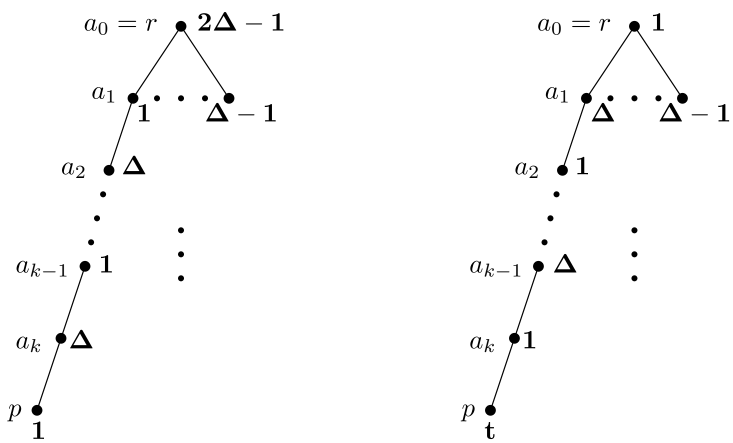

Figure 6: Case 2c, $N(p) \neq N(a_{k-1})\backslash\{a_{k-2}\}$. Initial and modified colouring.

We are now left with the situation in which $N(p) \neq N(a_{k-1})\backslash\{a_{k-2}\}$. If p has at least one child in T, then a_k and p's first child are both coloured Δ. If p has no children in T, then every vertex of $N(p)$ is in level k of T, and since $N(p) \neq N(a_{k-1})\backslash\{a_{k-2}\}$, we may choose to add vertices to T in an order that ensures p has at least one neighbour, other than a_k, with colour Δ. Thus p has at least two neighbours coloured Δ, so there exists a colour t, $\Delta + 1 \leq t \leq 2\Delta - 2$ that does not occur on any vertex of $N(p)$. We now modify the colouring as follows (see Figure 6): set $c(p) = t$, $c(r) = 1$, and $c(a_j) = 1$ if j is even, $c(a_j) = \Delta$ if j is odd (exchanging colours 1 and Δ along the path $a_1 a_2 \ldots a_k$). The colouring is still proper. To see that this colouring is distinguishing, observe that r is the only vertex of degree $\Delta - 1$ in Y that has colour 1, and so r is pinned. The remaining vertices of G are pinned because, in T, the children of any vertex are coloured with distinct colours.

Now suppose that $d(r, p)$ is even, i.e., $p \in Y$ and $c(p) = \Delta$. We proceed as when $d(r, p)$ is odd, up to the point that $N(p) \neq N(a_{k-1})\backslash\{a_{k-2}\}$. An analogous argument allows us to conclude that p has at least two neighbours coloured 1, and hence there is some t, $2 \leq t \leq \Delta - 1$ that does not occur on any vertex of $N(p)$. The remainder of the proof is identical to $d(r, p)$ odd.

Case 3. $\delta = \Delta$.

In this case G is a Δ-regular bipartite graph, and is not a tree, so has at least one cycle. The length of a shortest cycle in G is the *girth* of G, and is denoted g; because G is bipartite, g is even. Let C be any cycle of length g, and choose r (the root of the breadth-first search spanning tree T) to be a vertex of C. Because T is a breadth-first search spanning tree and r lies in a cycle of length $g = 2k$ (for some integer $k \geq 2$), there is an edge of C that is not in T but joins a vertex on level $k - 1$ of T to a vertex on level k of T. Furthermore, for $1 \leq j \leq k - 1$, no edge in $E(G)\backslash E(T)$ joins a vertex in level $j - 1$ to a vertex in level j. We consider two cases, depending on g.

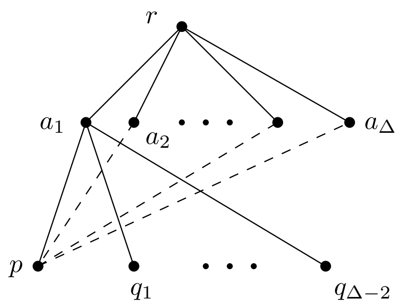

Figure 7: Case 3a(i).

Case 3a. $g = 4$.

If $G \cong K_{\Delta+1,\Delta+1} - M$, where M is a perfect matching in $K_{\Delta+1,\Delta+1}$, then by Theorem 1, $\chi_D(G) = \left\lceil 2\sqrt{\Delta+1}\,\right\rceil$. For $\Delta \geq 3$, $\left\lceil 2\sqrt{\Delta+1}\,\right\rceil \leq 2\Delta - 2$, and hence $\chi_D(G) \leq 2\Delta - 2$. Thus, in what follows, we assume $G \ncong K_{\Delta+1,\Delta+1} - M$.

Case 3a(i). There exist two vertices in G with same neighbourhood.

Without loss of generality, we may assume that we have $r, p \in V(G)$ such that $N(r) = N(p) = \{a_1, a_2, \ldots, a_\Delta\}$. This case is almost identical to the situation in Case 2b when $N(p) = N(r)$; the only difference is that, in this case, $d(r) = d(p) = \Delta$, whereas in Case 2b, $d(r) = d(p) = \Delta - 1$. Since $G \ncong K_{\Delta,\Delta}$, there exist i and j so that $N(a_i) \neq N(a_j)$; again without loss of generality, we may assume that $N(a_{\Delta-1}) \neq N(a_\Delta)$.

With r as usual the root of the breadth-first search spanning tree, T, we may assume that the vertices of $N(r)$ are added to T in the order $a_1, a_2, \ldots, a_\Delta$, from left to right, and that p is the first child of a_1 added to T (see Figure 7). Let $p, q_1, q_2, \ldots, q_{\Delta-2}$ denote the children of a_1 in the order that are added to T (from left to right). Since $G \ncong K_{\Delta,\Delta}$, there exists some j so that $N(q_j) \neq N(p)$ $(= N(r))$; we may assume that the vertices are added to T in an order so that $N(q_1) \neq N(p)$. In the initial colouring (with at most $2\Delta - 1$ colours), we have $c(r) = 2\Delta - 1$; $c(a_j) = j$ for $1 \leq j \leq \Delta - 1$, and $c(a_\Delta) = c(a_{\Delta-1}) = \Delta - 1$; $c(p) = \Delta$, and $c(q_j) = \Delta + j$ for $1 \leq j \leq \Delta - 2$.

We now recolour by setting $c(p) = c(q_1) = \Delta + 1$, $c(a_1) = \Delta$ and $c(r) = 1$. It is routine to verify that the colouring is still proper. To see that it is distinguishing, note that r is the only vertex in Y of colour 1, and this pins r. If the vertices in $N(r)$ were not pinned, then there would exist a nontrivial colour preserving automorphism, σ, exchanging $a_{\Delta-1}$ and a_Δ; but this would have been a colour preserving automorphism before p, a_1 and r were recoloured, a contradiction. Therefore, the vertices in $N(r)$ are pinned. It follows that p is pinned, since $N(p) = N(r)$ which is pinned, and the only other vertices that could have neighbourhood $N(p)$ are $r, q_2, \ldots, q_{\Delta-2}$, none of which have colour $c(p) = \Delta + 1$. It now follows that the remaining vertices of G are pinned, and so we have successfully eliminated colour $2\Delta - 1$.

Case 3a(ii). Any two distinct vertices have distinct neighbourhoods.

Recall that r, the root of the breadth-first search spanning tree T, is a vertex on a cycle C of length four. Denote the vertices of $N(r)$ by $a_1, a_2, \ldots, a_\Delta$, from left to right, in

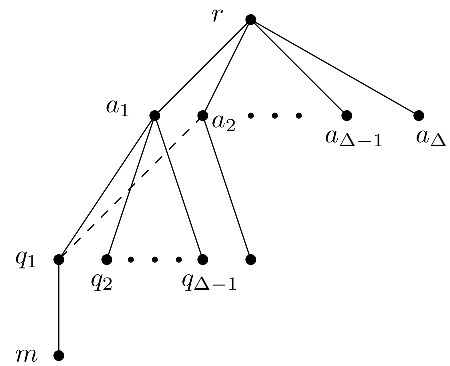

Figure 8: Case 3a(ii).

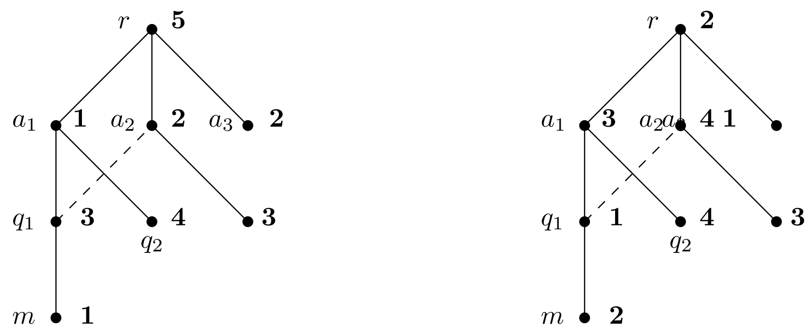

Figure 9: Case 3a(ii). Initial and final colouring when $\Delta = 3$.

the order that they are added to T, and denote the children of a_1 in T by $q_1, q_2, \ldots, q_{\Delta-1}$, from left to right, in the order that they are added to T. Without loss of generality, we may assume that the neighbours of r and a_1 are added so that $C = ra_1q_1a_2r$ is a cycle of length four in G. This situation is depicted in Figure 8. Since $N(a_2) \neq N(a_1)$, we may assume that $q_{\Delta-1} \notin N(a_2)$, and that a_2 has at least one child in T. Also, since $N(q_1) \neq N(r)$, q_1 has at least one child in T; let m denote the first child of q_1 in T.

Two key observations are: (1) q_1 has (at least) two neighbours in level 1 of T, and thus has no child coloured $\Delta - 1$; (2) a_2 is adjacent to q_1 in G, implying it has at most $\Delta - 2$ children in T, so a_2 has no child coloured $2\Delta - 2$; since a_2 is not adjacent to $q_{\Delta-1}$, it follow that a_2 has no neighbour in G with colour $2\Delta - 2$.

For $\Delta > 3$, recolour by setting $c(r) = 2$ and $c(a_2) = 2\Delta - 2$. This is a proper colouring since $\Delta > 3$ (so r is not connected to any vertex of colour 2), and distinguishing since r is pinned (as the only vertex in Y of colour 2).

If $\Delta = 3$, then recolour by setting $c(m) = \Delta - 1$ (=2), $c(q_1) = 1$, $c(a_1) = \Delta(= 3)$, $c(a_2) = 2\Delta - 2$, $c(r) = 2$, and $c(a_3) = 1$ (see Figure 9). Since $N(q_1) \neq N(r)$, $q_1 \notin N(a_3)$ and the resulting colouring is proper. To see that this colouring is distinguishing, note that r is the only vertex in Y that has colour 2, q_1 is the only vertex in Y that has colour

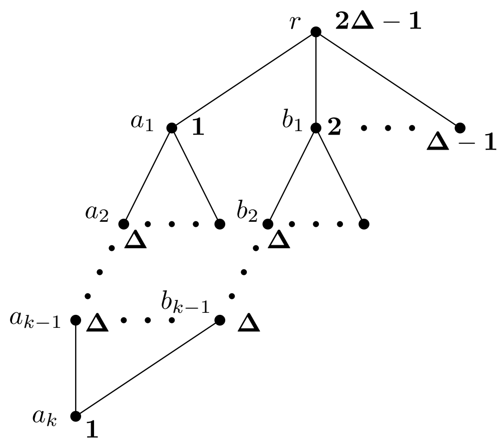

Figure 10: $g = 2k$, k odd. Initial and modified colouring.

1, a_1 is the only vertex in X that has colour Δ, a_2 is the only vertex in X that has colour $2\Delta - 2$. Thus, vertices r, q_1, a_1 and a_2 are all pinned, unless there is a colour preserving automorphism, σ that exchanges X and Y; this is only possible if $|X| = |Y| = 4$, and σ exchanges q_1 with a_3, m with r, q_2 with a_2, and a_1 with some vertex $p \in Y$. It follows that $G \cong K_{4,4} - M$, a contradiction. Therefore, r is pinned, and this is sufficient to pin the remaining vertices in G.

Case 3b. $g \geq 6$.

We may assume, without loss of generality, that vertices are added to T in an order that ensures that

$$C = ra_1a_2 \dots a_{k-1}a_kb_{k-1}b_{k-2} \dots b_2b_1r,$$

where a_1, b_1 are the two leftmost children of r in T, a_j is the leftmost descendant of a_1 on level j ($2 \leq j \leq k$), and b_j is the leftmost descendant of b_1 on level j ($2 \leq j \leq k-1$). The proper distinguishing colouring scheme described in Section 3 ensures that $c(a_1) = 1$, $c(b_1) = 2$, $c(a_j) = c(b_j) = 1$ if $j > 1$ is odd, and $c(a_j) = c(b_j) = \Delta$ if j is even.

Suppose first that k is odd (so $k \geq 3$); this situation is depicted in Figure 10. Consider the vertex a_k; then $c(a_k) = 1$ and $c(a_{k-1}) = c(b_{k-1}) = \Delta$. Let $S \subseteq V(G)$ consist of the union of all the vertices in levels 0 through $k-1$ of T. Because of our choice of C and r, $G[S] \cong T[S]$, with all leaves of $G[S]$ at level $k-1$. Since G has girth at least 6, a_k is not joined to two vertices in level $k-1$ that have a common parent. Therefore, we may choose to add vertices to T in an order that ensures that every level $k-1$ neighbour of a_k (in G) receives colour Δ. Any other neighbour of a_k in G must be a child of a_k in T; since a_k has at most $\Delta - 2$ children, no child of a_k is coloured $2\Delta - 2$. Therefore, no neighbour of a_k (in G) is coloured $2\Delta - 2$.

We now recolour vertices $r, a_1, a_2, \dots, a_k$ as follows: set $c(a_k) = 2\Delta - 2$; for j odd, $1 \leq j \leq k-2$, set $c(a_j) = \Delta$; for j even, $2 \leq j \leq k-1$, set $c(a_j) = 1$; set $c(r) = 1$. This colouring is still proper: recolouring a_k with colour $2\Delta - 2$ presents no problems, and the effect of the rest of the recolouring is to exchange colours 1 and Δ along the path (in T) from a_1 to a_{k-1}, making colour 1 available for r. To see that the colouring is distinguishing, observe that $r, a_2, a_4, \ldots, a_{k-1}$ are the only vertices of colour 1 in the partition Y, r has two neighbours coloured $\Delta - 1$ (the two rightmost children of r in T), and each of $a_2, a_4, \ldots, a_{k-1}$ has only one neighbour coloured $\Delta - 1$. This pins r, and is sufficient to pin all the remaining vertices in G.

The case when k is even is analogous, but because of a parity shift, we begin with $c(a_k) = \Delta$ and $c(a_{k-1}) = c(b_{k-1}) = 1$. In this case, vertices can be added to T in an order that ensures that no neighbour of a_k (in G) is coloured $\Delta - 1$. The recolouring is as in the case k odd, except a_k is coloured $\Delta - 1$. The remainder of the argument follows, and this completes Case 3b.

In all cases we have successfully eliminated colour $2\Delta - 1$, thus showing $\chi_D(G) \leq 2\Delta - 2$. □

5 Conclusion

We have proven that if G is a connected bipartite graph with $\Delta \geq 3$, then $\chi_D(G) \leq 2\Delta - 2$, unless $G \cong K_{\Delta-1,\Delta}$ (in which case $\chi_D(G) = 2\Delta - 1$) or $G \cong K_{\Delta,\Delta}$ (in which case $\chi_D(G) = 2\Delta$). There are infinitely many graphs for which $\chi_D = 2\Delta - 2$; Collins and Trenk [3] give a construction for an infinite class of such graphs, and by choosing appropriately, it is easy to construct infinitely many bipartite graphs with $\chi_D = 2\Delta - 2$. However, as Collins and Trenk [3] observe, this construction relies heavily on subgraphs isomorphic to $K_{\Delta-1,\Delta-1}$. This leads to the problem of characterizing those connected bipartite graphs for which $\chi_D = 2\Delta - 2$. In particular, does every such graph contain an induced subgraph isomorphic to $K_{\Delta-1,\Delta-1}$?

Our focus has been on bipartite graphs: we have shown that the only *bipartite* graph with $\Delta \geq 3$ and $\chi_D = 2\Delta - 1$ is $K_{\Delta-1,\Delta}$. It is natural to ask: do there exist non-bipartite connected graphs with $\Delta \geq 3$ and $\chi_D = 2\Delta - 1$?

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